Transportation mode choice

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Abstract. This paper presents a dynamic model of transportation mode choice and evolution of public transportation service based on some simple assumptions of individual behavior and economic necessities for providing transportation service. Critical values are shown to exist for the fares charged, for the cost of providing service, for the demand and supply of transportation, and for other parameters at which the system will bifurcate to different possible states of the system; critical thresholds must be reached in the quality of the network to observe its growth. Also shown is the role of history and the role that fluctuations in individual behavior and mode strategy play in the way the system structures, that is, in the evolution of the relative number of users of each mode and in the level of service obtained.

1 Introduction
In a previous paper (Deneubourg et al, 1979), hereafter referred to as paper 1, we developed a model of transportation mode choice of two variables, the number of people who choose the automobile, and the number of people who choose public transportation. That model showed clearly how the system structures in response to individual behavior and showed the role that fluctuations play on the response of the system when different mode choices are made. However, the model did not take account of transportation costs and service.

In this paper, we extend the previous model by taking into account the service offered by the public transportation mode which is assumed to be a function of the number of users of the mode, of the fares charged by the mode, and of the costs of offering the service. By so doing, we shall find here the occurrence of an optimum level of service which is related to the fares charged. We also find the occurrence of critical values for the fares, for the cost of maintaining a level of service, for the demand for transportation, and for the publicity and strength of imitative behavior which will be chosen as bifurcation parameters for the evolution of the system.

As in the previous paper 1, we make no pretense at developing a model of transportation mode choice which captures all the decisions which go into such mode choices. The intention, rather, is to show how some decisions by individuals and by the public transportation company interact in a dynamic system and how fluctuations can lead to different evolutionary paths.

We also point out that a parallel approach has been developed by Wilson (1979) in the framework of catastrophe theory.

2 The model
As in paper 1, we make certain assumptions on individual behavior (or, more precisely, on the average behavior of a group of individuals) and through the nonlinearities in this behavior we shall find out how the system structures when thresholds for bifurcation are reached (that is, how the system evolves with regard to the relative number of users of each mode and to the level of service obtained).
For the bus attractiveness, \( A_2^{(1)} \), we take the form
\[
A_2 = \frac{L}{\nu^2(\theta + \alpha_2 y)}
\]  
which states that the attractiveness is proportional to the service offered, \( L \), the publicity or information, \( \theta \) (assumed positive), and the importance of imitative behavior measured by \( \alpha_2 y \), and inversely proportional to the second power of the fares, \( \nu \), charged. We note that the form used for the dependence of \( A_2 \) on the various parameters will affect the structure of the system (because of the dependence of the structure on the nonlinearities). However, as it is not our intention here to reproduce an experimental result, and only to show how the system structures when nonlinearities are present, we present these nonlinearities (dependence of \( A_2 \) on the parameters) as only reasonable possibilities. Though, we should point out that it is not difficult to alter these dependencies when sufficiently valid data justifies this.

If we further simplify the problem as in paper 1 by taking \( a = 0 \), and \( b = 1 \) in equation (4), which is equivalent to assuming a constant attractiveness for the car, we obtain as our system of dynamical equations
\[
\begin{align*}
\dot{x} &= D\alpha_1 \left[ \frac{\alpha_1 + \frac{L}{\nu^2}(\theta + \alpha_2 y)}{\nu^2(\theta + \alpha_2 y)} \right] - x, \\
\dot{y} &= \frac{DL}{\nu^2(\theta + \alpha_2 y)} \left[ \frac{\alpha_1 + \frac{L}{\nu^2}(\theta + \alpha_2 y)}{\nu^2(\theta + \alpha_2 y)} - y \right] - y, \\
\dot{L} &= \nu y - KL.
\end{align*}
\]  
This system may be solved analytically. One stationary state \((\dot{x} = 0, \dot{y} = 0, \dot{L} = 0)\) of the system is
\[
x = D, \quad y = 0, \quad L = 0,
\]  
which states that the total demand for transportation is provided by the automobile.

