TRANSPORTATION MODE CHOICE AND
CITY-SUBURBAN PUBLIC TRANSPORTATION
SERVICE

D. KAHN,
U.S. Department of Transportation, Transportation Systems Center, Cambridge, MA 02142, U.S.A.

and

J. L. DENEBOURG and A. de PALMA
Chimie-Physique II, Université Libre de Bruxelles, Brussels 1050, Belgium

(Received 13 June 1981; in revised form 28 February 1982)

Abstract—A model is presented to describe the dynamics of transportation mode choice in which the
interaction between transportation users and a public transportation authority results in self-organization.
The model illustrates that a sufficient number of connections between a central city and its suburbs are
required for self-organization to occur whereby public transportation use and service will grow.

INTRODUCTION

This paper presents a new model of transportation mode choice which illustrates the use of
non-linear differential equations exhibiting instabilities. The starting point is the application of
the physical-mathematical concept of bifurcation. This concept has been applied previously to
the study of self organizing processes by the Brussels groups (Nicolis, 1977). Several studies
have extended the applications to biological (Babloyantz, 1977, 1979; Erneux, 1978; and
Prigogine, 1972), ecological (Deneubourg, 1977) and social systems (Allen, 1978–1980; de Palma,
1982). A number of other applications may be found in the annals of the New York Academy of
Sciences Volume 316, 1979. An important aspect of many of these studies is the attempt to
describe the formation of system structures as a function of exchanges of matter, energy and
information with the external environment. The evolution of the system then proceeds as a
result of the interactions among the various actors of the system. Under certain conditions
(Sattiguer, 1973), the system will evolve towards a stable regime corresponding to a stationary
state in which the observed variables remain constant with time even if actors change choices.
The stability of these stationary states under changing system constraints and under conditions
of changing behavior are of particular interest in these studies.

Recently, this concept has been applied to transportation mode choice (Deneubourg, 1979;
Kahn, 1980) which, in this paper, is given a spatial dependence by considering a policy of
establishing public transportation connections between a central city and the suburbs surrounding
it. The question asked here is how this policy initiative may serve to enhance public transportation
ridership and how large an investment is needed for the initiative to "catch on".

While the particular model developed here is not intended to give actual numerical values, it
does provide a theoretical framework so that such questions may be realistically addressed with
the proper elaboration of spatial detail and with an expansion of the factors which influence
behavioral actions.

The model, like those in Deneubourg (1979) and Kahn (1980) postulate certain basic
behavioral actions on the part of the actors, namely, the users of transportation and the public
transportation authority. The interaction among these elements, each with its own behavioral
criteria for action, permits different evolutions of the system depending on the strength of the
interaction. This, in turn, depends upon such factors as fares charged and costs of running the
transportation system. Other factors will also be discussed in the next section. What is
important in this description is that a particular evolutionary path may become significantly
altered if the strength of any one of these factors becomes great enough (exceeds a critical
threshold).

In this case, the system may become unstable to the “stimulus” being imposed, for
example, the fares charged and jump to another state which is stable. This phenomenon is a
simple example of order through fluctuation (Nicolis, 1977), where the fluctuation here is a
change in strength in the factors affecting the behavior of the actors. For systems like those studied in this paper, in addition to finding very stable regimes of behavior where only a prohibitively large change will affect the system's stability, other regimes exist where even small changes will markedly alter the system's behavior when the system is near one of its bifurcation points (Deneubourg, 1979; Kahn, 1980).

The factor we shall examine in this paper is a policy initiative of adding or subtracting public transportation lines that connect a central city with the surrounding suburbs. Even in the very simple model offered here, we shall find the existence of critical points also called bifurcation points around which the addition of only one more connecting line can profoundly alter the system's response, resulting in a significant increase in public transportation usage over that which existed previously. It is around such points of bifurcation where policy initiatives will be most effective in influencing the response of the system.

