*Transpn. Res.-B* Vol. 19, No. 2, pp. 143–153, 1985
Printed in the U.S.A.

0191–2615/85 $3.00 + .00
Pergamon Press Ltd.

# NOISY DEMAND AND MODE CHOICE

DAVID KAHN
Transportation Systems Center, Cambridge, MA 02142, U.S.A.

ANDRÉ DE PALMA
McMaster University, Hamilton, Ontario, Canada

and

JEAN LOUIS DENEUBOURG
Université Libre de Bruxelles, Bruxelles, Belgique

(*Received* 24 *February* 1984; *in revised form* 18 *May* 1984)

**Abstract**—Mode choice under stochastically varying demand is studied via a dynamic mathematical model which describes the behavioural interactions between population groups. The model is developed by assuming competing attractivity functions for automobile and public transit which motivate their use subject to an overall demand for transportation. When this demand is allowed to vary stochastically, a set of stochastic differential equations describing the model are obtained. These are solved for their steady-state values. It is found that noisy demand can structure the system qualitatively differently than when the demand is fixed. The noise is found to generally reduce the level of public transit ridership, but it also changes the values of the threshold at which new regimes occur and, most interestingly, it induces new steady-state solutions for ridership at critical values of the variance of demand. In the latter case, noise becomes a source of new possibilities in the system by triggering a steady-state solution not present in the noise-free environment.

## INTRODUCTION

A large class of transportation models are dynamic, describing, for example, the time evolution of the velocity pattern in traffic flow problems (Haight, 1963). Another example of dynamic representation which will be our concern here is the individual's demand for transportation, as in the mode choice problem (Hartgen, 1974; Wilson, 1979; Deneubourg *et al.*, 1979; de Palma and Lefevre, 1983; Kahn *et al.*, 1983). An important reason for using a dynamic representation is to capture the effect of changing relative attractivities of the modes as the mode choices are being made. This cannot be adequately done with a static representation. The values of the parameters introduced into these models are determined exogeneously or are estimated, assuming them to be constant. In many situations, however, it is well recognized that this is not true. For example, trip times will vary from hour to hour and from day to day depending upon such random factors as weather conditions, the percentage of vehicle mix at a given moment or the occurrence of an accident.

It is therefore worthwhile to examine the effect of fluctuations in the parameters of a transportation model on its behavior. The fluctuations affecting the parameters could take into account seasonal variations and variations among individuals in their resistance to mode change (Goodwin, 1977), as well as random fluctuations.

For linear systems (Arnold, 1974), if the fluctuations are rapid enough with regard to the deterministic time scale of the model, then the average values of the parameters are sufficient to determine the qualitative behavior of the system. For nonlinear systems, however, the fluctuations may induce qualitative changes in the system: Such phenomena are called noise-induced transitions (Lefever *et al.*, 1979; Horsthemke and Lefèver, 1980).

In Section 2 we recall a dynamic choice model for transportation modes. In Section 3 we derive the stochastic version of this model using some basic results from the theory of stochastic differential equations; a simple version of this model will be analyzed. Finally, in Section 4, we present the results of numerical solutions of a more complex version of the model and discuss some extensions.

## THE DETERMINISTIC MODEL

Recently, systems of nonlinear differential equations (Deneubourg *et al.*, 1979; Kahn *et al.*, 1981, 1983) have been developed to describe the temporal evolution of different modes of transportation in competition with each other for riders. These models are treated as choice models among the different transportation modes.

We consider here (see Deneubourg *et al.*, 1979) the case of a choice between a private mode, the car, and a public one, the bus.

Let $X$ and $Y$ be the number of individuals who choose the car and the bus, respectively. $D$ represents the total number $X + Y$ of individuals who are facing the mode choice problem and is assumed to be constant for the present time (unelastic demand). Let $V_C(X, Y)$ and $V_B(X, Y)$ be the observable utility functions of the car and the bus. The fraction of individuals using the car and the bus is assumed to be derivable from the logit model (McFadden, 1981) as follows:

$$\frac{X}{D} = \frac{e^{V_C(X,\,Y)}}{e^{V_C(X,\,Y)} + e^{V_B(X,\,Y)}}; \quad \frac{Y}{D} = 1 - \frac{X}{D}. \tag{1}$$

The derivation of this equation in a dynamical context has been presented in de Palma and Lefèvre (1983). These authors have proposed a dynamical adjustment process which describes how individuals select a mode choice (here a transportation mode) as a function of time (i.e. from day to day). As a function of time, the equations of evolution of the car and bus users are (Ben Akiva *et al.*, 1984; de Palma and Lefèvre, 1983)

$$\frac{\mathrm{d}X}{\mathrm{d}t} = D\,\frac{e^{V_C(X,\,Y)}}{e^{V_C(X,\,Y)} + e^{V_B(X,\,Y)}} - X; \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = -\frac{\mathrm{d}X}{\mathrm{d}t}. \tag{2}$$

