Public goods, participation constraints, and democracy:
A possibility theorem*

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Abstract It is well known that ex post efficient mechanisms for the provision of indivisible public goods are not interim individually rational. However, the corresponding literature assumes that agents who veto a mechanism can enforce a situation in which the public good is never provided. This paper instead considers majority voting with uniform cost sharing as the relevant status quo. Efficient mechanisms may then exist, which also satisfy all agents’ interim participation constraints. In this case, ex post inefficient voting mechanisms can be replaced by efficient ones without reducing any individual’s expected utility. Intuitively, agents with a low willingness to pay have to contribute more under majority rule than under an efficient mechanism with a balanced budget. This possibility theorem is not universal in the sense of Schweizer (Games and Economic Behavior, 2005).

Keywords: Public goods, ex post efficiency, participation constraints, majority voting, possibility theorem.

JEL classifications: D02, D61, D71, H41.
1 Introduction

This paper studies potential improvements of decision procedures for public projects. It investigates whether ex post inefficient voting mechanisms can be replaced by ex post efficient ones without reducing any individual’s expected payoff. Expected payoffs are evaluated at the interim stage, where individuals already hold private information about their willingness to pay for the public project. I show that there are cases in which, at the interim stage, all agents unanimously prefer an ex post efficient mechanism to simple majority voting.

Mechanism design theory has produced several very useful procedures for collective decision making. Two prominent examples are the second price auction and, as a generalization, the Vickrey Clarke Groves (VCG) mechanism. While the second price auction is frequently used for the allocation of indivisible private goods, the VCG mechanism for indivisible public projects can rarely be found in practice.

A standard argument why VCG mechanisms are not used for decisions about public projects is that they are too costly for some citizens. In deed, it is a well known result that ex post efficient mechanisms for the provision of indivisible public goods are generally not interim individually rational\(^1\). Under a VCG mechanism, agents who do not care about the public good may have to contribute to finance its provision or, alternatively, they have to pay for the externality that they generate when the good is not provided. However, the corresponding literature assumes that agents who veto a mechanism can enforce a situation in which the public good is never provided. Today, in most economies public goods are provided and mechanisms for the provision of public goods are used. Therefore, the question whether existing mechanisms can be replaced by ex post efficient ones can not be answered on the grounds of the existing literature.

Frequently, decisions about public goods are based on some variant of a voting mechanism. Such voting mechanisms also violate some agents’ participation constraints when the cost of the public good has to be shared by all agents. In practice, the procurement of a public good will either lead to an increase of the tax burden for all individuals or to a reduction of the budget for future expenditure. This budget effect creates an opportunity cost of public spending. In this paper, I assume that the cost of the public good is

shared equally among all agents. Agents decide whether they are willing to replace an existing voting procedure by an efficient mechanism after learning their own type, but before playing the voting game.

A voting mechanism asks all players for a binary signal and monotonically maps these signals into a decision. Not all voting mechanisms yield a possibility result. The possibility result holds for simple majority rule. This mechanism requires that a majority of agents favors the procurement of the public good, while, in case of a tie, the decision is made randomly. With such a mechanism in place, an alternative mechanism may exist, which implements the efficient project choice, satisfies all types’ interim participation constraints and has a balanced budget. Intuitively, the reason is that agents with a low willingness to pay have to contribute more under majority rule than under an efficient mechanism with a balanced budget.

In section 2, I analyze the basic case of two agents with uncorrelated, uniformly distributed types. This case yields a simple analytical solution for agents’ interim utility. Section 2 also has the main result - a possibility theorem for intermediate cost parameters. The paper then studies in detail the robustness of this possibility result. It turns out that the possibility theorem is not universal in the sense of Schweizer (2005). Universality would require that the participation constraints can be fulfilled independently of the distribution of types. In section 2 I show that it depends on the cost of the public project whether the possibility theorem holds or not. Section 3 presents additional possibility results for normally distributed types and larger populations.

In Sections 4, I further discuss the robustness of these findings in various settings. I first elaborate on the case of a large population. Possibility results for skewed distributions disappear when the population increases. This may also be the case when types are distributed symmetrically.

When the distribution of types is common knowledge, and when the number of agents is large, one could always directly implement the efficient decision. This is not possible when types are correlated. In section 4.2, I study three examples with correlated types. When types are correlated, the probability of provision is no longer monotonic in an agent’s type. The first example extends previous possibility results in a straightforward way to a class of correlated distributions. Skewed distributions instead yield impossibility results for large populations. The reason is that agents with a low willingness to pay

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2 With majority voting as a status quo, the type with the smallest interim gain from a new mechanism need not be located at the boundary of the type space. This is why Schweizer’s (2005) method to derive (universal) possibility theorems can not be applied to the present problem.
can rely on the fact that a majority opposes the provision of the public good when the median of the distribution of types is smaller than the mean. However, when the shape of the actual distribution is not known to the agents in advance, a possibility result with positive welfare gains may obtain - even in large populations.

Related literature

The present paper contributes to the literature on the ex post efficient provision of public goods, including in particular the papers by Güth and Hellwig (1986), Mailath and Postlewaite (1992), and Hellwig (2003). Güth and Hellwig (1986) prove an impossibility result à la Myerson Satterthwaite (1883) in the public goods context. Mailaith and Postlewaite (1992) consider the case with a large number of agents when the size of the project varies with the size of the population. They find that the second best surplus of the economy goes to zero when the population size grows. Hellwig (2003) instead considers the case where the size of the project does not grow unboundedly. Positive (possibility) results can be found in papers which assume correlated types such as McAfee and Reny (1985). However, if one introduces stricter equilibrium concepts such as Bergemann and Morris’s (2005) robustness criterion, the results are again negative (Bierbrauer and Hellwig, 2008).

Another way of guaranteeing participation in efficient mechanisms which has recently been studied in the literature, is the bundling of various decisions (Casella, 2005, Fang and Norman 2005, 2008, and Jackson and Sonnenschein, 2007).