PRESENTATION OF THE MODEL

To help us develop a model of interaction between the users of transportation and policy implementations of a public transport mode, we imagine a central city surrounded by suburbs, shown schematically in Fig. 1.

Within the central city there are automobile users, designated \( x \) and public transportation users, designated \( y \) (for specificity we shall take the public transportation mode to be the bus, though with a suitable change of scale this could be a subway system). We have derived the equations governing the time rate of change of individuals facing a discrete set of choices in a previous paper (de Palma, 1982). This same paper showed the connection between this approach and the disaggregated behavioral approach developed in McFadden, 1973 and applied by various authors to transportation (see, for example, Ben-Akiva, 1976). For the case of two choices (automobile and public transit) these equations reduce to (see also Deneubourg, 1979; and Kahn, 1980 for additional discussion of the equations):

\[
\dot{x} = \frac{DA_x}{A_x + A_y} - x; \quad \dot{y} = -\frac{DA_y}{A_x + A_y} - y. \tag{1}
\]

![Fig. 1. Schematic showing the central city and its links to the suburban areas.](image)
Here, $D$ is the demand for transportation which is assumed to vary slowly compared with the changes of $x$ and $y$, so that it may be considered constant (no new users are brought into the system during the time of interest). $A_c$ and $A_v$ are the attractiveness for car and bus, respectively, and the terms containing them in Equations (1) express the fraction of the total demand that is attracted to the car or to the bus. The explicit form of these attractivities will be discussed after we introduce the other variables of the system.

Also within the central city, the variable describing the quality of bus service is designated as $L$, whose time rate of change is given by:

$$\dot{L} = v y + \sum_j v y_j - K L$$

(2)

where $v$ is the fare ($v y$ then is the revenue received), $K$ is the cost per unit of service offered ($K L$ then represents the costs of providing service); $y_i$ is the number of residents of suburb $j$ who use the bus in the central city ($\Sigma_j v y_j$ then represents the revenue that the city bus system receives from the users $y_j$). The quantity $\Sigma_j v y_j - KL$ represents the amount of funds available and thus eqn (2) states an hypothesis of linearity which may be interpreted as a first order approximation. Clearly, other terms could be added to eqn (2) to more realistically express the dynamics of the public transportation mode, for example, a state subsidy term could be added. However, this will not be done here since eqn (2) is sufficient to demonstrate the feedback phenomena responsible for bifurcation as discussed in the Introduction.

The variable describing the commuter connection between the central city and the suburb $j$ is designated $C_j$ whose time rate of change is given by:

$$\dot{C}_j = v' y_j - K' C_j$$

(3)

which is similar in form to eqn (2). The quality of service between the central city and the suburb $j$ is assumed to be given simply by the net income: revenues received, $v' y_j$, less maintenance costs, $K' C_j$, where $v'$ and $K'$ are the fares and unit maintenance costs, respectively.

The equations for $y_j$ and $x_j$, representing people who live in suburb $j$ and use the bus in the central city and people who live in $j$ and commute by car, respectively, have the same form as eqns (1):

$$\dot{x}_j = \frac{D_j A_c'}{A_c' + A_v'} - x_j; \quad \dot{y}_j = \frac{D_j A_v'}{A_c' + A_v'} - y_j$$

(4)

where $D_j$ is the demand for transportation between suburb $j$ and central city. $A_c'$ and $A_v'$ are the attractiveness for car and bus commuting, respectively.