The dynamic mode choice model, introduced in Deneubourg *et al.* (1979) is summarized below. For the car it is assumed that the utility is constant:

$$V_C = \ln a_1 \tag{3}$$

For the bus, the utility is assumed to be a function of the "publicity" used to promote bus usage (factor $\Theta_2$) and of the "imitative" behaviour (or bandwagon effect) resulting in bus usage (see de Palma and Lefèvre, 1983):

$$V_B = \ln(\Theta_2 y + a_2 y^2). \tag{4}$$

The variables $X$ and $Y$ in eqn (2) may approach stationary values where they will no longer change with further changes in time. The stationary solutions of eqns (2), defined by $\mathrm{d}X/\mathrm{d}t = 0$ and $\mathrm{d}Y/\mathrm{d}t = 0$, are precisely the formulas given in eqn (1). For the functional forms (3) and (4) selected, we obtain, for the stationary state,

$$\frac{X}{D} = \frac{a_1}{a_1 + \Theta_2 Y + a_2 Y^2}; \quad Y = D - X. \tag{5}$$

The solutions of eqn (5) are given by

$$Y = 0; \quad X = D \tag{6}$$

and

$$a_2 Y^2 + (\Theta_2 - a_2 D)Y + (a_1 - D\Theta_2) = 0. \tag{7}$$

There are qualitatively two different bifurcation diagrams displaying the stationary states as a function of $D$ for the $Y$ solutions of eqn (7). The bifurcation diagrams will yield information

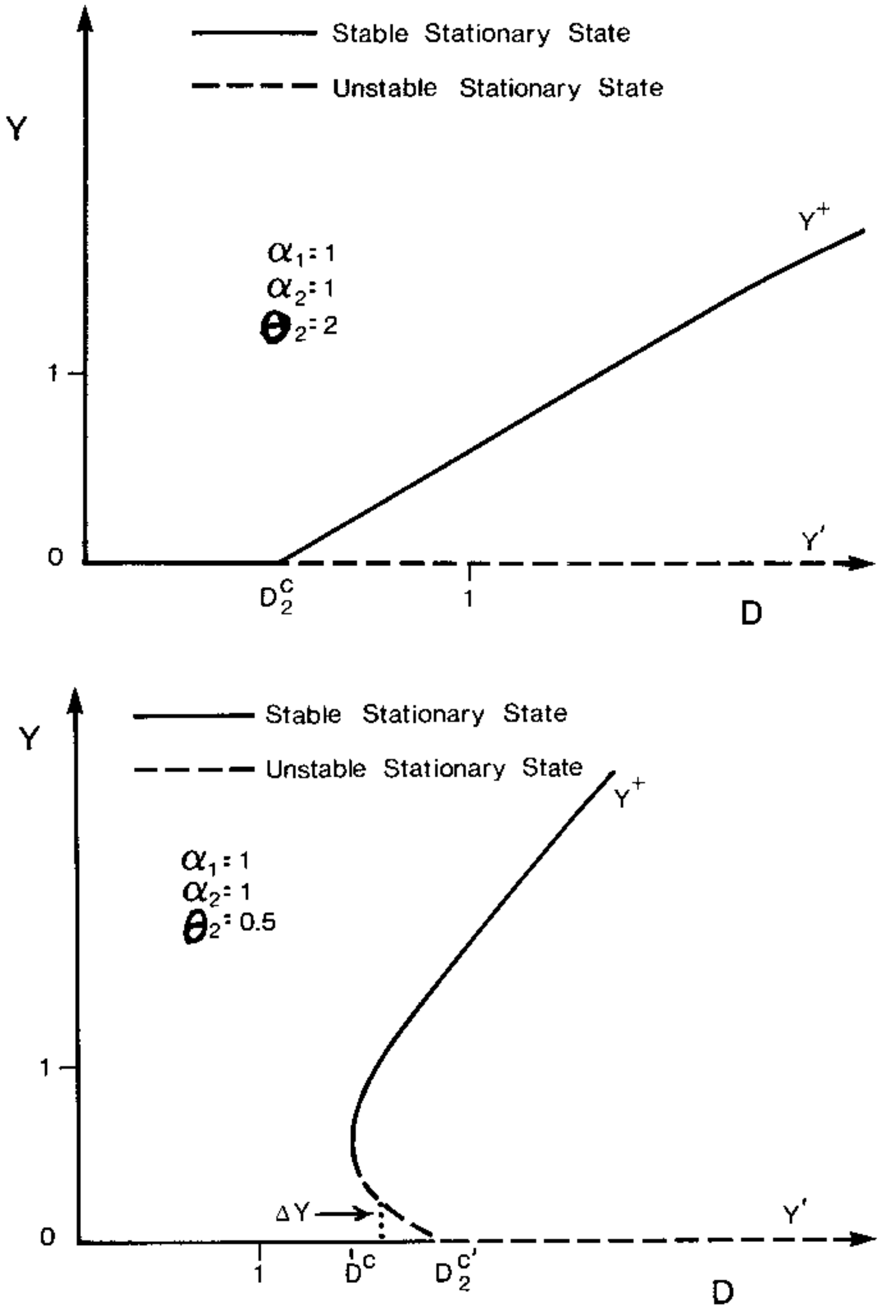


Fig. 1. Bifurcation diagrams for the case when (a) $\Theta_2 > \sqrt{a_1, a_2}$ and (b) $\Theta_2 < \sqrt{a_1, a_2}$.

on which of the possible solutions are accepted by the system, as well as information on their stability properties when subjected to perturbations in $X$ and $Y$. The diagrams are shown in Fig. 1. In Fig. 1 we have introduced critical densities defined by

$$D_2^c = \frac{a_1}{\Theta_2} \tag{8}$$

and

$$D^c = \frac{(4a_1a_2)^{1/2} - \Theta_2}{a_2}. \tag{9}$$

For Fig. 1(a), the publicity factor $\Theta_2$ has a high value $[\Theta_2 > (a_1\, a_2)^{1/2}]$. In this case we see that no bus service can be initiated if the demand $D$ is less than the critical demand $D_2^c$.