When a stability analysis is performed, with system (6) subjected to perturbations \( \delta x, \delta y, \delta L \) (see the appendix for the details of this analysis), it is found that the stationary state given by equations (7) is stable only when the costs, \( K \), for providing bus service, are above a certain critical value \( K_c \) given by
\[
K > K_c = \frac{D\theta}{\nu\alpha_1},
\]  
(8a)

since there is then no incentive for instituting bus service; or if the fares \( \nu \) charged are above a critical value \( \nu_c \), where
\[
\nu > \nu_c = \frac{D\theta}{\alpha_1 K},
\]  
(8b)

which represents the maximum fare people are willing to pay for the bus journey. Equations (7) will also be stable if the demand for transportation, \( D \), is below a certain critical value \( D_c \) given by
\[
D < D_c = \frac{\nu\alpha_1 K}{\theta},
\]  
(8c)

(1) In paper 1, the attractiveness for bus usage was given as proportional to the number of users, \( y \), the idea being that the more users there are, the more frequent would be the bus service and hence the shorter the waiting time. Here we take the attractiveness to be directly proportional to the quality of service, \( L \).
since an insufficient demand does not justify the initiation of a bus service. We also find that the all-car system (7) is stable when the imitative behavior for car usage, \( \alpha_1 \), is above a critical value \( \alpha_{1c} \) given by

\[
\alpha_{1c} > \alpha_{1c} = \frac{D \theta}{\bar{v} K},
\]

or the advertising for bus usage, \( \theta \), is below a threshold \( \theta_{c} \),

\[
\theta < \theta_{c} = \frac{\rho \theta K}{D}.
\]

In addition to the stationary state given by equations (7), there are two other possible stationary states of the system (6), given by

\[
y^x = \left( \frac{D - \theta \alpha_2}{\alpha_2} \right) \pm \frac{1}{2} \left[ \left( \frac{D - \theta \alpha_2}{\alpha_2} \right)^2 - \frac{4 \alpha_2 \rho K}{\alpha_2} \right]^{1/2},
\]

\[
x^x = \left( \frac{D + \theta \alpha_2}{\alpha_2} \right) \pm \frac{1}{2} \left[ \left( \frac{D + \theta \alpha_2}{\alpha_2} \right)^2 - \frac{4 \alpha_2 \rho K}{\alpha_2} \right]^{1/2},
\]

\[
L^x = \frac{\rho \theta}{K}.
\]

The stability analysis (see the appendix) shows that when these solutions are real and positive, the \((x^x, y^x, L^x)\) solution is stable and the \((x^c, y^c, L^c)\) solution is unstable. The implications of this for causing transitions between stable states will be seen in the next section when we discuss the results of a numerical example.

Solution (9) will be real (positive or negative) if the cost \( K \) for providing bus service is below a critical value \( K^* \):

\[
K < K^* = \frac{\alpha_2 (D + \theta \alpha_2)^2}{4 \rho \theta}.
\]

Both solutions will exist physically if \( K > K_c = DB/\alpha_1 \bar{v} \) [see equation (8a)] and \( D > \theta \alpha_2 \). If \( K < K_c \) with \( D > \theta \alpha_2 \) only the stable solution \((x^c, y^c, L^c)\) will be positive (will exist physically). This is illustrated in the schematic of figure 1.

Similarly, the solution (9) will be real if the fares \( \nu \) are less than a critical value \( \nu^c \):

\[
\nu < \nu^c = \frac{\alpha_2 (D + \theta \alpha_2)^2}{4 \rho \theta}.
\]

**Figure 1. Schematic showing different regions for stability and sign of roots.**

\( D \) is the demand for transportation which is assumed to be so slowly varying compared to the time variation of \( x \) and \( y \) that it may be considered as constant, that is to say, no new users are brought into the system during the time of interest.

\( A_1 \) is the attractiveness for the automobile which we take to be its speed, with the imitative behavior of people included in this term as well; and

\( A_2 \) is the attractiveness for the bus which will involve the service offered by the bus mode, the fares charged, the amount of advertisement for bus usage, and also the imitative behavior of individuals.

The rationale of these dynamical equations is fully explained in paper 1, and we only note here that \( x + y = D - (x + y) \), so that in the steady state one has \( D = x + y \). The third equation to complete our system is for the evolution of bus service and is assumed to be given by:

\[
L = \nu v - K L,
\]

where \( \nu \) is the fare charged (and thus \( \nu v \) is the revenue received) and \( K \) is the maintenance cost per unit of service offered. The equation simply states that bus service will grow in time if revenues exceed the cost of providing service.