It is now necessary to give explicit representation to the attractiveness $A_c$, $A_v$, $A_c'$, and $A_v'$. We shall use very simple non-trivial assumptions (that is, all attractivities are not assumed to be constant) and will see that they still give rise to bifurcation phenomena. We assume a constant attractiveness for car use within the central city, given by $g$, and another constant attractiveness for car use between suburb and central city, given by $g_j$:

$$A_c = g; \quad A_c' = g_j$$

(5)

For bus usage, we assume that the attractiveness depend upon the quality of service offered. Hence, for people who live in the central city, we assume

$$A_v = L$$

(6)

and for people who live in a suburb $j$ and who commute via public transportation to the city where they use the central city bus system, we have:

$$A_v' = L + C_j$$

(7)
We note, in passing, that this form (7) is only one of several that may be used. For example, the product of \( \bar{C}_i \) with \( L \) could be used to describe the situation where the absence of either \( C_i \) or \( L \) would preclude growth of \( y \).

Using these attractiveness in eqns (1) and (4) give for the evolution of car and bus users:

\[
\dot{x} = \frac{Dg}{L+g} - x; \quad \dot{y} = \frac{DL}{L+g} - y
\]

(8)

\[
\dot{x}_i = \frac{Dg_i}{C_i + L + g_i} - x_i
\]

\[
\dot{y}_i = \frac{D(C_i + L)}{C_i + L + g_i} - y_i
\]

(9)

In words, these equations state that the growth in bus and car users depends upon the relative attraction of the bus or the car mode, which in turn, depends upon the relative quality of service offered. In the case of central city residents this relative quality of service is \( L/(L + g) \) for the bus and \( g(L + g) \) for the car, while for suburban residents it is \( (L + C_i)/(L + C_i + g_i) \) for the bus and \( g/(L + C_i + g_i) \) for the car.

Relative growth rate assumption

The system of eqns (2) and (3) for the growth of the central city bus system and the growth of the commuter lines, respectively, together with eqns (8) and (9) for the growth of transportation users, describe the model. However, we shall now use the fact (Lerman, 1975) that any central city bus or commuter line change occurs much more slowly than does the reaction of people to mode choice. This remark allows us to replace eqns (8) and (9) with their stationary state values, while retaining the time rate of change form for the bus and commuter line equations (that is, we assume that there is a time long enough so that the \( y \)’s and the \( x \)’s have attained a stationary state while the \( L \) and \( C_i \) still have not). With this assumption, eqns (8) and (9) reduce to algebraic expressions which upon substitution into eqns (2) and (3) yield:

\[
\dot{L} = \frac{vDL}{L + g} + \sum_i \frac{vD(C_i + L)}{C_i + L + g_i} - KL
\]

(10)

\[
\dot{C}_i = \frac{vD(C_i + L)}{C_i + L + g_i} - KC_i
\]

(11)

as the model equations for the time evolution of the central city bus service, \( L \) and the commuter lines, \( C_i \).

ANALYTICAL SOLUTIONS OF THE MODEL

A simplified version of the model

The equations for the growth of central city bus service (10) and for the growth of suburban-central city commuter lines (11), may be easily solved analytically for its stationary state values if we make the simplifying assumptions that each suburb has the same transportation demand, denoted by \( D_c \) and that each suburban-central city automobile attractiveness term, \( g_i \) is the same, denoted by \( g_c \). We further simplify the calculations by taking the fares and maintenance costs to be the same and denoted, respectively, by \( v \) and \( K \).

We point out that these simplifying assumptions do not allow us to distinguish between suburbs, nor allow us to investigate the effect of the installation of different levels of commuter service to the different suburbs. This effect will be investigated in a future paper. The present paper will be restricted to the investigation of the effect of adding or subtracting commuter line connections between "equal" suburbs and the central city. That is, we shall be

\[\text{For completeness we may note that we have implicitly assumed that there is a reaction rate } \epsilon = 1 \text{ which multiplies the right hand side of eqn (10). This reaction rate } \epsilon = 1 \text{, where } \gamma \text{ would be the corresponding reaction rate to mode choice in eqns (8), (9). This idea is shown by use of the stationary state values for the } y \text{ and } x.\]
Transportation mode choice and city-suburban public transportation service

Concerned here only with the effect of the number of lines on transportation usage and growth.†

