

## Dynamic models of competition between transportation modes

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**Abstract.** We present a methodology to study the choice of transportation mode. It is dynamic and allows inherent fluctuations in individual behavior to play a role. The methodology has a deterministic aspect which yields the time evolution of the number of users for a transportation mode and it is based on certain assumptions about the attractivities of each mode, but it is always subject to fluctuations in human behavior. The problem then is to find the solutions of the deterministic equations describing the system and to examine the evolutionary path the system will take. Two models of competition between transportation-mode choice are developed to illustrate the methodology and to justify the dynamic approach to travel choice.

### 1 Introduction

The models which we shall present are models of choice among different transportation modes. Our purpose here is not to develop a model of transportation choice which reflects the actual and complex decisions which go into such choices by different groups of individuals. Rather it is to present a methodology which is dynamic and which allows inherent fluctuations in the behavior of the individuals to play a role (and often it is a fundamental one) in the determination of the way the system responds to different mode choices.

We consider the case of choice between two modes of transportation. Let  $x$  and  $y$  be the number of individuals who choose transportation modes 1 and 2 respectively. Let  $A_1$  and  $A_2$  be the attractivities of transportation modes 1 and 2. Let  $D$  be the estimated number of people who want to go from point A to point B, and assume in this first approach that  $D$  is constant (though in reality  $D$  is a function of the locational processes). When the system reaches a stationary state we have

$$\frac{x}{D} = \frac{A_1}{A_1 + A_2} = P_1 \quad \text{and} \quad \frac{y}{D} = \frac{A_2}{A_1 + A_2} = P_2 \quad (1)$$

In general  $A_1$  and  $A_2$  are functions of  $x$  and  $y$ . System (1) can have more than one solution. It is not possible to know without some additional information which solution the system will adopt. In fact we must provide the system with information on its dynamic evolution in order for it to integrate different historical occurrences which will determine the final solution adopted by that system. In general the system will remember its initial conditions ( $x_0$  and  $y_0$ ) and the perturbations, both external and internal, which have occurred in its history. In this case only the densities of  $x$  and  $y$  will be subjected to perturbations.

### 2 Development of the dynamic equations

Taking an approach often used in ecology, we write the law of evolution of the variable  $z$ , defined by

$$z = x + y \quad (2)$$

as

$$\dot{z} = D - z, \quad (3)$$

where  $\dot{z} = dz/dt$  ( $t$  denotes time). In equation (3) the parameter  $D$ , which is a constant, is usually called the 'carrying capacity'. The value of the carrying capacity determines the final state reached by the system.

For the variables  $x$  and  $y$  we assume the same form for the equations of evolution:

$$\dot{x} = D_1 - x \quad \text{and} \quad \dot{y} = D_2 - y, \quad (4)$$

$D_1$  and  $D_2$  being unknown functions which must be determined. Since equation (2) holds for all times  $t$ , we have

$$D_1 + D_2 = D. \quad (5)$$

The equations of evolution of the variables  $x$  and  $y$ , given equations (1), will therefore be

$$\dot{x} = \frac{DA_1}{A_1 + A_2} - x \quad \text{and} \quad \dot{y} = \frac{DA_2}{A_1 + A_2} - y. \quad (6)$$

Note that the carrying capacities in the equations for the evolution of the variables  $x$  and  $y$  are functions of time.

In order to illustrate the behavior of the equations (6) we have sketched the evolution of the variables  $x$  and  $y$  for the case when  $P_1 = P_1(x, y)$  is given (figure 1). In general  $P_1(x, y)$  is not known. Figure 1 shows graphically that  $z$  tends asymptotically to  $D$ , as is obtained by integration of equation (3) over time. The curve represented by  $DP_1(t)$  is taken as given and the curve labelled  $DP_2(t)$  is obtained from this by use of the fact that  $P_1 + P_2 = 1$ . The curves for  $x(t)$  and  $y(t)$ , which may be obtained by numerical integration of equations (6), are also sketched in the figure. It is apparent that the solutions  $x(t)$  and  $y(t)$  tend asymptotically to  $DP_1$  and  $DP_2$  respectively, and, for all times  $t$ ,  $x + y = z$ .