The present analysis follows closely an idea that has been developed in Cramton, Gibbons and Klemperer (1987). Their paper shows in a trade context that initial conditions are key in determining whether or not a possibility or an impossibility theorem à la Myerson and Satterthwaite holds. In another related paper, Schmitz (2002) also considers the case of a public project. He shows that a possibility theorem may obtain when the status quo is a stochastic decision with an exogenously fixed provision probability. In such a case, an agents’ interim status quo utility is linear in his type while the VCG interim utility is strictly convex. A possibility theorem holds because the VCG interim utility of the indifferent type (i.e. the type whose gross willingness to pay equals the average cost) is positive. This agent’s payoff equals the average externality that he generates. As will become clear below, the status quo utility in the present paper is piecewise linear instead which makes the formal analysis more complicated.
2 The case of a uniform distribution

2.1 The setup

This section studies the case with \( n = 2 \) agents with independently and uniformly distributed types. This case has a simple analytical solution. Cases with a larger number of agents will be discussed in sections 3 and 4. The notation introduced here will also be used in the rest of the paper. Consider two agents, \( i = 1, 2 \) who decide on the provision of a non-rival indivisible public good. The decision is denoted by \( x \in \{0, 1\} \) and agents’ payoff derived from the consumption of the good is

\[
    u_i(x, \theta_i) = \theta_i x. \tag{1}
\]

The parameter \( \theta_i \) is distributed uniformly on the unit interval. It is private information of agent \( i \). An agent’s total payoff \( v_i \) is additively separable in \( u_i(x, \theta_i) \) and a monetary transfer \( t_i \) that is paid to the agent.

\[
    v_i(x, \theta_i, t) = u_i(x, \theta_i) + t_i. \tag{2}
\]

A planner can procure the good at a known cost of \( C = n \cdot c < 2 \).

2.2 The status quo

I consider four alternative status quo situations. First, the conventional one, called no provision. In this situation the public good is never provided - independently of the realization of both agents’ private information.

Under the second status quo mechanism (maximum rule) both agents simultaneously and independently cast a vote \( \gamma_i \in \{0, 1\} \). The good is bought if and only if at least one agent votes in favor of provision, i.e. \( x = \max \{\gamma_2, \gamma_2\} \). Moreover, in case of provision, each agent has to cover half of the cost, \( c \). The corresponding direct mechanism would ask both agents for their type and buys the good if and only if one agent’s valuation is at least \( c \). This mechanism is incentive compatible but, in general, it is not ex post efficient.

Under the third mechanism (majority rule with tie breaking) agents also cast votes \( \gamma_i \in \{0, 1\} \). The good is bought if both agents vote in favor of provision, it is not bought when both agents vote against. In case of a tie the good is provided with probability \( 1/2 \). When the good is provided, each agent has to cover half of the cost, \( c \). Finally, I consider provision under unanimity rule, i.e. \( x = \min \{\gamma_2, \gamma_2\} \).
2.3 Efficient Mechanisms

As an alternative to the status quo, I consider a mechanism which implements the ex post efficient decision in a Bayesian Nash equilibrium. I denote by \( x = f (\theta_1, \theta_2) \) the ex post efficient project choice and by \( U_i (\theta_i) \) the interim expected utility of agent \( i \) when \( f (\theta_1, \theta_2) \) gets implemented with a balanced budget. An AGV (Arrow, 1979, and d’Aspremont Gérard-Varet, 1979) mechanism balances the social planner’s budget for all realizations of agents’ types, a VCG mechanism can only balance the budget in expected terms. Both mechanisms yield identical interim expected payoffs.

The interim probability that the good is provided is given by

\[
\pi (\theta_i) = \begin{cases} 
0 & \text{if } \theta_i < 2c - 1 \\
1 - (2c - \theta_i) & \text{if } 2c - 1 \leq \theta_i \leq 2c \\
1 & \text{if } \theta_i > 2c
\end{cases}
\]  

\[
(3)
\]

2.4 How the analysis proceeds

The analysis in this section proceeds as follows. First, a well known result in mechanism design theory is used to calculate the difference \( U_i (\theta_i) - U_i (0) \) using \( \pi (\theta_i) \). Next, the expected payoff (excluding transfers) \( E_{\theta_{-i}}u_i (f (\theta_i, \theta_{-i}), \theta_i) \) for an efficient project choice is calculated. The difference gives us the transfers paid to each type \( \theta_i \),

\[
E_{\theta_{-i}}t_i = U_i (\theta_i) - E_{\theta_{-i}}u_i (f (\theta_i, \theta_{-i}), \theta_i) = U_i (0) + (U_i (\theta_i) - U_i (0)) - E_{\theta_{-i}}u_i (f (\theta_i, \theta_{-i}), \theta_i) .
\]

\[
(4)
\]

\[
(5)
\]

Requiring that \( E_{\theta_i} E_{\theta_{-i}}t_i = 0 \) yields the value of \( U_i (0) \) which balances the budget,

\[
U_i (0) = E_{\theta_i, \theta_{-i}} (u_i (f (\theta_i, \theta_{-i}), \theta_i) - (U_i (\theta_i) - U_i (0))) .
\]

\[
(6)
\]

Finally, the balanced budget interim utility \( U_i (\theta_i) \) is compared to the interim utility under majority voting \( M (\theta_i) \).

Two cases with low and with high costs \( c \) have to be distinguished.

2.4.1 The case where \( 2c \leq 1 \)

When \( 2c \leq 1 \) it may be the case that a single agent has a high enough willingness to pay \( (\theta_i \geq 2c) \) to make sure that the good is provided. Using a standard result, for \( \theta_i \leq 2c \) the interim expected utility (including transfers) can be calculated as
\[ U_i(\theta_i) : = E_{\theta_i} v_i(x(\theta_i, \theta_i, t(\theta))) \]
\[ = U_i(0) + \int_0^{\theta_i} (1 - (2c - s)) \, ds \]
\[ = U_i(0) + (1 - 2c) \, \theta_i + \frac{1}{2} \theta_i^2. \]

For values \( \theta_i > 2c \), \( U_i(\theta_i) \) increases linearly with slope 1 because the good is provided with certainty. To summarize, interim utility is given by the following expression.

\[ U_i(\theta_i) = \begin{cases} 
U_i(0) + (1 - 2c) \, \theta_i + \frac{1}{2} \theta_i^2 & \text{if } \theta_i < 2c \\
U_i(0) - \frac{1}{2} (2c)^2 + \theta_i & \text{if } \theta_i \geq 2c.
\end{cases} \]

Under an efficient mechanism, agent \( i \) derives an interim expected utility of

\[ E_{\theta_i} u_i(x, \theta_i) = \begin{cases} 
(1 - 2c) \, \theta_i + \theta_i^2 & \text{if } \theta_i < 2c \\
\theta_i & \text{if } \theta_i \geq 2c
\end{cases} \]

from the choice of the project. Taking differences, we obtain that an agent receives expected transfers of

\[ E_{\theta_i} t_i = \begin{cases} 
U_i(0) - \frac{1}{2} \theta_i^2 & \text{if } \theta_i < 2c \\
U_i(0) - 2c^2 & \text{if } \theta_i \geq 2c
\end{cases} \]