It now remains to give explicit representations to the attractiveness \( A_1 \) and \( A_2 \) in equation (1). For the automobile attractiveness function, \( A_1 \), we assume as in paper 1,

\[
A_1 = v_1 \alpha_1 x
\]

where \( v_1 \) is the automobile speed and \( \alpha_1 \) measures the strength of the imitative term \( \alpha_1 x \). Also, as in paper 1, we take the speed to be an inverse function of \( x \) (congestion effect see Haight, 1963; Herman and Prigogine, 1979), and neglect the traffic interaction between cars and buses:

\[
\frac{1}{a + bx}
\]

where \( a \) and \( b \) are positive constants.
and will be positive if \( \nu > \nu_c = D\theta/\alpha_1 K \) and \( D > \theta/\alpha_2 \). When \( \nu < \nu_c \) only the stable solution will physically exist.

In terms of the demand for transportation, \( D \), the condition that the solution (9) be real is that there be a sufficient demand \( D^c \),

\[
D > D^c = \left( \frac{4\alpha_1 \nu K}{\alpha_2} \right)^{\frac{1}{3}} - \frac{\theta}{\alpha_2},
\]

(10c)

and when the demand \( D > \theta/\alpha_2 \), but less than \( D_c = \alpha_1 \nu K/\theta \), both solutions will exist. If \( D > D_c \) only the stable solution will be positive.

In terms of imitative behavior for the car, \( \alpha_1 \), there will be two real roots when

\[
\alpha_1 < \alpha_1^c = \alpha_2 \left( \frac{D + \frac{\theta}{\alpha_2}}{4\nu K} \right)^{\frac{1}{2}} \]

(10d)

with both roots positive when \( D > \theta/\alpha_2 \) and \( \alpha_1 > \alpha_1^c = D\theta/\alpha_2 \), and one positive and one negative root when \( D > \theta/\alpha_2 \) but \( \alpha_1 < \alpha_1^c \).

Finally, in terms of the bus publicity term \( \theta \), there will be two real roots when

\[
\theta > \theta^c = (4\alpha_1 \alpha_2 \nu K)^{\frac{1}{3}} - \alpha_2 D
\]

(10e)

with both roots positive when \( \theta < D/\alpha_2 \) and \( \theta < \theta_c = \alpha_1 \nu K/D \), and only one positive root if \( \theta > \theta_c \).

These conditions, in terms of parameter \( K \), are summarized in the schematic of figure 1. The implications of this kind of system structure will be illustrated in the next section with a numerical example.

3 Discussion of a numerical example
In this section we discuss the structure of the system as a function of the parameters of the system, and the numerical values of the parameters will be given in the figures.

3.1 Fares
Figure 2 shows the three variables of the system, the number of people choosing the car mode, \( x \), the number choosing the bus mode, \( y \), and the service offered by the bus mode, \( L \), as a function of increasing fares \( \nu \) charged by the bus company.

As fares increase but remain below a critical value only one stable stationary state is possible, the \((x^*, y^*, L^*)\) state. The other stationary state \((x = D, y = 0, L = 0)\) exists but is unstable in this low fare regime, so that any perturbation from this state no matter how slight, will cause the system to jump to the stable state (and we note that in the real world perturbations always are occurring). In this regime, the

![Figure 2. Mode choice and quality of service versus fares.](image-url)
number of bus users decrease (because of the increasing fares), the bus service
improves (because of increasing revenues), and the number of automobile users
increases (corresponding to the decrease in the number of bus users).

As fares continue to increase beyond \( v_c \), but still remain below \( v^* \), two stable
stationary states and one unstable stationary state exist in the system. In the stable
\((x^-, y^*, L^+)\) state the number of bus users continues to decline with increasing fares,
and there is a corresponding increase in the number of car users. Bus service
continues to improve with increasing fares, but then reaches a maximum and begins
to rapidly deteriorate because the increasing fares being charged cannot make up for
the resultant loss of passengers.

The optimum fare \( v_m \) that should be charged for the best service may be computed
analytically. This is most easily done by first finding the optimum number of
passengers for producing maximum service. This is obtained from

\[
\frac{\partial L}{\partial y} = \frac{\nu}{K} + \frac{y \partial \nu}{K} = 0.
\]

We obtain for the optimum number \( y_m \)

\[
y_m = \frac{1}{2} \left\{ D - \frac{\theta}{\alpha_2} + \left[ \left( D - \frac{\theta}{\alpha_2} \right)^2 + \frac{3D \theta}{\alpha_2} \right]^{1/2} \right\}.
\]

The optimum fare \( v_m \) is then given by

\[
v_m = \frac{\alpha_2}{\alpha_1 K} \left[ D \theta + y_m \left( D - \frac{\theta}{\alpha_2} \right) - y_m^2 \right]^{-1/2}
\]

where we have used equation \((9)\) to obtain \( \nu \) as a function of \( y \). We also note that
equation \((11)\) may be put into the form

\[
\frac{dy}{y} = \frac{dv}{\nu} = -1,
\]

which shows that the maximum \( L \) is achieved when the elasticity is \(-1\).