Note that when  $P_1$  and  $P_2$  are given by equations (1) it is not possible in general to compute analytically the solutions  $x(t)$  and  $y(t)$  of equations (6). However, mathematical techniques do exist which provide information on the evolution of the system. For our purposes we may use bifurcation analysis, which yields information on which possible final solutions are accepted by the system as well as information on their stability properties when subjected to perturbations in the densities of  $x$  and  $y$ .

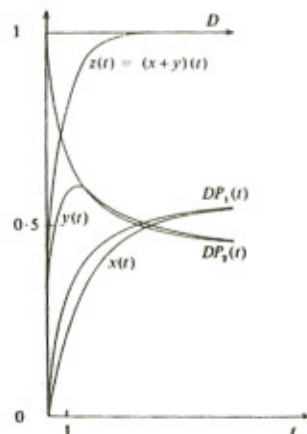


Figure 1. Evolution of  $x$  and  $y$  with time.

### 3 The theoretical models

The models presented here do not pretend to take into consideration all decisions affecting mode choice. We prefer instead to develop simpler models which may be computed analytically. This is done in order to be able to show some of the interesting properties which appear when  $P_1$  and  $P_2$  are functions of the state of the system [see equations (1)].

#### 3.1 The first model

In order to define the model we must give a particular form to the attractivity functions,  $A_1$  and  $A_2$ . We suppose that these functions depend solely on the speed of transport. We assume that

$$A_1 = v_1^p \quad \text{and} \quad A_2 = v_2^q, \quad (7)$$

where  $v_1$  and  $v_2$  are the velocities of modes 1 and 2 respectively, and  $p$  and  $q$  are positive exponents. We suppose, for example, that mode 1 corresponds to the automobile and mode 2 to the bus. We further assume, in this example, that there is no interaction between the two transportation modes (it is not difficult, however, to remove this restriction).

Figure 2(a) shows the assumed dependence of the velocity of cars on their density (the congestion effect—see Haight, 1963). For the velocity-density relationship for buses we assume in addition that the supply tends to adjust to the demand; that is, as more people demand bus transit, more buses are put into service, which results in a reduced overall time of transit (waiting plus riding time). This relationship is sketched in figure 2(b).

We may fit the curves of figures 2(a) and 2(b) respectively by the following analytic expressions:

$$v_1 = \frac{1}{a + bx} \quad \text{and} \quad v_2 = \frac{dy^n}{c + y^r s}, \quad (8)$$

where  $a, b, c, d, s, n$ , and  $r$  are positive constants and  $n \leq r$ .

For example, if we take for the velocities of mode 1 (the car) and mode 2 (the bus) the following:

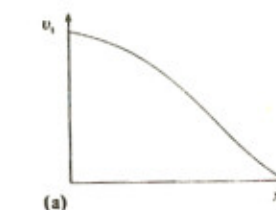
$$v_1 = \frac{1}{a + x} \quad \text{and} \quad v_2 = \frac{dy}{c + y}; \quad (9)$$

and for the exponents of equations (7) the following:

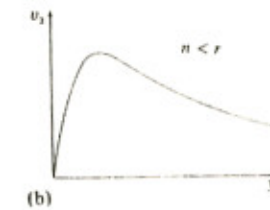
$$p = q = 1; \quad (10)$$

the equations for the time evolution of transportation modes 1 and 2, equations (6) then become

$$\dot{x} = \frac{D}{a+x} \left( \frac{1}{a+x} + \frac{dy}{c+y} \right) - x \quad \text{and} \quad \dot{y} = \frac{Ddy}{c+y} \left( \frac{1}{a+x} + \frac{dy}{c+y} \right) - y. \quad (11)$$



(a)



(b)