Total expected transfer payments to one agent are

\[ E_{(\theta_1, \theta_2)} t_i = U_i(0) + \int_0^{2c} -\frac{1}{2} \theta_i^2 \, d\theta_i + (1 - 2c) \left( -2c^2 \right) \]
\[ = U_i(0) + \frac{16}{6} c^3 - 2c^2. \]

The ex-ante probability that the good is provided is

\[ p(c) = \begin{cases} 
1 - \frac{(2c)^2}{2} & \text{if } 2c < 1 \\
2 (1 - c)^2 & \text{if } 2c \geq 1
\end{cases} \]

Hence, for \( 2c < 1 \) the budget is balanced in expected terms when

\[ -2U_i(0) - \frac{32}{6} c^3 + 4c^2 - \left( 1 - \frac{(2c)^2}{2} \right) 2c = 0, \]

where the last term represents the expected cost of provision. This last condition yields:

\[ U_i(0) = \frac{-2}{3} c^3 + 2c^2 - c. \]

Interim expected utility is given by (7) and (14).
2.4.2 The case where $2c > 1$

When $2c > 1$ it may be the case that a single agent’s willingness to pay is so low ($\theta_i < 2c - 1$) that the public good is never provided. The interim expected utility is given by

$$U_i(\theta_i) = \begin{cases} U_i(0) & \text{if } \theta_i < 2c - 1 \\ U_i(0) + \int_{2c-1}^{\theta_i} (1 - (2c - s)) \, ds & \text{if } \theta_i \geq 2c - 1 \end{cases}. \quad (18)$$

Under an efficient mechanism, agent $i$ derives an interim expected utility of

$$E_{\theta_i} u_i(x, \theta_i) = \begin{cases} 0 & \text{if } \theta_i < 2c - 1 \\ (1 - 2c) \theta_i + \theta_i^2 & \text{if } \theta_i \geq 2c - 1 \end{cases} \quad (19)$$

from the choice of the project. Taking differences, we get that he receives expected transfers of

$$E_{\theta_i} t_i = \begin{cases} U_i(0) & \text{if } \theta_i < 2c - 1 \\ U_i(0) - \frac{1}{2} \theta_i^2 + \frac{1}{2} (2c - 1)^2 & \text{if } \theta_i \geq 2c - 1 \end{cases}. \quad (20)$$

Total expected transfers made to one agent are

$$E_{(\theta_1, \theta_2)} t_i = U_i(0) + \int_{2c-1}^{1} -\frac{1}{2} \theta_i^2 + \frac{1}{2} (2c - 1)^2 \, d\theta_i \quad (21)$$

$$= U_i(0) - \frac{1}{6} + \frac{1}{6} (2c - 1)^3 + (2c - 1)^2 (1 - c) \quad (22)$$

The planner’s budget constraint yields:

$$U_i(0) = \frac{2}{3} c^3 - 2c^2 + 2c - \frac{2}{3}. \quad (23)$$

One can summarize the previous results as follows.

**Lemma 1** An agent’s interim expected utility under a balanced budget mechanism which implements the efficient project choice is given by

$$U_i(\theta_i) = \begin{cases} -\frac{2}{3} c^3 + 2c^2 - c + (1 - 2c) \theta_i + \frac{1}{2} \theta_i^2 & \text{if } c \leq 1/2 \text{ and } \theta_i < 2c \\ -\frac{2}{3} c^3 + 2c^2 - c - \frac{1}{2} (2c)^2 + \theta_i & \text{if } c \leq 1/2 \text{ and } \theta_i > 2c \\ \frac{2}{3} c^3 - 2c^2 + 2c - \frac{2}{3} & \text{if } c > 1/2 \text{ and } \theta_i < 2c - 1 \\ \frac{2}{3} c^3 - 2c^2 + 2c - \frac{2}{3} & \text{if } c > 1/2 \text{ and } \theta_i > 2c - 1 \\ + (1 - 2c) \theta_i + \frac{1}{2} \theta_i^2 + \frac{1}{2} (2c - 1)^2 & \text{if } c > 1/2 \text{ and } \theta_i > 2c - 1 \end{cases}. \quad (24)$$

Figure 1 displays the lowest interim utility $U_i(0)$ for various values of $c$. It shows that agents with the lowest valuation ($\theta_i = 0$) are never willing to participate in an ex post
efficient mechanism when the outside option is the no-provision decision. The highest loss arises for intermediate values of \( c \) because higher costs also reduce the efficient interim probability of provision.

Figure 1: Minimum interim utility \( U_i(0) \) as a function of the cost parameter \( c \).

### 2.5 A possibility theorem

I now compare agents’ interim utility under the VCG mechanism and under the three different voting rules. Under unanimity rule, interim utility \( M_i(\theta) \) is given by

\[
M_i(\theta_i) = \begin{cases} 
0 & \theta_i < c \\
(1 - c) (\theta_i - c) & \theta_i \geq c
\end{cases}
\]  

(25)

It immediately follows that \( U_i(0) < M(0) \). A consequence is the following impossibility result.

**Proposition 1** When the status quo is a vote under unanimity rule, the efficient project choice, a balanced budget and interim participation cannot simultaneously be satisfied.

A similar result holds under the maximum rule. Under this rule, interim utility \( \overline{M}(\theta_i) \) is also piecewise linear.

\[
\overline{M}(\theta_i) = \begin{cases} 
(1 - c) (\theta_i - c) & \text{if } \theta < c \\
(\theta_i - c) & \text{if } \theta \geq c
\end{cases}
\]  

(26)

This status quo gives an advantage to high types who can enforce their desired outcome alone. High types prefer this status quo to an efficient mechanism.
Proposition 2  When the status quo is a vote under the maximum rule, the highest type always opposes any balanced budget mechanism which implements the efficient project choice while the lowest type favors any such mechanism.

Proof  See appendix.

Consider next the interim utility under majority voting with a tie breaking rule. Interim expected utility is piecewise linear and given by

\[ M(\theta_i) = \begin{cases} \frac{1}{2} (1 - c) (\theta_i - c) & \theta < c \\ (1 - \frac{c}{2}) \ (\theta_i - c) & \theta \geq c \end{cases} \]  

The main result of this paper is that, for intermediate values of the cost parameter \( c \), a possibility theorem holds.