If the publicity is low [Fig. 1(b)] $[\Theta_2 < (a_1\, a_2)^{1/2}]$, a larger demand is necessary before service is possible ($D > D^C > D_2^c$). However, even in this case, if $D^C < D < D_2^{C'}$, bus service may not be observed, since two stable regimes coexist, one with zero ridership ($Y^1 = 0$) and one with positive ridership ($Y^+ > 0$).

## A STOCHASTIC VERSION OF THE MODEL

In the preceding analysis, the total demand $D$ was assumed to be constant. In fact, for a number of reasons $D$ varies as a function of time. $D$ will vary because of seasonal variations, price modifications which affect the elasticity of demand for transportation and other external factors. These include employment conditions, weather conditions and similar externalities influencing the demand for transportation. The summation of these effects implies that the demand $D$ appears to be fluctuating.

Let us now define a stochastic demand function

$$D_t = D + \xi_t, \tag{10}$$

where $D$ is the average demand and $\xi_t$ is the fluctuating component of the demand. We will assume that $\xi_t$ is a Markovian stationary stochastic process. More precisely we assume that $\xi_t$ is given by white noise with zero mean and a variance denoted by $\sigma^2$.

Equation (10) then becomes

$$\mathrm{d}D_t = D\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{11}$$

where $W_t$ is a Wiener process. It can be shown (Arnold, 1974) that $Y_t$ will then obey the following Ito linear non-stationary stochastic differential equation

$$\mathrm{d}Y_t = [-Y_t + DG(X_t, Y_t)]\,\mathrm{d}t + \sigma G(X_t\, Y_t)\,\mathrm{d}W_t. \tag{12}$$

The probability density $P(Y_t, t)$ of the $Y_t$ process is the solution of a Fokker–Planck equation. The stationary distribution can be explicitly determined (Arnold, 1974) and is given as

$$P(Y) = NG(Y)^{-1} \exp\left\{\frac{2}{\sigma^2}\int^Y [DG(Z) - Z]G(Z)^{-2}\,\mathrm{d}Z, \tag{13}$$

where $N$ is a positive normalization factor ensuring the conservation property of the probability function.

The extrema of the stationary distribution for $Y \neq 0$ are solutions of the equation

$$[DG(Z) - Z] - \frac{\sigma^2}{2} G(Z)G'(Z) = 0, \tag{14}$$

where $G'(Z)$ is the derivative of $G(Z)$ with respect to $Z$. The extrema of the stationary distribution correspond to the points where it is more probable that the state of the system will be found and correspond to the macroscopic stationary states (stable stationary states). Of course it is easily verified that, when $\sigma^2 = 0$, the maxima of eqn (13) are identical to the stable stationary states of eqns (2). For linear systems and for systems with additive noise defined by $G(Z)$ = constant or $G'(Z) = 0$, the peaks of the stationary distribution $P(Y)$ are broadened and damped only when the intensity of the noise is increasing. This is not a general result, as we shall see with the aid of an example using a multiplicative noise, $G'(Z) \neq 0$.

The case $Y = 0$ will be studied directly from eqn (13).

*The white noise limit and* $a_2 = 0$.

We shall now examine the model as given by the stochastic differential eqn (12) and by the attractivity (or utility) eqns (3) and (4) for which $a_2 = 0$. In this case the equation for the extrema [eqn (14)] becomes

$$-1 + \frac{D\Theta_2}{a_1 + \Theta_2 y} - \frac{\sigma^2}{2}\frac{a_1\Theta_2^2}{(a_1 + \Theta_2 y)^3} = 0. \tag{15}$$

This equation may be rewritten as

$$F(Y) \equiv -\frac{2}{a_1\Theta_2^2}(a_1 + \Theta_2 Y)^2 (Y\Theta_2 + a_1 - D\Theta_2) = \sigma^2. \tag{16}$$

We first look at the case when

$$D \leqslant \frac{a_1}{\Theta_2}. \tag{17}$$

In this case

$$F(Y) < 0, \quad \left.\frac{dP(Y)}{dY}\right|_{\varepsilon} < 0, \quad 0 < \varepsilon \ll D \tag{18}$$

so that the probability function $P(Y)$ is maximum at $Y = 0$ and is always decreasing (see Fig. 2). This result is consistent with the analysis done in the deterministic case.

Let us now consider the case when

$$D > \frac{a_1}{\Theta_2}. \tag{19}$$

In this case the derivative of the probability, $dP(Y)/dY$, follows the sign of the function $\sigma_c^2 - \sigma^2$, where $\sigma_c^2 \equiv (2a_1/\Theta_2^2)(D\Theta_2 - a_1)$.