Rewriting eqns (10) and (11) to reflect the above assumptions gives:

\[ \dot{L} = \frac{vDL}{L + g} + \sum \frac{vD_i(C_i + L)}{C_i + L + g} - KL \]  
(12)

\[ \dot{C_i} = \frac{vD_i(C_i + L)}{C_i + L + g} - KC_i \]  
(13)

Analysis of the model

Introductory analysis. Before studying the model eqns (12) and (13) we introduce in this section an analysis of a much simpler case in order to help see more clearly how the suburb and central city systems influence each other in producing growth of public transportation. We shall begin with a central city only, adding first one commuter line and then another to see how their introduction influences public transportation in the central city. Following this simple case we will then return to the more complicated, general set of coupled central city-suburban eqns (12) and (13).

Beginning with only a central city and no connection to the suburbs, the equation which governs the time evolution of central city bus service is (from 12)

\[ \dot{L} = \frac{vDL}{L + g} - KL. \]  
(12a)

We are interested in the conditions for growth of central city bus service. From (12a) we find that there will be growth \( \dot{L} > 0 \) if \( vD/g > K \). When this condition is met, any perturbation \( dL \) around the zero stationary state solution of (12a), \( L = 0 \), will grow in time. Thus \( L = 0 \) is an unstable solution whenever \( K < vD/g \), and bus service will grow. Physically, this is clear: \( L \) will grow whenever the costs \( K \) of providing service are sufficiently small (or the demand \( D \) for transportation is sufficiently great).

We now ask, similarly, what the conditions for commuter line growth may be if central city bus service is nonexistent. The equation describing commuter service when \( L = 0 \) is (from eqn 13).

\[ \dot{C_i} = \frac{vD_iC_i}{C_i + g} - KC_i. \]  
(13a)

We are interested in the conditions for growth of commuter line service. From (13a) we find that \( C_i = 0 \) is unstable whenever \( K < vD_i/g_i \); when costs are low (or demand high) the system will grow.

We now wish to see the effect of adding a commuter link on central city bus service. The effect of such a link is to enhance central city bus service as can be seen by noting that the non-trivial stationary state solution for commuter service is given by (from eqn 13a):

\[ C_i = \frac{vD_i}{K} - g_i \]  
(13b)

which, as a physical solution is positive, and thereby produces growth \( L > 0 \) in the central city bus system. And, similarly, if the central city bus service has reached its non-trivial stationary state:

\[ L = \frac{vD}{K} - g \]  
(12b)

†We also note that while this paper considers only one central city, it is also of interest to consider and compare a spatial structure with more than one central city (one or more central business districts) serving a number of suburban locations.
this positive term will produce growth $\dot{C} > 0$ in the commuter line system as is evident from eqn (13).

We illustrate this phenomenon of mutual reinforcement in Fig. 2 which shows the time history of the system when two commuter lines are successively introduced.

The figure shows the system starting off with no commuter links $C = 0$ when a perturbation $dL \neq 0$ is introduced. $L$ grows since the costs $K$ have been made sufficiently small ($K < vD/g$). This growth approaches a steady state (given by eqn 12b). At some later time, $t = 9$ units, a commuter line is introduced into the system. We observe the growth of that commuter line for costs $K < vD/g$, (sufficiently small to allow growth to occur) and observe its rather strong effect on the subsequent growth of central city bus service $L$. The growth in $L$ and $C$ continues until new steady states are reached for these quantities. Finally, the figure shows the introduction of a second commuter line at a still later time. We observe its growth to a steady state as well as the growth of central city bus service $L$ which it helped stimulate.

For completeness, Fig. 3 illustrates the concomitant time history of bus users, as central city service and commuter service grow. We observe the increase in bus ridership corresponding to growth in $L$ until a steady state is reached and the ridership saturates. However, the introduction of a commuter line causes a further increase in bus ridership until that too saturates. The figure shows for two such commuter lines, the consequent increase in ridership.