Proposition 3  For intermediate cost parameters \( c \in [c, \bar{c}] \) with \( 0 < c < 1/2 < \bar{c} < 1 \) we have:

(i) The efficient project choice can not be implemented with a balanced budget when all agents have to participate voluntarily at the interim stage and when the outside option is no provision,

(ii) The efficient project choice can be implemented with a balanced budget and satisfying the interim participation constraints when the outside option is a vote under simple majority rule.

(iii) For cost parameters \( c \) that are too high or too low, an impossibility theorem holds instead.

Proof  See the appendix.

Figure 2 compares the two interim utilities under a VCG mechanism and under majority rule for the case \( c = 1/2 \). The relative gain in expected welfare from a VCG mechanism is 33.3\%. Note that this would be the welfare gain with respect to all three voting mechanisms. However, it can only be realized when the status quo is simple majority rule. It is also worth noting that the expected Benthamian welfare in a democracy with simple majority rule exceeds welfare in the no provision status quo. Hence, the welfare gain from implementing the ex post efficient decision is higher under the no provision status quo, while implementation is possible in the democracy. Another interesting observation in the case \( c = 1/2 \) is that no participation constraint is binding. Accordingly, the planner could even raise extra revenues by organizing the public decision efficiently.

The previous results can be summarized as follows. When the status quo is a voting procedure, it may be possible to improve the decision process further by introducing...
a VCG mechanism. This yields a higher surplus and, for intermediate costs, a higher interim utility for all agents. Very strong or very weak majority requirements change this result. Under unanimity rule, everybody has to agree to the production of the public good. Therefore, the interim utility of low types is zero and the known impossibility theorem applies. When only one positive vote is required, the high types oppose the VCG mechanism.

Figure 2: Agents’ interim utility in a democracy (piecewise linear in the type \( \theta_i \)) and under a VCG mechanism (convex) for the case \( c = 1/2 \).

3 Normally distributed types

A possibility theorem also holds for another symmetric distribution, the normal distribution. In this section, I report some additional possibility results for normally distributed types. I consider \( n = m + 1 \) agents whose net willingness to pay \( \tilde{\theta}_i := \theta_i - c \) is normally distributed with mean zero and variance 1.\(^3\)

The analysis in this section does not require the case distinctions of the previous section. I use the following more convenient method to check whether a possibility theorem holds. First, I calculate the interim expected probability of provision \( \pi(\tilde{\theta}_i) \) for each net valuation \( \tilde{\theta}_i \). This probability yields the expected utility from the decision,

\[
E_{\theta_i} u_i(x, \tilde{\theta}_i) = \pi(\tilde{\theta}_i) \cdot \tilde{\theta}_i.
\]  

\(^3\)This implies that, ex ante, there is an expected benefit from consumption \((E\theta_i > 0)\), while some agents may actually dislike the public good at the interim stage \((\theta_i < 0)\).
Next, I calculate the expected externality generated by each type, \( h(\tilde{\theta}_i) \). An expected externality mechanism can redistribute the average expected revenues \( r = E_{\tilde{\theta}_i}(h(\tilde{\theta}_i)) \) among all agents. This yields the balanced budget interim utility \( \pi(\tilde{\theta}_i) \cdot \tilde{\theta}_i - h(\tilde{\theta}_i) + r \). Finally, this value is compared to the interim utility under majority rule, \( M(\tilde{\theta}_i) \).

With normally distributed types, the cumulative density of the sum of the \( m \) other agents’ net willingness to pay, \( \tilde{\theta}_{-i} \), is

\[
F(\tilde{\theta}_{-i}) = \int_{-\infty}^{\tilde{\theta}} \frac{1}{(m \cdot 2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2} \frac{\tilde{s}^2}{m}} ds = \frac{1}{2} \text{erf} \left( \frac{1}{2m} \sqrt{2m} \tilde{\theta} \right) + \frac{1}{2}. \tag{29}
\]

The interim probability of provision of the public good for a given net valuation \( \tilde{\theta}_i \) can then be expressed as follows.

\[
\pi(\tilde{\theta}_i) = 1 - F(-\tilde{\theta}_i) = \frac{1}{2} - \frac{1}{2} \text{erf} \left( \frac{1}{2m} \sqrt{2m} \tilde{\theta}_i \right). \tag{30}
\]

Agents’ direct utility from the decision is

\[
\pi(\tilde{\theta}_i) \cdot \tilde{\theta}_i = \left( \frac{1}{2} - \frac{1}{2} \text{erf} \left( \frac{1}{2m} \sqrt{2m} \tilde{\theta}_i \right) \right) \cdot \tilde{\theta}_i. \tag{31}
\]

An agent with type \( \tilde{\theta}_i \) generates an expected externality of

\[
h(\tilde{\theta}_i) = \int_{0}^{\tilde{\theta}_i} \frac{1}{(m \cdot 2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2} \frac{s^2}{m}} ds. \tag{32}
\]

Figure 3 displays the expected externalities for different values of \( m \) and different types \( \theta \).
The planner’s expected revenue per agent is

\[ r := \int_{-\infty}^{\infty} \left( \frac{1}{(m \cdot 2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2} \frac{s^2}{m}} \cdot h(\tilde{\theta}_i) \right) ds \quad (33) \]

This expected revenue can be redistributed to all agents. Hence, the interim expected payoff of an agent is given by \( \pi(\theta) \cdot \theta - h(\tilde{\theta}_i) + r \).

For odd values of \( n \), the interim payoff under majority vote is given by

\[
M(\tilde{\theta}_i) = \begin{cases} 
\left( \frac{1}{2} \right)^n \sum_{i=\frac{n}{2}+1}^{n} \binom{n}{i} \tilde{\theta}_i & \tilde{\theta}_i < 0 \\
\left( \frac{1}{2} \right)^n \sum_{i=\frac{n}{2}}^{n} \binom{n}{i} \tilde{\theta}_i & \tilde{\theta}_i \geq 0 
\end{cases}
\quad (34)
\]

and for even \( n \) it is given by

\[
M(\tilde{\theta}_i) = \begin{cases} 
\left( \frac{1}{2} \right)^n \left( \sum_{i=\frac{n+3}{2}}^{n} \binom{n}{i} \tilde{\theta}_i + \binom{n+1}{\frac{n}{2}} \tilde{\theta}_i \right) & \tilde{\theta}_i < 0 \\
\left( \frac{1}{2} \right)^n \left( \sum_{i=\frac{n+1}{2}}^{n} \binom{n}{i} \tilde{\theta}_i + \binom{n+1}{\frac{n}{2}} \tilde{\theta}_i \right) & \tilde{\theta}_i \geq 0 
\end{cases}
\quad (35)
\]

Figure 4 makes clear why possibility theorems may hold in this particular setup. It shows the interim decision utility net of costs \( E_{\tilde{\theta}_i} u_i(x, \tilde{\theta}_i) \) as a function of an agent’s types. The utility is not monotonic because very low types expect that their willingness to pay may change the decision in their favor. This is different under a voting mechanisms where the interim payoff is strictly monotonic and unbounded from below.