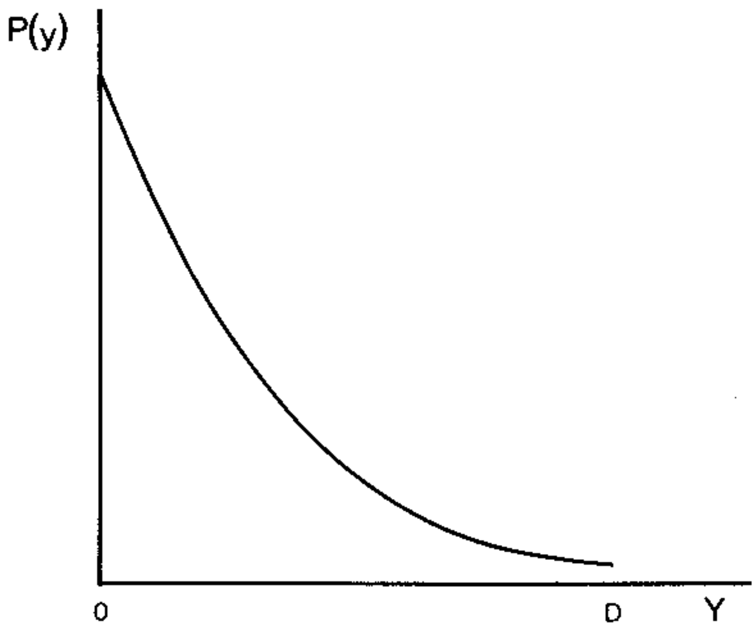


Fig. 2. Stationary probability distribution; $D \leq a_1/\Theta_2$.

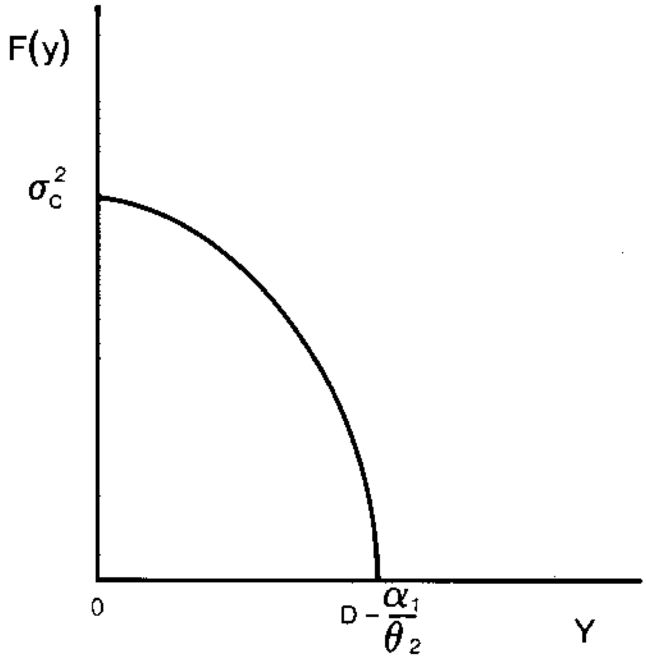


Fig. 3. Display of function $F(X)$; $D\varepsilon]$ $(a_1/\Theta_2)$, $\frac{3}{2}$ $(a_1/\Theta_2)]$.

There are two qualitatively different situations in this case:

(i) If $D\varepsilon]$ $a_1/\Theta$, $\frac{3}{2}$ $a_1/\Theta_2]$, then $F(Y)$ is a monotonically decreasing function of $Y$ with $F(0) = (2a_1/\Theta_2^2)(D\Theta_2 - a_1)$ and $F(D - a_1/\Theta_2) = 0$ (see Fig. 3). Thus the probability $P(Y)$ will either exhibit a maximum (for $\sigma^2 \leqslant \sigma_c^2$) or be monotonically decreasing (when $\sigma^2 > \sigma_c^2$). This is shown in Fig. 4(a) and (b). We thus see that the introduction of fluctuations in the demand has the effect of shifting the maximum of $P(Y)$ from the deterministic value $Y = D - a_1/\Theta_2$ (for $\sigma^2 = 0$) to zero for $\sigma^2 > \sigma_c^2$ (see Figs. 3 and 4).

(ii) The second situation is when $D > \frac{3}{2}$ $a_1/\Theta_2$. Then $F(Y)$ is an increasing function of $Y$ for $Y < 2D/3 - a_1/\Theta_2$, reaching a maximum at $Y = 2D/3 - a_1/\Theta_2$, and is a decreasing function of $Y$ for $Y > 2D/3 - a_1/\Theta_2$. This situation is depicted in Fig. 5. Thus the probability $P(Y)$ has a unique maximum when $\sigma^2 \leqslant \sigma_c^2$ exhibits both a minimum and a maximum when $\sigma_c^2 < \sigma^2 < (2D/3)^3\Theta_2/a_1$ and is a monotonically decreasing function of $Y$ when $\sigma^2 \geqslant (2D/3)^3\Theta_2/a_1$. These three different cases are shown in Fig. 6. In the cases depicted in Fig. 6(a) and 6(c), fluctuations in the demand have introduced similar phenomena already discussed with reference to Fig. 4(a) and (b). However, in the case depicted in Fig. 6(b) a qualitatively new

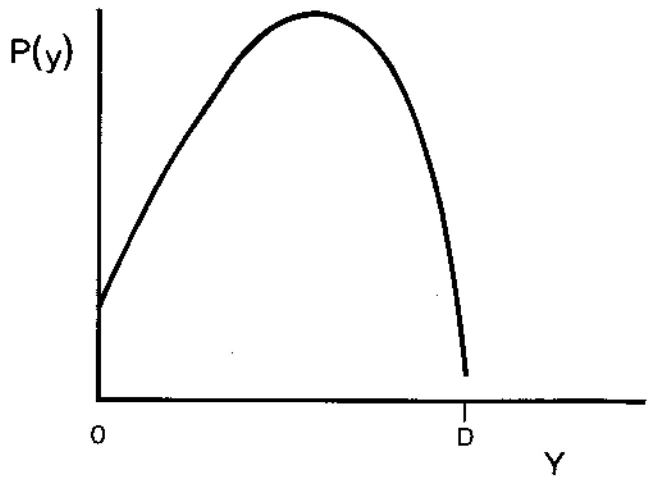


Fig. 4. Stationary probability distribution; $D\varepsilon]$ $(a_1/\Theta_2$, $\frac{3}{2}$ $(a_1/\Theta_2)]$.