It is easily verified that the final stationary states of the system ( $\dot{x} = 0, \dot{y} = 0$ ) are such that

$$x_s + y_s = D \tag{12}$$

Using this relationship to find the values of the stationary states, we see from equations (11) that the state ( $x = D, y = 0$ ) is a stationary state. We can therefore write the equation giving the values of the other stationary states in the following form:

$$dx^2 + (1+da)x - (c+D) = 0 \tag{13}$$

Then system (11) has the following three stationary solutions:

$$x' = D, \quad y' = 0; \tag{14}$$

$$x^+ = \frac{-(da+1) + [da+1]^2 + 4(c+D)d]^{1/2}}{2d}; \tag{15}$$

$$x^- < 0. \tag{16}$$

The solution ( $x^-, y^-$ ) is physically not acceptable because  $x$  is positive (or zero). The solution ( $x^+, y^+$ ) is physically acceptable only if [see equation (12)]

$$D \geq x^+. \tag{17}$$

We may put this condition in the form

$$D \geq \frac{1}{2} \left[ -a + \left( a^2 + \frac{4c}{d} \right)^{1/2} \right]. \tag{18}$$

Solutions (14) and (15) are represented graphically in figure 3.

We thus see that, for a sufficiently large transit density,  $D$ , the system accepts a solution other than given by  $x' = D, y' = 0$  (all cars). In fact, in this example, the system can tell us which solution will be adopted even if we do not know its historical evolution. The system will adopt a solution only if it is sufficiently stable to fluctuations in the densities  $x$  and  $y$ . The laws introduced describe only the average behavior of the densities, but perturbations around this average behavior are inevitable. In our example a stability calculation (Nicolis and Prigogine, 1977) shows that the solution ( $x', y'$ ) becomes unstable if the density,  $D$ , becomes large enough (whether caused by fluctuations or by other means). All perturbations ( $\delta x, \delta y$ ) around a stationary state are assumed to vary with time according to the function  $\exp(ut)$ . The stability of the stationary state will depend on the sign of  $u$ . If it is positive the system is unstable to perturbations; if it is negative the system is stable

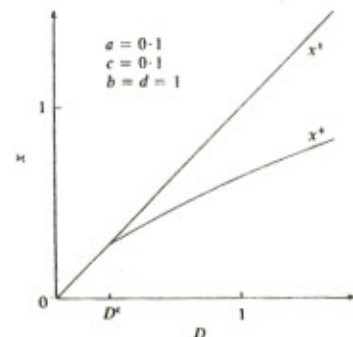


Figure 3. Solutions  $x^1$  and  $x^+$  of equations (14) and (15).

to perturbations. Then the solution ( $x', y'$ ) becomes unstable when

$$D \geq \frac{1}{2} \left[ -a + \left( a^2 + \frac{4c}{d} \right)^{1/2} \right] = D^c. \tag{19}$$

Note that this stability condition is identical to the condition for existence of the solution  $x^+$  in this case.

Figure 4 presents the bifurcation diagrams showing the different final solutions the system may adopt and their stability as a function of the parameter  $D$ . We see in figure 4(b) that if  $D < D^c$  then the transit density is not sufficiently large for the initiation of a bus service: the only stationary state permitted by the system is the state ( $x' = D, y' = 0$ ). However, for higher densities  $D > D^c$ , the share of people taking the bus mode,  $y/D$ , increases.

Note that internal perturbations near a stationary state ( $x_s, y_s$ ) are such that

$$\delta x \ll x_s \quad \text{and} \quad \delta y \ll y_s. \tag{20}$$

In the case where  $y_s = 0$ , the perturbation must be introduced as an external factor (corresponding to a new transportation mode). The theory developed here thus tells us the conditions under which the system becomes unstable with regard to the introduction of a new transportation mode. The condition in which the system accepts this new mode of transportation ( $D > D^c$ ) obviously depends on the characteristics

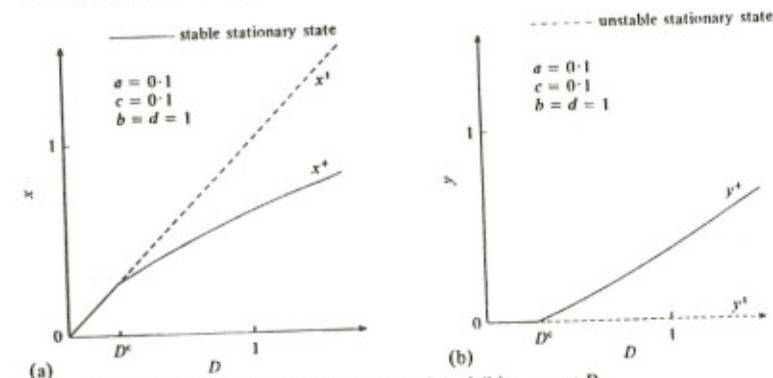


Figure 4. Bifurcation diagrams of (a)  $x$  versus  $D$  and (b)  $y$  versus  $D$ .