![Figure 4](image-url)

Figure 4: Interim decision utility \( \pi \left( \tilde{\theta}_i \right) \cdot \tilde{\theta}_i \). Types are normally distributed, and \( n = 2 \).

I numerically approximated the values of the per capita expected revenue \( r \) given by (33) for various values of \( m \) using the lower integral sum. The values are reported in Table 1.
<table>
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<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>10</th>
</tr>
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<td>0.16502</td>
<td>0.19971</td>
<td>0.22401</td>
<td>0.47325</td>
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</table>

Table 1: Expected revenues from the expected externalities (per individual).

The following figures compare the interim utilities under majority voting and under the balanced budget VCG mechanism for a selection of population sizes $n$, taking these values into account. In the cases considered in Table 1, a possibility result holds.

Figure 5a: Interim payoffs for normally distributed types $\tilde{\theta}_i$, $n = 2$.

Figure 5b: Interim payoffs for normally distributed types $\tilde{\theta}_i$, $n = 3$.

Figure 5c: Interim payoffs for normally distributed types $\tilde{\theta}_i$, $n = 4$. 
4 Robustness

4.1 Large populations

The previous analysis shows that possibility results may hold in small populations when simple majority voting is the relevant status quo. How relevant are these results for larger populations? And what are the consequences when agents’ types are correlated? In this and in the following section, I discuss both questions in various settings.

When the number of agents grows, it may happen that a possibility result turns into an impossibility result and vice versa. To understand why this is the case, it is first useful to consider the case of a continuum of agents. When all agents’ types are drawn independently from the same density distribution, the actual distribution of types coincides with the underlying probability distribution. Moreover, the position of the distribution’s mean and median determine the outcome under majority voting and under a VCG mechanism. With a continuum of agents there are no externalities in an equilibrium of a VCG mechanism. Therefore, the allocation of any balanced budget VCG mechanism and that of the majority voting mechanism coincide whenever the median and the mean of the distribution \( \phi (\theta_i) \) are on the same side of the cost parameter \( c \). Hence, a (trivial) possibility theorem holds if and only if the mean \( \bar{\theta} \) and median \( \theta^M \) of the distribution of types are on the same side of the cost parameter \( c \).

A consequence of this and Proposition 3 (iii) is that the same distribution of types \( \phi (\theta_i) \) may yield a possibility result for small \( n \) but not for large \( n \). On the other hand, as \( n \) goes to infinity and when \( \theta^M = 1/2 < c \), the impossibility result in Proposition 3 (iii) turns into a (trivial) possibility result.

One can also show that, even when the distribution of types is symmetric, a possibility result need not be robust when one increases the number of agents. Figure 6 shows the difference of the interim payoffs for the case of a symmetric and uniform distribution of types and \( n = 3 \) agents (the calculations for this case are very similar to the ones in section 2). Some types above and some types below \( \theta_i = c = 0.5 \) prefer the status quo to the ex-post efficient mechanism. However, Figure 7 also makes clear that, even though the participation constraints do not hold for all types, the largest losses are small compared to the one that obtains with the conventional status quo. The highest loss of an agent amounts to about 0.8 % of the per capita cost \( c \) whereas it equals 50 percent in the case with a no-provision status quo.
4.2 Correlated types

When the distribution of types is common knowledge, and when the number of agents is large, one could always directly choose to implement the appropriate decision. This is why this case is of little interest. In the rest of this paper, I concentrate on the more interesting case of correlated types.

When agents’ types are correlated, the conventional formula (8) for agents’ interim utility can no longer be applied. Agents with a low willingness to pay can conclude that it is more likely that the other agents also have a low willingness to pay. This affects their beliefs about the outcomes of the majority voting mechanism and the efficient mechanism.

In this section, I present three examples with correlated types. All three examples consider the implementation of a VCG mechanisms which balances the social planner’s budget in expected terms. I first present a general possibility result for correlated types.
I then turn to one particular distribution of agents’ net willingness to pay, for which an impossibility theorem holds. Finally, I present a case with a continuum of agents and a possibility result.

4.2.1 Example 1: Hybrid densities

Consider a population of size \( n \) and a given uncorrelated distribution of agents’ net willingness to pay \( \tilde{\theta}_i := \theta_i - c \) with density \( \phi(\tilde{\theta}_i) \) for which a possibility theorem holds. Call \( \tilde{\theta} \) the vector of types. This density can be modified as follows to encompass cases with correlated types. Let \( 1 - \alpha \) be the probability with which the joint distribution is drawn independently according to \( \phi(\tilde{\theta}_i) \). Moreover, let the types of all agents coincide with probability \( \alpha \), i.e. \( \tilde{\theta}_i = \tilde{\theta}_j \) for all \( i, j = 1..n \). In the latter case, again assume that the \( \tilde{\theta}_i \) are drawn according to \( \phi(\cdot) \).

**Definition 1** A tupel \( (\phi(\tilde{\theta}_i), \alpha) \) of a density function \( \phi(\tilde{\theta}_i) \) and a correlation parameter \( \alpha \) is called a hybrid density.

A possibility theorem trivially applies for \( \alpha = 1 \) (perfect correlation). By assumption, a possibility theorem also holds for the case of independent types, \( \alpha = 0 \). For intermediate values of \( \alpha \), one has to show that the interim utility under an expected balanced budget VCG mechanism \( \hat{U}_i(\tilde{\theta}_i, \alpha) \) exceeds the interim utility under majority voting \( \hat{M}_i(\tilde{\theta}_i, \alpha) \) for all types.