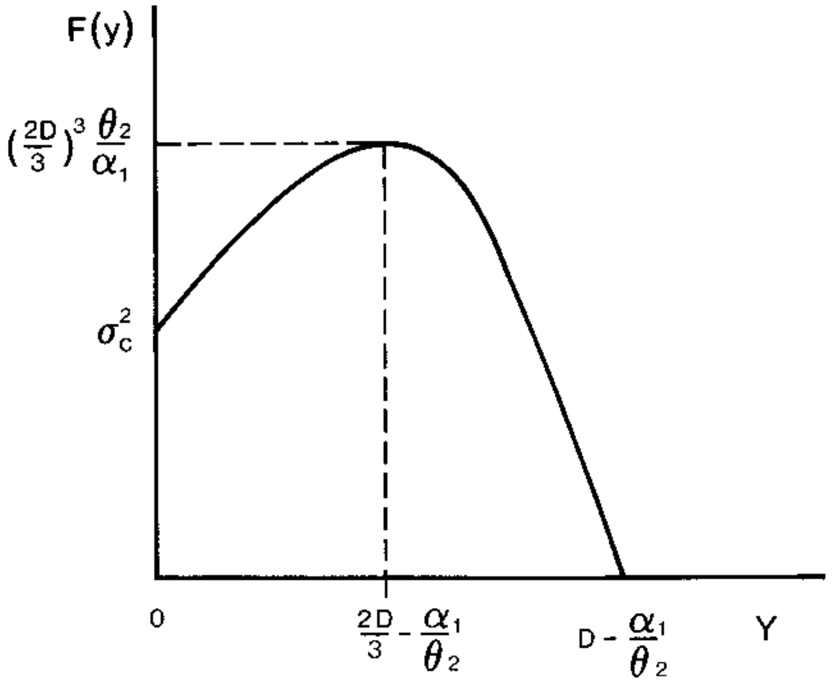


Fig. 5. Display of function $F(Y)$; $D > \frac{3}{2}(a_1/\Theta_2)$.

behavior is exhibited in the system which now admits two maxima. This is an example whereby the noise has introduced a peak-splitting transition. This would correspond in the macroscopic description to a new observable state of the system.

## NUMERICAL RESULTS

In this section we present numerical results for the more complex model $a_2 \neq 0$ treated in the deterministic case in Deneubourg *et al.* (1979). This will enable us to compare directly the deterministic case in which the demand $D$ is constant with the stochastic case treated here where the demand fluctuates. The equation for the extrema is given by

$$-1 + \frac{D(\Theta_2 + a_2Y)}{a_1 + \Theta_2Y + a_2Y^2} - \frac{\sigma^2}{2}\frac{a_1(\Theta_2 + a_2Y)(\Theta_2 + 2a_2Y)}{(a_1 + \Theta_2Y + a_2Y^2)^3} = 0, \qquad (20)$$

which reduces to eqn (15) when $a_2 = 0$.

We remind the reader that the extrema $Y$ of the stationary probability distribution $P(Y)$ [see eqn (13)] are identified with the observable macroscopic states of the system. The maxima of $P(Y)$ where the process spends most of its time are the stable steady states, while the mimima where relatively little time is spent are the unstable states. In the limit $\sigma^2 \to 0$, the extrema are simply the solutions of the deterministic equation [already obtained in Deneubourg (1979)]. We now compare the deterministic solutions with solutions from the stochastic differential eqn (12) as given by eqn (20). This equation is solved for $Y$ as a function of the demand $D$ and as a function of the variance $\sigma^2$ for the same numerical values of the parameters as in the deterministic case. The results are shown in Figs. 7 and 8. In these figures, the solid lines represent the maxima of the probability function $P(Y)$ corresponding to observable macroscopic states, and the dashed lines represent the minima corresponding to unobservable macroscopic states.

Figure 7, in which $Y$ is plotted as a function of $D$ for different values of $\sigma^2$, exhibits all three phenomena induced by a fluctuating demand as discussed in the previous section. We see that the noisy demand has caused *peak damping*, forcing the maximum value of $P(Y)$ to become depressed for all finite values of $\sigma^2$ compared to the deterministic case where $\sigma^2 = 0$.