We now point out the consequences of this kind of structure of two stable
stationary states and one unstable state on the response of the system to fluctuations.

As the fares continue to increase in this regime \( v_c < v < v^* \), it becomes more and
more likely that a fluctuation in the number of car users or of the number of bus users
will be found that will cause the system to jump to the state that has zero bus users.

Finally, for still higher fares with \( v \) exceeding the critical value \( v^* \), the system
becomes insensitive to perturbations, and adopts the \((x = D, y = 0, L = 0)\)
stationary state which is stable in this regime of high fares.

3.2 Cost of providing service

Figure 3 shows the bifurcation diagram as a function of costs of providing services \( K \).

When the costs are below the critical value \( K_c \), there is one stable stationary state
in which buses and cars coexist. The all-car solution is unstable in this regime of low
costs.

As might be expected, the level of bus service decreases with increasing costs \((2)\),
and hence the number of bus users decreases with a corresponding increase in the
number of car users.

As costs continue to rise, in the range \( K_c < K < K^* \), the system can exist in one
of two possible stable stationary states. If the system is in the \((x^-, y^*, L^+)\) state, the
chances of remaining there diminish with increasing costs, as the strength of a

\((2)\) Because of the form of the steady state solution \( L = \nu y / K \), a rather unrealistic result is obtained
for very low costs as \( K \to 0 \).
perturbation which could cause the system to jump to the \((x = D, y = 0, L = 0)\) state decreases with increasing costs.

When costs exceed the critical value \(K_c\), no one chooses the bus and only the \((x = D, y = 0, L = 0)\) stationary state is stable.

![Diagram](image_url)

**Figure 3.** Mode choice and quality of service versus costs.

3.3 **Publicity for the bus**

Figure 4 shows the relationship between \(x, y,\) and \(L\) and the amount of publicity or advertising for bus usage.

The system has one stable stationary state below a critical value \(\theta^c\), two stable stationary states in the range \(\theta^c < \theta < \theta_c\), and one stable stationary state for \(\theta\) beyond \(\theta_c\). The figure points to the need to exceed a critical value \(\theta^c\) before people will choose the bus mode, but once this critical value is exceeded, further publicity has very little effect on increasing ridership. However, if the system happens to be in the all-car state, increasing publicity does have a strong effect on increasing the likelihood that a fluctuation will cause the system to change to the mixed mode state. And when \(\theta > \theta_c\) any perturbation will cause the system to jump to the mixed mode state.

![Diagram](image_url)

**Figure 4.** Mode choice and quality of service versus publicity.
same high level of publicity for the system to remain in this state, as is evident from
the figure. This is an example of the phenomenon of hysteresis.

3.4 Demand for transportation
Figure 5 shows the effect of total demand for transportation $D$ on mode choice $x$
and $y$ and on bus service $L$.

When there is an insufficient total demand for transportation, no bus service is
offered. Not until a critical value of demand $D^e$ is exceeded is a bus service offered.
The zero bus service state, however, is still a possible stable stationary state in the
range of demand $D^e < D < D_c$, but becomes increasingly less likely because smaller
and smaller fluctuations can cause the system to jump to the mixed mode solution.

For still higher demand, $D > D_c$, the zero bus users state becomes unstable. In
the stable stationary state, the number of car users declines with increasing demand
as congestion effects become more pronounced, the number of bus users increases as
people leave their cars, and the bus service improves as more revenues are received
because of the increased bus usage.

![Graph showing mode choice and quality of service versus total demand for transportation]

**Figure 5.** Mode choice and quality of service versus total demand for transportation.

3.5 Imitative terms
Figure 6 shows the bifurcation diagram as a function of the strength $\alpha_1$, of the
imitative behavior for car usage.

Below a critical strength $\alpha_{1,c}$, the mixed mode solution is the only stable
stationary state in which car usage, as expected, increases with increasing $\alpha_1$.

In the range $\alpha_{1,c} < \alpha_1 < \alpha_1^*$, the all-car mode solution also becomes a stable state
and when $\alpha_1 > \alpha_1^*$, the all-car mode becomes the only possible stationary state of the
system.