Call \( \hat{u}_i(\tilde{\theta}_i, \alpha) \) the expected value of the interim decision utility \( u_i(f(\tilde{\theta}), \tilde{\theta}_i) \) of agent \( i \) under a VCG mechanism when the correlation parameter is \( \alpha \). It is not difficult to show that for all agents with valuations \( \tilde{\theta}_i \leq c \), \( \hat{u}_i(\tilde{\theta}_i, \alpha) = (1 - \alpha) \hat{u}_i(\tilde{\theta}_i, 0) \). This is so because the probability of provision is multiplied by the factor \( 1 - \alpha \). Moreover, no externality arises when all types coincide. Therefore, the expected Clarke tax paid by agent \( i \), \( \hat{h}_i(\tilde{\theta}_i, \alpha) \), is reduced by a factor of \( (1 - \alpha) \),

\[
\hat{h}(\tilde{\theta}_i, \alpha) = (1 - \alpha) \hat{h}(\tilde{\theta}_i, 0). \tag{36}
\]

The interim utility under majority voting \( \hat{M}_i(\tilde{\theta}_i, \alpha) \) also satisfies \( \hat{M}_i(\tilde{\theta}_i, \alpha) = (1 - \alpha) \hat{M}_i(\tilde{\theta}_i, 0) \) because with probability \( \alpha \) all agents have identical types and vote against provision of the public good.

Similarly, the interim decision utility of agents with valuations \( \tilde{\theta}_i > c \) is \( \hat{u}_i(\tilde{\theta}_i, \alpha) = (1 - \alpha) \hat{u}_i(\tilde{\theta}_i, 0) + \alpha \tilde{\theta}_i \). The probability of provision equals \( (1 - \alpha) \) times the original probability with an i.i.d. distribution of types plus \( \alpha \). The interim utility under majority
voting $\hat{M}_i(\tilde{\theta}, \alpha)$ then satisfies $\hat{M}_i(\tilde{\theta}, \alpha) = (1 - \alpha) \hat{M}_i(\tilde{\theta}, 0) + \alpha \tilde{\theta}_i$. Under the VCG mechanism, no taxes have to be paid when all types coincide. Therefore, conditional on $\tilde{\theta}_i > c$, the aggregate average revenues from a Clarke tax are again multiplied with a factor of $1 - \alpha$.

We may conclude that the expected average Clarke-tax revenue for a given correlation parameter $\alpha$ is $\hat{r}(\alpha) = (1 - \alpha) \hat{r}(0)$. This revenue can be redistributed to all agents. Consequently, for $\alpha < 1$, and $\tilde{\theta}_i \leq c$,

$$\hat{U}_i(\tilde{\theta}, \alpha) \geq \hat{M}_i(\tilde{\theta}, \alpha) \iff$$

$$(1 - \alpha) \left( E_u_i(\tilde{\theta}, 0) - \tilde{h}(\tilde{\theta}, 0) + \hat{r}(0) \right) \geq (1 - \alpha) \hat{M}_i(\tilde{\theta}, 0) \iff$$

$$(1 - \alpha) \hat{U}_i(\tilde{\theta}, 0) \geq (1 - \alpha) \hat{M}_i(\tilde{\theta}, 0) \iff$$

$$\hat{U}_i(\tilde{\theta}, 0) \geq \hat{M}_i(\tilde{\theta}, 0).$$

For $\alpha < 1$, and $\tilde{\theta}_i > c$,

$$\hat{U}_i(\tilde{\theta}, \alpha) \geq \hat{M}_i(\tilde{\theta}, \alpha) \iff$$

$$(1 - \alpha) \left( E_u_i(\tilde{\theta}, 0) - \tilde{h}(\tilde{\theta}, 0) + \hat{r}(0) \right) + \alpha \tilde{\theta}_i \geq (1 - \alpha) \hat{M}_i(\tilde{\theta}, 0) + \alpha \tilde{\theta}_i \iff$$

$$(1 - \alpha) \hat{U}_i(\tilde{\theta}, 0) + \alpha \tilde{\theta}_i \geq (1 - \alpha) \hat{M}_i(\tilde{\theta}, 0) + \alpha \tilde{\theta}_i \iff$$

$$\hat{U}_i(\tilde{\theta}, 0) \geq \hat{M}_i(\tilde{\theta}, 0).$$

To summarize we have:

**Proposition 4** Consider majority voting as the relevant status quo. Assume that the joint distribution of types is characterized by a hybrid density $(\phi(\tilde{\theta}_i), \alpha)$ with correlation parameter $\alpha < 1$. A possibility theorem for uncorrelated types $(\alpha = 0)$ holds if and only if a possibility theorem holds for all cases with correlated types $(\alpha > 0)$.

4.2.2 Example 2: Shifting densities

The second example with correlated types considers a different stochastic structure. Consider the case with two agents and the following specification of the joint probability distribution of types. Both agent’s types $\theta_i$ $(i = 1, 2)$ are drawn from the same uniform probability distribution $\phi(\theta_i | s)$ with mean $s$ and a support of size $2a$ with $a \in [0, 1]$. The parameter $s$ in turn is drawn from another uniform distribution which has a support of size $2 \cdot (1 - a)$ and mean $\tilde{\theta} = c = 1$. The support of the entire type space is the interval $[0, 2]$. This specification includes as special cases a setup with uncorrelated types $(a = 0)$.
and perfectly correlated types \((a = 1)\). One can easily see, that a possibility theorem holds for both cases. At \(a = 0\), the possibility theorem from section 3 applies. At \(a = 1\), a possibility theorem also holds because the allocation under an expected externality mechanism is the same as the one under majority voting.

I show by example that there may be intermediate values of \(a\) for which an impossibility theorem holds. Consider the case where \(a = 1/2\). For a symmetric distribution of types around \(c\), interim payoff differences of both mechanisms are also symmetric. We may restrict our analysis of interim utilities to the lower half of the support, \(\theta < 1\). Consider a VCG mechanism which redistributes expected revenues among both agents. For values \(\theta_i < \frac{1}{2}\) an agent can be sure that the public good will not be provided. The interim expected externality that such an agent generates can be calculated as

\[
\int_0^{\theta_i + \frac{1}{2}} \left( \int_1^{\min\{s + \frac{1}{2}, 2 - \theta_i\}} (x - 1) \, dx \right) \frac{1}{\theta_i} \, ds = \frac{1}{6} \theta_i^2. 
\]  

(45)

This expression takes into account that the relevant support of \(s\) is the interval \([\frac{1}{2}, \frac{1}{2} + \theta]\). Moreover, an externality arises only for types in the interval \([\frac{1}{2}, s + \frac{1}{2}]\).