The figure further shows the phenomenon of *peak shifting* by which the location of the transition points are shifted. The practical significance of this is that one needs a higher demand

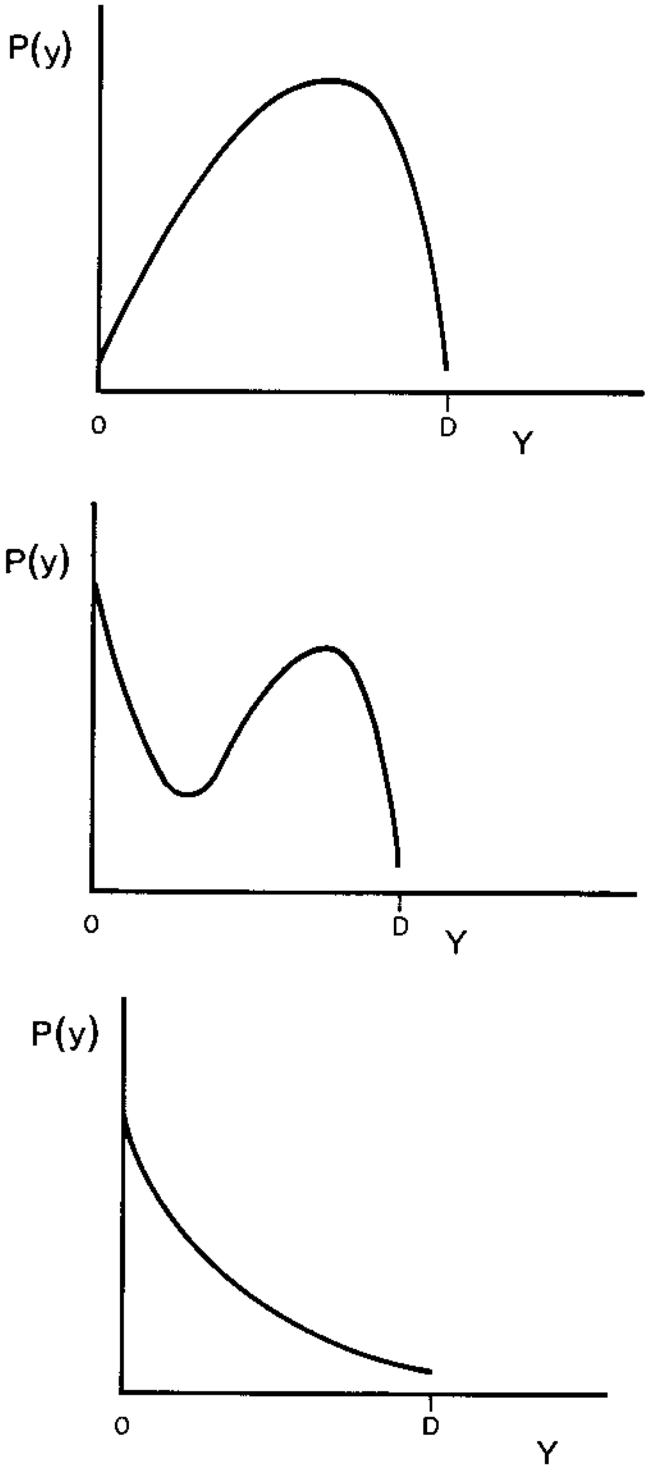


Fig. 6. Stationary probability distribution; $D > \frac{3}{8}(a_1/\Theta_2)$.

$D$ before bus service is initiated. The value of the demand necessary to initiate bus service for different values of $\sigma^2$ is computed by calculating the function $\partial D/\partial Y = 0$ and solving it for $Y$ to obtain the critical values of $D$.

Finally the phenomenon of *peak splitting* is also seen to occur, whereby a new macroscopically observable state appears in the system. For example, at $D = 5$ only one observable state appears in the deterministic case $\sigma^2 = 0$, whereas two observable states appear when $\sigma^2 = 10$ (see Fig. 7).

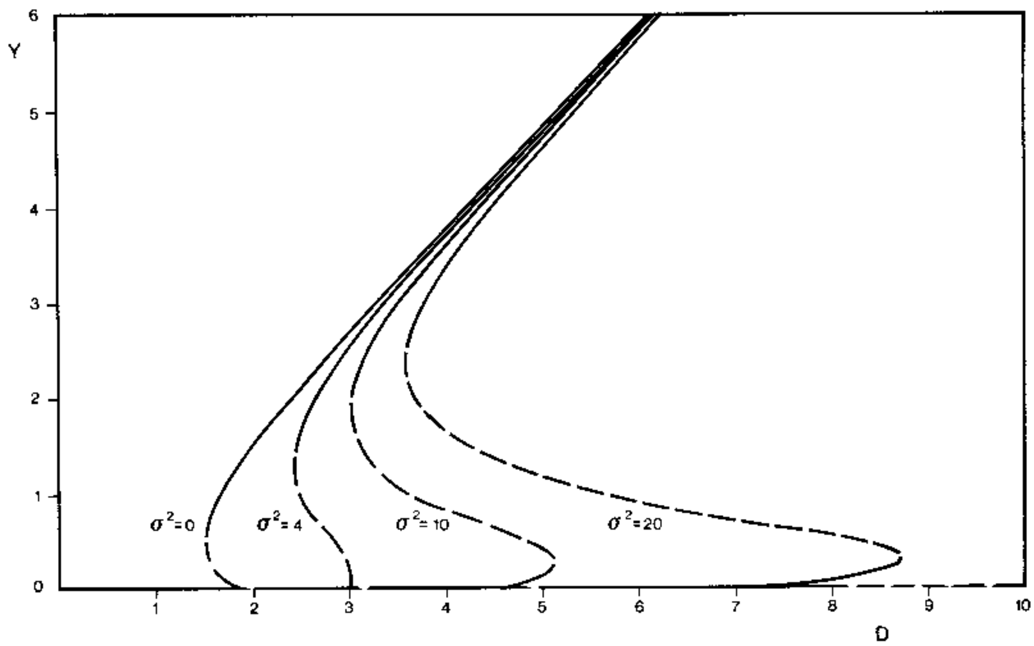


Fig. 7. Ridership versus demand.