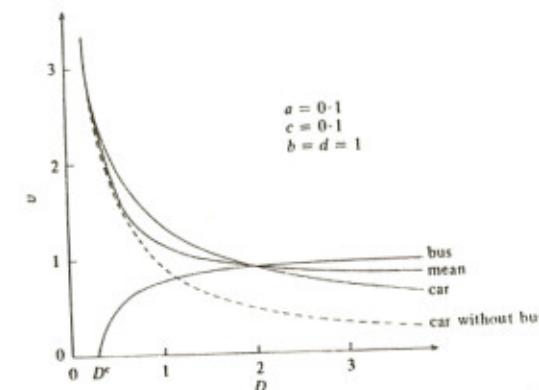


Figure 5. Car, bus, and average velocities in the system as functions of the transit density,  $D$ .

of the existing transportation mode (the parameter  $a$ ) and on the characteristics of the new one (the parameters  $c$  and  $d$ ) [see equations (9) and (19)]. The fundamental role played by the bifurcations has been illustrated by this example.

Figure 5 sketches the evolution of the velocities of each transportation mode and the average velocity in the system as functions of the density of transit,  $D$  (computed when the system has reached the stationary state).

### 3.2 The second model

In the first model we introduced a classical effect for the attractivity function, namely that, as the speed of a transportation mode increases, the attractivity of that mode increases. There are other factors, psychological for example, which also influence the choice. Publicity and increased information about a particular mode, for instance, may influence an individual's choice. In a similar vein the process of imitation for people taking a particular mode may partially explain some existing situations. We will show in this section that these kinds of effects can considerably increase the richness of the behavior of the system.

We now introduce into the attractivity functions, psychological factors,  $F_i$ , to obtain

$$A_1 = v_1^p F_1 \quad \text{and} \quad A_2 = v_2^q F_2. \quad (21)$$

For the functions  $F_i$  we take the following simple forms:

$$F_1 = \theta_1 + \alpha_1 x \quad \text{and} \quad F_2 = \theta_2 + \alpha_2 y, \quad (22)$$

where  $\theta_1$  and  $\theta_2$  are publicity terms and  $\alpha_1 x$  and  $\alpha_2 y$  are imitation terms. For the dependence on the velocity, we use the same forms as before [equations (8)], and in order not to complicate the problem these velocities will in this case be simplified to

$$v_1 = \frac{1}{x} \quad \text{and} \quad v_2 = y. \quad (23)$$

We also take

$$p = q = 1 \quad (24)$$

in equations (21). With these values the equations of evolution, equations (6), become

$$\left. \begin{aligned} \dot{x} &= D \left( \frac{\theta_1}{x} + \alpha_1 \right) \left( \frac{\theta_1}{x} + \alpha_1 + \theta_2 y + \alpha_2 y^2 \right) - x \\ \text{and} \\ \dot{y} &= D(\theta_2 y + \alpha_2 y^2) \left( \frac{\theta_1}{x} + \alpha_1 + \theta_2 y + \alpha_2 y^2 \right) - y. \end{aligned} \right\} \quad (25)$$

Note that we have taken all parameters to be positive (it is in fact certainly possible to have, for example, negative publicity terms). Figure 6 shows the attractivities of the two modes as functions of  $x$  and  $y$ .

We shall now discuss the case in which  $\theta_1 = 0$  (no publicity for the car). As can be seen from equations (21), (22), (23), and (24), this case yields a constant attractivity for the car mode,  $A_1 = \alpha_1$ .

Using equation (12) and the fact that  $(x' = D, y' = 0)$  is a stationary state, we find that the other stationary state of system (25) will be given by

$$\alpha_2 y^2 + (\theta_2 - \alpha_2 D)y + (\alpha_1 - D\theta_2) = 0. \quad (26)$$

This equation will have no, one, or two physically acceptable solutions with the following properties.