Similarly, the expected externality for types \(\theta_i \in [\frac{1}{2}, 1]\) is given by

\[
\int_{\theta_i}^{\frac{1}{2}} \left( \int_1^{\min\{s + \frac{1}{2}, 2 - \theta_i\}} (x - 1) \, dx \right) \frac{1}{\theta_i} \, ds
\]

(46)

\[
= \frac{1}{\theta_i} \left( \int_{\frac{1}{2}}^{2 - \theta_i} \left( \int_1^{s + \frac{1}{2}} (x - 1) \, dx \right) ds + \int_{\theta_i}^{\frac{1}{2}} \left( \int_1^{2 - \theta_i} (x - 1) \, dx \right) ds \right)
\]

(47)

\[
= \frac{1}{\theta_i} \left( \frac{1}{2} (\theta_i - 1)^2 (2\theta_i - 1) - \frac{1}{6} (\theta_i - 1)^3 \right). 
\]

(48)

Integration of (45) and (48) on the interval \([0, 1]\) yields one half of the total expected externality generated by one agent. The total expected externality per agent is:

\[
2 \cdot \left( \int_0^{\frac{1}{2}} \frac{1}{6} \theta_i^2 d\theta_i + \int_{\frac{1}{2}}^1 \frac{1}{\theta_i} \left( \frac{1}{2} (\theta_i - 1)^2 (2\theta_i - 1) - \frac{1}{6} (\theta_i - 1)^3 \right) d\theta_i \right) = \frac{1}{2} - \frac{2}{3} \ln 2.
\]

(49)

The public good is provided with an interim probability of

\[
\pi (\theta_i) = \begin{cases} 
0 & \theta_i < \frac{1}{2} \\
\theta_i - \frac{1}{2} & \frac{1}{2} \leq \theta_i \leq \frac{3}{2} \\
1 & \theta_i > \frac{3}{2}.
\end{cases}
\]

(50)
Utility from the decision is

$$\pi (\theta_i) \cdot (\theta_i - 1) = \begin{cases} 0 & \theta_i < \frac{1}{2} \\ (\theta_i - \frac{1}{2}) (\theta_i - 1) & \frac{1}{2} \leq \theta_i \leq \frac{3}{2} \\ 1 & \theta_i > \frac{3}{2} \end{cases} .$$  \quad (51)$$

Under majority voting, the probability of provision for $\theta_i < 1$ is instead given by

$$\frac{1}{2} \int_{\frac{1}{2}}^{\theta_i + \frac{1}{2}} \frac{(s + \frac{1}{2} - 1) \ ds}{\theta_i} = \frac{1}{4} \theta_i,$$  \quad (52)$$

and an agent’s interim utility is $\frac{1}{4} \theta_i (\theta_i - 1)$.

Figure 3 depicts the interim utilities for the voting mechanism and for the VCG mechanism. The efficient project choice can not be implemented with an expected externality mechanism when all agents have to participate voluntarily at the interim stage and when the outside option is the no-provision decision. The same holds for simple majority rule. Types which are close enough to the mean of the type space conclude, that it is more likely, that the ex-post efficient mechanisms yields a positive decision about the public good than a majority vote. When they are not too close to the average cost $c$, the redistributed revenues of the efficient VCG mechanism do not compensate for this. However, one can also see that the aggregate loss of the opponents to such a mechanism is comparably small. This indicates that the second best mechanism yields a much higher surplus when the status quo is a vote under majority rule.

Figure 8: Interim utility in a democracy (differentiable) and with a VCG mechanism for types on the interval [0, 1].
4.2.3 Example 3: Large populations

In the previous example, I considered a particular parameterization of the distribution of types. This distribution of types can be characterized as follows. There is a first move of nature that determines the location of all agents’ type spaces. This is the choice of the parameter $s$. Then, there is a second move of nature, where each agent’s type is drawn from some distribution $\phi(\theta|s)$. In such a setup, and when the population is large, the shape of the distribution $\phi(\theta|s)$ determines whether a possibility theorem holds or not. When the distribution’s median and mean coincide, a trivial possibility theorem holds and no welfare gain can be realized compared to majority voting. When the median of the distribution $\phi(\theta|s)$ is smaller than its mean and when the mean is below the cost $c$, an impossibility theorem holds. In this case, a majority of agents favors majority voting, because it knows that its preferred outcome is realized, while this is not the case under a balanced budget expected externality mechanism. Therefore, there is little hope for a meaningful possibility theorem to hold in large distributions when the shape of the conditional distribution $\phi(\theta|s)$ is known in advance.

In the third example, I consider the case where the shape of the realized distribution of types is not determined in advance. In this example, the VCG mechanism yields a strictly higher surplus than majority voting. Moreover, it may satisfy a slightly weaker concept of individual rationality. The concept that I use can be defined as follows.

**Definition 2** Call $U(\theta_i)$ and $M(\theta_i)$ the interim utility under a VCG mechanism with an (expected) balanced budget and under majority voting respectively. A mechanism is called $\epsilon$-individually rational if and only if, for all types $\theta_i$, $U(\theta_i) \geq M(\theta_i) - \epsilon$.

I will now show that, for large enough $n$ there always is a distribution of all agents’ types such that an ex post efficient mechanism exist which is $\epsilon$-individually rational for arbitrarily small values of $\epsilon$. At the same time, an expected balanced budget VCG mechanism mechanism increases aggregate surplus by a non-negligible positive amount.

The trick used in this example is that the final distribution of types is also skewed, but that it is not known in advance, in which direction it will be skewed.

**Proposition 5** Consider majority voting as the relevant status quo. For large $n$ and for all $\epsilon > 0$, there always is a continuous density with correlated types such that (i) the efficient project choice can be implemented with a VCG mechanism with an expected balanced budget which is $\epsilon$ individually rational and (ii) the ex post efficient allocation generates a strictly higher surplus than majority voting. Moreover, (iii) the VCG mechanism is not individually rational when the outside option is the no-provision decision.
Proof Consider a case with only four possible realizations of agents’ types: 0, \( c - \epsilon \), \( c + \epsilon \), and \( 2c \). Assume that with probability 1/2 a majority of 51 percent has a type below \( c \). With probability 1/2 a majority of 51 percent has a valuation above \( c \). Moreover, if the majority’s valuation is below \( c \), with probability 1/2 the corresponding probability mass can be found at \( \theta = 0 \). If the majority’s valuation is above \( c \), the corresponding probability mass is allocated to \( 2c \) with probability 1/2.

Consider now first the interim probability that a majority voting mechanism chooses \( x = 1 \). It is 1/2 for all realizations of the valuation \( \theta_i \). Consider next the interim probability of provision under a VCG mechanism. When the valuation \( \theta_i \) is zero, there only is a probability of 25 percent that the good is actually provided. The same holds for a valuation which is slightly above \( c \). The reason is that only when the low valuation is \( c - \epsilon \) the good is provided with probability 1/2. A similar reasoning can be applied to show that the remaining two interim probabilities of provision are 3/4. The rest of the proposition follows from the fact that, with a continuum of agents, the expected externality is zero for all types. Q.E.D.