Figure 8(a) and (b) show $Y$ as a function of $\sigma^2$ directly. Figure 8(a) corresponds to the previous Fig. 5. We note that beyond a critical noise level $\sigma_c^2$ no bus service is possible for the given level of demand, as discussed above.

Figure 8(b) shows the appearance of a new observable macroscopic state $Y \neq 0$ for a given value of the variance $\sigma^2$. This, of course, is qualitatively new, not possible in the deterministic case where $\sigma^2 = 0$. Thus, here two non-zero roots can coexist; one root is the continuation of the deterministic root (upper branch), while the other did not exist in the noise-free deterministic case.

## CONCLUSIONS

The importance of this work, we believe, lies in the new behavior exhibited by a transportation system subject to environmental noise under very broad conditions. As long as the noise is multiplicative and the system non-linear, which is the usual case for real transportation systems, the noise may induce quantitative as well as qualitative effects not present in the noise-free environment. This is true even if the system is in equilibrium in the sense that the random fluctuations or noise cancel out on the average.

The effects induced by the noise are of the following types: first, there is a general decrease in bus ridership as the fluctuations increase; secondly, the minimum demand necessary to have bus service increases with increasing fluctuations in demand; finally, the presence of noise can shift an initially high level of bus ridership operation into a low-level ridership regime.

Behaviourally, we interpret these results to signify people's reluctance to frequent a service which, though on average maintains a given level of service, is unpredictable from day to day. There is a mode switch to the car under these conditions of variability. Mathematically, from the assumptions of the model [see eqns (2)–(4)], the fraction of individuals taking the bus increases with increasing demand in a concave fashion (see Fig. 7). Thus, the downward fluctuations in total demand are more detrimental than the symmetric upward fluctuations in total demand.

From the point of view of a bus company which wishes to introduce a new line, the presence of a fluctuating demand increases the market necessary to support the line. Thus the company may wish to develop methods to reduce the impact of fluctuations as well as the fluctuations themselves. On a long-range or overall managerial level, the company may wish to consider policies and infrastructural changes which would help to reduce the impact of

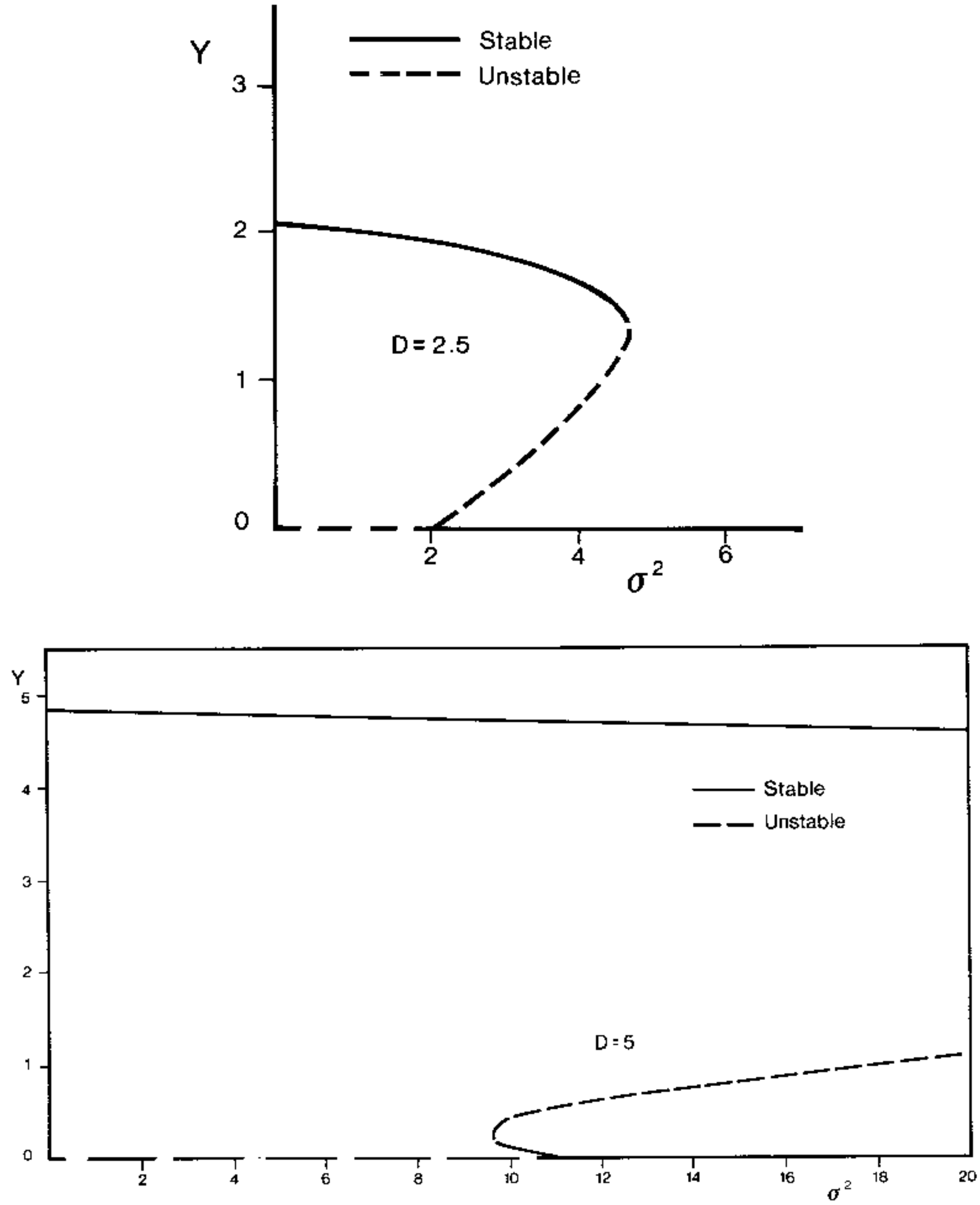


Fig. 8. Ridership versus variance of demand.

fluctuations in demand. One way to do this would be to make the velocity or frequency of bus service less sensitive to the demand.