1. If the publicity term for the bus is large enough,  $\theta_2 > (4\alpha_1\alpha_2)^{1/2}$ , equation (26) will have two real roots. For  $\theta_2 < (4\alpha_1\alpha_2)^{1/2}$ , equation (26) will have two real solutions only

if the transit density is high enough,  $D > D^c$ , where

$$D^c = \frac{(4\alpha_1\alpha_2)^{1/2} - \theta_2}{\alpha_2}. \quad (27)$$

2. If equation (26) has two real solutions,  $y^+$  and  $y^-$ , then the sign of these roots will depend upon the relative magnitude of the transit density with respect to the two critical densities  $D_1^c$  and  $D_2^c$  defined by

$$D_1^c = \frac{\theta_2}{\alpha_2} \quad \text{and} \quad D_2^c = \frac{\alpha_1}{\theta_2}. \quad (28)$$

If  $D < D_1^c$  and  $D < D_2^c$ , equation (26) has two negative roots;

$D_1^c < D < D_2^c$ , equation (26) has two positive roots;

$D > D_2^c$ , equation (26) has one positive root and one negative root.

3.  $D^c < D_2^c$  whatever the values of the parameters.

4. For large values of the publicity parameter for the bus,  $\theta_2 > (\alpha_1\alpha_2)^{1/2}$ , we have

$$D^c < D_1^c, \quad \text{and} \quad D_1^c > D_2^c. \quad (29)$$

For small values of the bus publicity parameter,  $\theta_2 < (\alpha_1\alpha_2)^{1/2}$ , we have

$$D_1^c < D_2^c \quad \text{and} \quad D^c > D_1^c. \quad (30)$$

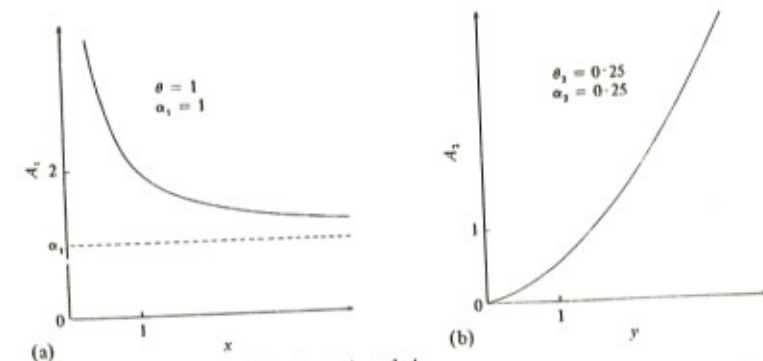


Figure 6. The attractivity functions,  $A_1$  and  $A_2$ .

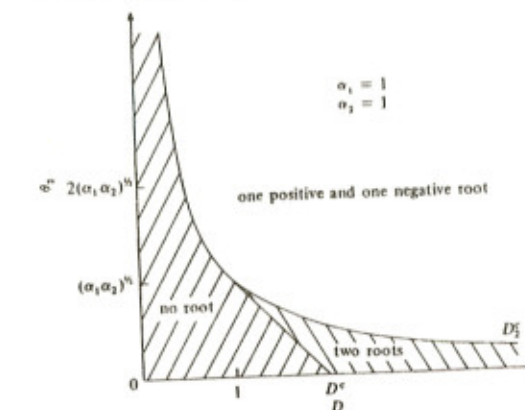


Figure 7. Conditions for solutions of equation (26).

Figure 7 summarizes these conditions, showing the various conditions for the solutions of equation (26).

There are qualitatively two different bifurcation diagrams for the solutions  $y$  of equation (26). These are shown in figure 8. Figure 8(a) is similar to figure 4(b) and so will not be discussed further. Figure 8(b), where  $\theta_2 < (\alpha_1 \alpha_2)^{1/2}$ , however, represents a qualitatively new situation. In this case we might say that the nonlinear term is more important than the linear term. For a density  $D^c < D < D_2^*$ , the system can accept two stationary states,  $y^1$  and  $y^+$ . (The stationary state  $y^-$  is unstable and cannot therefore be considered physically as a final state since perturbations will always cause the system to move away from this state.)