Figure 7 shows the effects of the strength $\alpha_2$ of imitative behavior for bus use on
mode choice.

Below a critical value $\alpha_2^*$ only one solution is possible, namely the all-car state;
not until sufficiently strong imitative behavior for bus usage exists do people choose
the bus mode.

With increasing strength of imitative behavior for the bus, $\alpha_2 > \alpha_2^*$, two solutions
become possible—the all-car state and the mixed mode state. Although, in this case,
the all-car state remains a stable stationary state, it becomes less and less likely that the
system, if in this state, will remain there because as $\alpha_2$ becomes stronger only
small positive fluctuations in the number of bus users or in the bus service are needed to cause the system to move from the all-car state to the mixed mode state.

solid lines: stable stationary state
dashed lines: unstable stationary state
parameter values: $\alpha_e = 2$, $\theta = 30$, $\nu = 45$, $K = 25$, $D = 100$
critical values: $\alpha_{1,e} = 2.7$, $\alpha_1^* = 5.9$

Figure 6. Mode choice and quality of service versus imitative behavior for car usage.

solid lines: stable stationary state
dashed lines: unstable stationary state
parameter values: $\alpha_1 = 5$, $\theta = 30$, $\nu = 45$, $K = 25$, $D = 100$
critical value: $\alpha_2^* = 1.6$

Figure 7. Mode choice and quality of service versus imitative behavior for public transportation.

3.6 Trajectories
The different situations discussed in the previous paragraphs are depicted in an $(L, y)$ phase space to help visualize trajectories in the three qualitatively different situations which can occur.

Figure 8 depicts the situation when there is one stable node at $(0, 0)$; figure 9 shows the case when there is one stable node at $(0, 0)$ and two nontrivial stationary states: a stable node $(x^*, y^*, L^*)$ and a saddle point $(x^*, y^*, L^-)$; and figure 10 depicts the situation when there is a saddle point $(0, 0)$ and a nontrivial stable node $(x^*, y^*, L^*)$. We note that the nontrivial stationary states lie on the line $y = kL/\nu$.

In the case of figure 8, all trajectories converge to the stable node $(0, 0)$ so that once the system reaches this stationary state, it loses all memory of perturbations and of its initial state.

In the situation of figure 9, we have drawn the separatix, S, which separates two regions, A and B. A point in region B is within the 'sphere of influence' of the stable node $(0, 0)$: whatever the initial conditions in this region, the system will evolve to the stable node $(0, 0)$. 
Figure 8. Trajectories in $(L, y)$ phase space, with one stable node at $(0, 0)$.

graph showing trajectories with parameter values: $\alpha_1 = 5, \alpha_2 = 2, \theta = 30, \nu = 45, K = 40, D = 100$

Figure 9. Trajectories in $(L, y)$ phase space, with one stable node at $(0, 0)$, one stable node at $(x^n, y^n, L^n)$, and one saddle point at $(x^s, y^s, L^s)$.

graph showing trajectories with parameter values: $\alpha_1 = 5, \alpha_2 = 2, \theta = 30, \nu = 45, K = 23, D = 100$

Figure 10. Trajectories in $(L, y)$ phase space, with one saddle point at $(0, 0)$ and one stable node at $(x^n, y^n, L^n)$.

graph showing trajectories with parameter values: $\alpha_1 = 5, \alpha_2 = 2, \theta = 30, \nu = 45, K = 8, D = 100$
If, however, the system is in the \((0, 0)\) stationary state, a critical size perturbation which will bring the system beyond the separatix into region A is necessary to reach the nontrivial stationary state \((x^*, y^*, L^*)\).

We remark that this says that a critical size investment in public transport is necessary to bring the system away from the 'all' car situation; below that critical value, the investment is lost.

Figure 10 may be considered as a limiting case of figure 9: in figure 10 whatever the size of the perturbation from \((0, 0)\), the system will evolve to the nontrivial stationary state \((x^*, y^*, L^*)\).

4 Conclusions
In this paper we have presented a dynamic model of transportation mode choice in which mode choice was based on individual behavioral characteristics and on the service offered by the mode.

In examining the stationary states of the system and the stability of these states to fluctuations, we found the existence of critical values of the parameters (fares charged, cost of providing service, demand for transportation, etc) at which the system bifurcated to a new solution.

For some range of the parameters we found that only one of two possible states of the system was stable to fluctuations and hence in this range the system would adopt one of the two possible states (the stable one).