On another more operational level, the company may reduce the fluctuations directly by such stratagems as the use of back-up buses. A less capital-intensive strategy for reducing the fluctuations in demand would be to provide transportation users with information concerning tie-ups or bottlenecks. The idea here is to relocate the local demand from one line to another as traffic conditions change. In fact there is a program which is now attempting to determine the benefits of providing such information to potential users (Transportation Systems Center, 1981). This program is called Computerized Rider Information System (CRIS) and is being tested in Pittsburgh, Erie, Albany, San Diego, Salt Lake City and Columbus. While the details of the program differ from city to city, the essential common feature is the provision of information to a telephone caller on the time of arrival of a bus at stops specified by that caller. This information is expected to reduce the amount and the variability of wait time at bus stops. It is hoped that this, in turn, will lead to increased ridership. To the extent that a reduction in wait-time variability will result in an associated reduction in the variability of total demand, this paper would predict an increase in transit ridership.

The transit service reliability demonstration project in Minneapolis–St. Paul could provide data needed to test the model. The reader is referred to the Minneapolis–St. Paul Transit Service Reliability Demonstration report (Multiplications, 1983) for detailed information on the actual demonstration results. This type of demonstration may be used for model testing by obtaining "before" data on the demand for transportation between two areas serviced by a bus route (their route 5, say). The daily variance of this demand would be obtained as an average over several weeks. Bus ridership levels would also be obtained for this period for this route. Following these before data collection activities, real-time holding strategies would be used to increase bus reliability through increased adherence to scheduled departure times from the various stops along the bus route. If it is assumed that improvement actually occurs (as it did in the demonstration), "after" data will be obtained on demand, on the daily variance of demand and on bus ridership. After accounting for ridership changes due to other factors (the externalities), the data would be analyzed for relationships between ridership changes and changes in demand and demand variability and then compared with model predictions, such as shown, for example, in Figs. 7 and 8.

*Acknowledgements*—The authors would like to thank Moshe Ben-Akiva, R. Lefèver and R. Arnolt for useful suggestions. We also take pleasure in acknowledging I. Prigogine for suggesting the application of the stochastic method for the study of social phenomena.

## REFERENCES

Arnold L. (1974) *Stochastic Differential Equations*. Wiley-Interscience, New York.

Ben-Akiva M., Cyna M. and de Palma A. (1984) Dynamic model of peak period congestion. *Transpn. Res.* **18B**, 339–355.

Deneubourg J. L., de Palma A. and Kahn D. (1979) Dynamic models of competition between transportation modes. *Environment Plann.* **11**, 665–673.

de Palma A. (1981) Models stochastiques des compartements collectifs dans les systemes complexes. Ph.D. Dissertation, Université Libre de Bruxelles, Bruxelles.

de Palma A. and Lefèvre C. (1983) Individual decision-making in dynamic collective systems. *J. Math. Soc.*, **9**, 103–124.

Goodwin P. B. (1977) *Urban Stud.* **14**, 95–98.

Haight F. A. (1963) *Mathematical Theories of Traffic Flow*. Academic Press, New York.

Hartgen D. T. (1974) *Transportation* **3**, 45–58.

Kahn D., Deneubourg J. L. and de Palma A. (1981) Transportation mode choice. *Environment Plann. A* **13**, 1163–1174.

Kahn D., Deneubourg J. L. and de Palma A. (1983) Transportation mode choice and city-suburban public transportation service. *Transpn. Res.* **17B**, 25–43.

Horsthemke W. and Lefèver R. (1980) A perturbation expansion for external wide band Markovian noise: Applications to transitions induced by Ornstein-Uhlenbeck Noise, *Z. Physik B—Condensed Matter* **40**, 241–247.

Lefever R. and Horsthemke W. (1979) Multiple transitions induced by light intensity fluctuations in illuminated chemical systems. *Proc. Natn. Acad. Sci. U.S.A.* **76**(6), 2490–2497.

McFadden D. (1981) Econometric models of probabilistic choice. In Manski C. F. and McFadden D. (eds), *Structural Analysis of Discrete Data with Econometric Applications*. MIT Press, Cambridge, Mass.

Multiplications (1983) *Minneapolis-St. Paul Transit Service Reliability Demonstration*, Final Report, Cambridge, Massachusetts, for the Urban Mass Transit Administration through contract number DOT-TSC-1756.

Transportation Systems Center (1981) Service and Methods Demonstration Program Report, UMTA-MA-06-0049-81-12, Computerized Rider Information Systems. Boston, Mass.

Wilson A. G. (1979) *Aspects of Catastrophe Theory and Bifurcation Theory in Regional Science*, WP-249, School of Geography, University of Leeds, Leeds, England.