Let us say that a perturbation in the density  $y$ , of value  $\Delta y_1$  (for a given value of the traffic density,  $D_1$ ), is needed to bring the system from the stationary state  $(x^1, y^1)$  to the stationary state  $(x^+, y^+)$ . In figure 8(b) we see that the value of the perturbation,  $\Delta y_1$ , for  $D^c < D < D_2^*$ , needed for this transformation decreases as density of transit,  $D_1$ , increases. This points to the role of history in determining which stationary state the system adopts. Further, if  $D_1 > D_2^*$ , whatever the value of the perturbation ( $\Delta y > 0$ ), the system will spontaneously go to the stationary state  $(x^+, y^+)$  since the state  $(x^1, y^1)$  is unstable when  $D > D_2^*$ . We note that the bifurcation parameter  $D$  measures the feedback effect in the system. When the feedback parameter is sufficiently small,  $D < D^c$ , the system has only one stationary state. However, if  $D$  is sufficiently large,  $D > D^c$ , a qualitatively new stationary state appears in the system. In general, as  $D$  increases, the number of possible stationary states of the system increases.

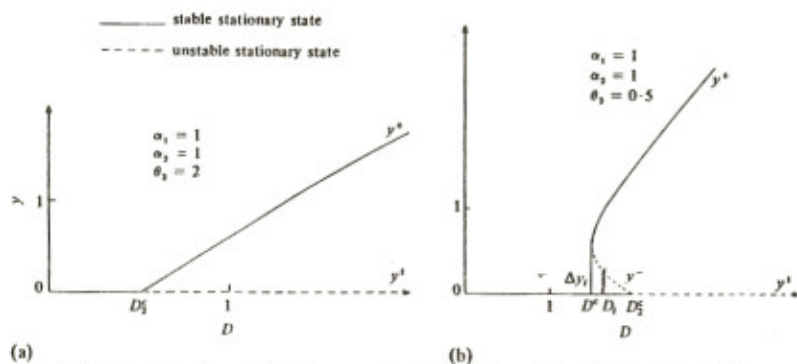


Figure 8. Bifurcation diagrams for the cases when (a)  $\theta_2 > (\alpha_1 \alpha_2)^{1/2}$  and (b)  $\theta_2 < (\alpha_1 \alpha_2)^{1/2}$ .

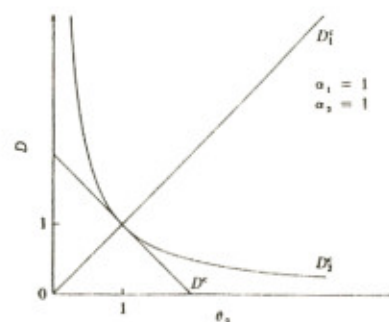


Figure 9. Density,  $D$ , versus the bus publicity factor,  $\theta_2$ .

Finally, figure 9 shows under what conditions there may be coexistence between the two modes of transportation. For constant  $\alpha_1$  and  $\alpha_2$ , a state of coexistence between the bus and the car will appear first (for small values of  $D$ ) for the case when there is good publicity,  $\theta_2 > (\alpha_1 \alpha_2)^{1/2}$ , and only later for the case when there is little bus publicity,  $\theta_2 < (\alpha_1 \alpha_2)^{1/2}$ , which requires larger values of the traffic density,  $D$ , for there to be coexistence.

#### 4 Conclusions

The methodology and models presented in this paper have illustrated the importance of behavioral fluctuations in determining the stability of competing modes of transportation. The bifurcation diagrams introduced in the text to illustrate the feedback effects resulting when travel choice is allowed to be a function of the state of the system provide information on the stability of the system to such fluctuations in human behavior. Some stationary states are seen to be unstable even to small fluctuations, whereas others, though locally stable, would become unstable if a sufficiently large fluctuation occurred. The system would then adopt a new solution which is stable to perturbation. This adaptive emergence is one example of the concept of order by fluctuation (Nicolis and Prigogine, 1977), whereby a system reorganizes itself into a new mode of behavior when critical size thresholds for stability are exceeded.

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