In another range of values of the parameters we found the existence of two stable states separated by an unstable one. This kind of structure, in which two stable states exist, points to the role of history (through the initial states and through fluctuations) in determining which state the system will adopt (since either one is theoretically possible).

Further, this kind of structure also points to the importance of fluctuations in influencing the behavior of a system as sufficiently strong fluctuations can cause the system to jump from one stable state to another. The size of the fluctuation needed depended on the closeness of the unstable state to one of the stable ones which, in turn, depended upon the values of the parameters of the system.

We point out that the concept of self-organization which appears under certain conditions involving the feedback between a system and its environment, springs from the work done in nonlinear thermodynamics (Nicolis and Prigogine, 1977) and has found specific applications in biology, ecology, and the social sciences.

References
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Appendix: Stability analysis
The dynamical equations of our system are (see section 2)
\[
\begin{align*}
\dot{x} &= D\alpha_1 \left[ \alpha_1 + \frac{L}{v^3}(\theta + \alpha_2 y) \right] - x, \\
\dot{y} &= \frac{DL}{v^3}(\theta + \alpha_2 y) \left[ \alpha_1 + \frac{L}{v^3}(\theta + \alpha_2 y) \right] - y, \\
\dot{L} &= vy - KL.
\end{align*}
\]
We do a linear stability analysis, subjecting this system to perturbations \( \delta x, \delta y, \) and \( \delta L \) around a stationary state. We assume the variables to have a time dependence of the form \( \exp(i\omega t) \). The stability of the stationary state will depend upon the sign of the real part of \( \omega \): if \( \text{Re}(\omega) > 0 \) the system will amplify the fluctuation and hence be unstable to fluctuations; if \( \text{Re}(\omega) < 0 \) the fluctuations are damped and the system is stable to the fluctuation.

The resulting set of equations when we only consider the case \( \delta x = -\delta y \) are:

\[
\begin{align*}
(\omega + 1)\delta y &= \frac{D\alpha_1}{\nu^2}[\theta (\alpha_2 + \alpha_2 y) \delta L + \alpha_2 L \delta y] \left( \frac{\theta L}{\nu^2} + \frac{\alpha_2 y L}{\nu^2} \right)^2, \\
(\omega + K)\delta L &= \nu \delta y.
\end{align*}
\]

One stationary state of system (A1) is given by \((x = D, y = 0, L = 0)\). In this case, equations (A2) reduce to

\[
\begin{align*}
(\omega + 1)\delta y &= \frac{D\theta}{\alpha_1 \nu^2} \delta L, \\
(\omega + K)\delta L &= \nu \delta y.
\end{align*}
\]

When these are solved simultaneously, we find as the equation for \( \omega \)

\[
\omega^2 + \omega(K + 1) + K - \frac{D\theta}{\nu \alpha_1} = 0.
\]

Hence we find that the stationary state \((x = D, y = 0, L = 0)\) is a stable node if

\[
K > \frac{D\theta}{\nu \alpha_1},
\]

and a saddle point if

\[
K < \frac{D\theta}{\nu \alpha_1},
\]

as given in equation (8a) of section 2. This same stability relationship gives the conditions of stability in terms of the other parameters of the system [see equations (8b)–(8e)].

The stability of the other stationary states is obtained in a similar manner, they are the solutions of

\[
\begin{align*}
y^2 + y(D - \theta \alpha_2) + \frac{D\theta}{\alpha_2} - \frac{\alpha_1 \nu K}{\alpha_2} = 0, \\
x = D - y, \\
L = \frac{\nu y}{K}.
\end{align*}
\]

The solutions \((x^-, y^+, L^+)\) and \((x^+, y^-, L^-)\) are given in equations (9).

Solving equations (A2) simultaneously gives as the equation for \( \omega \):

\[
\omega^2 + \omega \left[ K + 1 - \alpha_1 \alpha_2 D y + \frac{D\theta}{\nu K} \left( \frac{\theta y}{\nu K} + \frac{\alpha_2 y^2}{\nu K} \right)^2 \right] + K - D\alpha_1 (\theta + 2 \alpha_2 y) \left( \frac{\theta y}{\nu K} + \frac{\alpha_2 y^2}{\nu K} \right)^2 = 0.
\]

For positive values of the parameters, the stationary state given by \((x^-, y^+, L^+)\) of equations (9) is a stable node \([\text{Re}(\omega) < 0]\) whereas the stationary state \((x^+, y^-, L^-)\) is a saddle point whenever the variables \(x, y, L\) are positive.