5 Conclusion

This paper studied an institutional reform: the replacement of a voting mechanism for the provision of a public good by an efficient VCG mechanism. Voting mechanisms do not always efficiently decide on the provision of public goods. The present paper shows that there are cases in which all agents unanimously accept switching to the new ex post efficient mechanism. Therefore, one may be more optimistic about efficient institutional reforms when the status quo is a voting decision which involves coercion to contribute to the cost of collective goods.

The possibility theorem of section 2 of this paper does not hold for all distributions of independent types. Moreover, possibility results need not be robust when the size of the population increases. Skewed distributions are more likely to produce impossibility theorems - in particular when types are correlated and when the shape of the distribution is known by all agents. Nevertheless, this paper is more optimistic about possible welfare gains than previous related studies. Even when ex post efficiency can not be achieved, the status quo may be relevant for the economy’s second best welfare level.

Moreover, in some of the cases considered, the actual number of informed agents who oppose an ex-post efficient mechanism is relatively small. The same holds for the size of their monetary losses. This indicates that majority requirements below unanimity rule
may enable voting systems to switch to efficient institutions.\footnote{Note that democracy itself would also not be interim individually rational for all agents in the present setup.}

The insight of Cramton, Gibbons, and Klemperer’s (1987) paper is that any analysis of institutional reform under asymmetric information has to take the relevant status quo into account. The present paper applied this insight to the field of public economics. A useful extension of the present analysis is to study other more complex institutions such as representative democracies, voluntary provision schemes, or lobbying for collective provision.

A second interesting option for further research is to consider less complex reform proposals for collective decision procedures. VCG mechanisms are often criticized for being overly complicated and too unintuitive. This may in deed be an obstacle to their implementation in practice. It would be useful to find out whether more intuitive welfare enhancing mechanisms can also satisfy agents’ participation constrains.

6 Appendix: Proof of Propositions 2 and 3

Proof of Proposition 2 First consider the case where $c \leq 1/2$. Interim expected utility $U_i(\theta)$ under the VCG mechanism is increasing and convex with a slope of 1 for valuations above total cost. Low valuations prefer the efficient mechanism while high valuations don’t. We have

\begin{align}
U(0) & \geq \bar{M}(0) \\
\Leftrightarrow & -\frac{2}{3}c^3 + 2c^2 - c \geq -(1 - c)c \\
\Leftrightarrow & \frac{3}{2} \geq c,
\end{align}

and

\begin{align}
U(1) & \geq \bar{M}(1) \\
\Leftrightarrow & -\frac{2}{3}c^3 + 2c^2 - c - 2c^2 + 1 \geq (1 - c) \\
\Leftrightarrow & -\frac{2}{3}c^3 \geq 0.
\end{align}
This contradicts $c > 0$. Next consider the case where $c > 1/2$. We have

$$
U(0) \geq \bar{M}(0) \quad (59)
$$

$$
\iff \frac{2}{3}c^3 - 2c^2 + 2c - \frac{2}{3} \geq -(1-c)c \quad (60)
$$

$$
\iff c \in \left[0, \frac{1}{4}\sqrt{33} + \frac{7}{4}, \infty \right). \quad (61)
$$

and

$$
U(1) \geq \bar{M}(1) \quad (62)
$$

$$
\iff \frac{2}{3}c^3 - 2c^2 + 2c - \frac{2}{3} + (1-2c) + \frac{1}{2} + \frac{1}{2}(2c-1)^2 \geq 1 - c \quad (63)
$$

$$
\iff c \in \left[-\frac{1}{2}\sqrt{3} - \frac{1}{2}, 0.36603 \right] \cup \left[1, \infty \right). \quad (64)
$$

This contradicts $c \in [0.5, 1]$. Q.E.D.

**Proof of Proposition 3**

(i) is well known.

(ii) It suffices to provide an example. Consider the case of $c = 1/2$. In this case the interim utilities are $-\frac{2}{3} + \frac{1}{2}\theta_i^2$. For $\theta < c$ the participation constraint becomes

$$
-\frac{2}{3} + \frac{1}{2}\theta_i^2 > \frac{1}{4}\left(\theta_i - \frac{1}{2}\right) \iff \frac{1}{2}\theta_i^2 - \frac{1}{4}\theta_i > -\frac{1}{24}. \quad (65)
$$

The l.h.s. has its minimum at $\theta_i = \frac{1}{4}$ with a value of $-\frac{1}{32}$. Hence, this condition holds for all $\theta_i$. For $\theta \geq c$ the participation constraint becomes

$$
-\frac{2}{3} + \frac{1}{2}\theta_i^2 > \left(1 - \frac{1}{4}\right)\left(\theta_i - \frac{1}{2}\right) \iff \frac{1}{2}\theta_i^2 - \frac{3}{4}\theta_i > -\frac{7}{24}. \quad (66)
$$

The left hand side has its minimum at $\theta_i = \frac{3}{4}$ with a value of zero. Hence, this condition holds for all $\theta_i$.

(iii) For cost parameters $c$ that are too high or too low, an impossibility theorem applies. To see this, consider first the case where $c < 1/2$. We have

$$
U(0) \geq \bar{M}(0) \quad (67)
$$

$$
\iff -\frac{2}{3}c^3 + 2c^2 - c \geq -\frac{1}{2}(1-c)c \quad (68)
$$

$$
\iff c \in [0.40693, 1.8431].
$$

\]
and

\[
U(1) \geq M(1) \quad (69)
\]

\[
\iff \quad -\frac{2}{3}c^3 + 2c^2 - c - 2c^2 + 1 > \left(1 - \frac{c}{2}\right)(1 - c) \quad (70)
\]

\[
\iff \quad c \in (0, 0.56873) . \quad (71)
\]

Outside the intersection of both intervals, a possibility theorem does not hold.

Next consider the case where \( c > 1/2 \). In this case we have

\[
U(1) \geq M(1) \quad (72)
\]

\[
\iff \quad \frac{2}{3}c^3 - 2c^2 + 2c - \frac{2}{3} + (1 - 2c) + \frac{1}{2} + \frac{1}{2} (2c - 1)^2 \geq \left(1 - \frac{c}{2}\right)(1 - c) \quad (73)
\]

\[
\iff \quad c \in \left[-\frac{1}{8}\sqrt{33} - \frac{1}{8}, 0.59307\right] \cup [1, \infty) . \quad (74)
\]

This is the empty set. Q.E.D.

References