**Stability analysis.** Returning now to the model eqns (12) and (13) for the growth of the central city bus system and commuter lines, we find immediately the stationary state solutions:

$$L = 0, \quad C_i = 0$$  \hspace{1cm} (14)

which expresses the absence of any central city bus service and the absence of any commuter service. From eqns (8) and (9) taken at the stationary state, we see that it implies all car usage, that is:

$$\begin{cases} y = 0 \\ x = D \quad \{ x_j = D_j \end{cases}$$  \hspace{1cm} (14a)

Fig. 2. Time history of growth of bus service as two commuter links are successively introduced.
Though this is a solution of the model equations, its existence is not enough to insure that it will persist for all (or even for any) of the values of the parameters. If the system is unstable to perturbations around this state and a perturbation here would be equivalent to the introduction of a bus line into the system, then the state will not persist. On the other hand, if (14) is stable to the perturbations, then the zero state will persist even though the bus line was introduced. It is therefore important to know the range of values for which the stationary state is stable, and from the point of view of the decision maker, it is important to know the values of the parameters at which the system is most sensitive to change, so that a timely policy initiative may be made to effect a desired change and also to be aware when an unwanted change may be likely to occur. In other words, for which values of the parameters of the system (here, for example, level of fares, costs of maintaining the service, demand, number of commuter lines, etc.) should the decision maker be alerted to a likely significant change of system performance (change of state) and in what direction is this change going to occur. Also, what “perturbation” must be imposed upon the system to affect its performance in a desired way? The model equation's solutions, their stability properties and numerical example should illustrate the above points.

The stability analysis is done by subjecting the model eqns (12) and (13) to perturbations $dL$ and $dC_f$ around its stationary states. The first stationary state found was the trivial one given in eqn (14) in which the system permitted only automobile transportation. We want to investigate the stability of this “busless” state to perturbations in the quality of central city bus service, $dL$, and to perturbations in the quality of the service between central city and the suburbs $dC_f$.

We know (Sattinger, 1973) that the time dependence of the variables $L$ and $C_f$ vary as $\exp(\omega t)$; the model eqns (12) and (13) become, when subjected to perturbations around the trivial stationary state,

\begin{align}
(w + K - \frac{vD}{g_c})dL - \frac{vD}{g_c}dC_f &= 0 \quad (15) \\
(w + K - \frac{vD}{g_c})dC_f - \frac{vD}{g_c}dL &= 0. \quad (16)
\end{align}

Fig. 3. Time history of bus ridership as central city bus service grows and two new commuter links are introduced.
Non-trivial solutions are possible when \( w_1 \) and \( w_2 \) are the roots of the following quadratic equation (also called the characteristic equation):

\[
\begin{align*}
w^2 + \left(2K - \frac{vD}{g} - \frac{nvD_1}{g_c} - \frac{vD_2}{g_c}\right)w \\
+ \left(K - \frac{vD}{g} - \frac{nvD_1}{g_c}\right) \left(K - \frac{vD_2}{g_c}\right) - n \left(\frac{vD_2}{g_c}\right)^2 = 0
\end{align*}
\]

(17)

We now define a number \( n^c \) as

\[
n^c = (K - a)(K - b)/kb
\]

(18)

with

\[
a = vD/g, \quad b = vD_2/g_c
\]

(18a)

As a function of the values of the parameters \( K, a, b \) and \( n \), four different cases are possible. These, obtained using simple algebra, are summarized in the schematic below:

<table>
<thead>
<tr>
<th>Schematic regions of stability of ( L = 0, C_j = 0 ) solution</th>
<th>( K &lt; a + nb )</th>
<th>( K &gt; a + nb )</th>
</tr>
</thead>
<tbody>
<tr>
<td>if ( K &lt; b ) and ( n &lt; n^c )</td>
<td>if ( K &lt; b ) and ( n &gt; n^c )</td>
<td>if ( n &lt; n^c )</td>
</tr>
<tr>
<td>if ( b &lt; K &lt; a ) and ( n \in [0, n^c] )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>if ( a &lt; K &lt; a + nb ) and ( n &gt; n^c )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unstable Node</td>
<td>Saddle Point</td>
<td>Stable Node</td>
</tr>
<tr>
<td>( w_1 &gt; 0 )</td>
<td>( w_1 &gt; 0 )</td>
<td>( w_1 &lt; 0 )</td>
</tr>
<tr>
<td>( w_2 &gt; 0 )</td>
<td>( w_2 &lt; 0 )</td>
<td>( w_2 &lt; 0 )</td>
</tr>
<tr>
<td>Case 2</td>
<td>Case 4</td>
<td>Case 1</td>
</tr>
</tbody>
</table>

These four cases are shown in Fig. 4 which maps out the regions of stability/instability of the busless all car state in \((n, K)\) space.

This Figure shows the busless state to be always unstable (either an unstable node or a saddle point) when costs \( K \) are sufficiently low (in the Figure, for \( K < a \)). Bus service in this regime, if initiated, will grow as will be seen later when we discuss the non-trivial solutions of the model.

The effect of adding commuter service is most evident for costs \( K > a \). Here we see that even for high maintenance costs \( K > a \), if there is a sufficiently large number of connections \( n > n^c \), the busless state remains unstable to fluctuations such as the initiation of bus service. Not until both the costs \( K \) exceed a critical threshold, \( a + nb \), and the number of connections falls below a critical value \( n^c \) is the busless all car state a stable solution of the model.

Non-trivial solutions. In addition to the trivial stationary state solution (14), there are other stationary state solutions of the model (eqns (12) and (13)), which, after some straightforward algebra turn out to be given by the cubic equation:

\[
C_j^3 + a_1 C_j^2 + a_1 C_j + a_2 = 0
\]

(19)

with

\[
a_2 = \frac{1}{n + 1} \left\{ (n + 2)g_c + \frac{vD}{K} - (n + 1) \left(\frac{2vD_2}{K} + g\right) \right\}
\]

(20)
Fig. 4. Regions of stability and instability of the busless all-car state, $L = 0, C = 0.$

\[ a_1 = \frac{1}{n+1} \left\{ \frac{vD}{K} \left[ \frac{vD}{K} + 2g - (n+2)g_c - \frac{2vD}{K} \right] + g_c \left( g_c + \frac{vD}{K} - g \right) \right\} \]  

\[ a_0 = \frac{vD}{(n+1)K} \left\{ \frac{vD}{K} - (n+1)g + g_c \left( g - \frac{vD}{K} \right) \right\} \]  

where $n$ is the number of links between the central city and its surrounding suburbs (one link per suburb). The corresponding stationary state solution for $L$ is given by

\[ L = C \left[ \frac{g_c}{(vD/K) - C_c} - 1 \right] \]  

We note here that for a given $L$, eqn (23) is a quadratic equation in $C$ which has only one positive root, then to each value of $L$ corresponds a unique value of $C$; vice versa, a solution of the cubic eqn (19) will be acceptable if $L > 0$, that is if

\[ C > vD/K - g_c. \]  

In terms of the reduced variable $C_R = Ch/g_c$, the condition is

\[ C_R > 1 - h, h = g_c Kl/vD. \]  

DISCUSSION

Inspection of the cubic eqn (19) shows that there will be at least one positive (and hence physically acceptable) root if the coefficient $a_0 < 0$. This condition on $a_0$ is equivalent to the condition

\[ K^2 - K(a + (n + 1)b) + ab < 0. \]  

TR-B 178, No. 1–C
In terms of the number of connections, \( n \), this condition becomes

\[
\frac{(K-a)(K-b)}{Kb} > n^c.
\]  

(27)

The right hand side of (27) is precisely \( n^c \), the critical number of links that was found to determine the stability of the trivial state. Thus, just at the point \( n = n^c \) where the trivial solution becomes unstable a new solution emerges for positive central city and suburban bus service.

In fact, it can be shown that \( u_0 = 0 \) if \( n = n^c \); in this case \( C = 0 \) is a solution of eqn (12). As \( n \) increases, the solution \( C \) would also increase from that zero value to the stationary state value corresponding to the given \( n > n^c \).

Figure 5 depicts this situation: Below a critical number of connections \( n < n^c \), and above a critical cost \( K > K^c \), \((K^c, C^c)\) obtained as a solution of eqn (26) where the trivial solution is stable, there is no physically acceptable root to the cubic equation. Thus, the only stationary state in this regime is the busless one.

For a larger number of connections, \( n > n^c \), Fig. 5 shows the existence of one stable stationary state (the trivial solution is unstable in this regime). This stable stationary state grows in magnitude as the costs \( K \) continue to decrease and it can be shown that the solution of the cubic remains acceptable (real and positive) when \( n < n^c \) and \( K < K^c \).

We have pointed out that not all positive solutions \( C \) of the cubic are allowed since \( L \) must also be a positive quantity. (See eqns (24) and (25)).

All values of \( C_R \) that fall below the 1-h line in Fig. 5 give physically unacceptable negative values for \( L \). We also point out that there are no positive values of \( C_R \) that lie between the 1-h line and the given positive \( C_R \) root. This can be shown easily: when we substitute 1-h into the cubic equation we find that its value is negative, then there, of course, is no intersection because no point on the 1-h line is a solution of the cubic.

Summarizing Fig. 5 which is meant to provide a numerical example which illustrates the discussion of this section, we have plotted the reduced variable \( Ch/g_c \) vs \( K \), and find that there is no physically acceptable solution of the cubic for \( K > K^c \). However, for all values of the costs \( K < K^c \) (where the trivial solution is unstable), there is a physically acceptable solution which grows as costs decrease. This solution is stable. Other positive \( C_R \) roots fall below the 1-h line \((K^c, C^c)\) always less than \( b \) which are therefore unacceptable as previously discussed \((L < 0 \) there).

In the next section we provide additional simulations of the model.
Stationary states simulation

The analysis in the preceding sections is now further illustrated with the aid of several numerical examples.

In the first example, the stationary state solutions of eqns (19) were obtained, their stability to perturbations determined (from eqns A8–A10), and plotted as a function of an increasing number of central city-suburban public transportation links. The results are shown in Figs. 6–8. Also shown in the figures are the all car stationary state solutions.

Figure 6 shows that the only possible stationary state solution of the system is a stable all car solution (\( L = 0, C_j = 0 \)) for values of \( n < n^c \). The stationary state solutions of the cubic, eqn (19) are non-physical in this regime, and hence do not exist. However, beyond the critical number \( n^c \), the all car solution, while still a physical and hence possible solution of the system, becomes unstable to perturbations. Also, at this critical value, \( n^c \), one of the solutions to the cubic becomes realizable and is stable to perturbations. This means that for values of \( n > n^c \) any perturbation, no matter how slight, will cause the all car solution to jump to the stable branch. This stable stationary state grows with increasing \( n \). Specifically, as can be seen from Fig. 6, once \( n > n^c \), central city bus service \( L \) grows quite rapidly with increasing \( n \) as does, though less rapidly, the quality of commuter service between central city and the suburbs, \( C_j \).

The corresponding increase in bus ridership is shown in Fig. 7. We note that continued investment in links \( n \), though increasingly improving bus service \( L \), does not result in as rapid an increase in bus ridership, which begins to flatten out. In other words, there is diminishing return on investment.

Also as this bus ridership increases, there is a corresponding decrease in car usage as can be seen in Fig. 8. This figure shows, as well, that central city residents \( x \), give up their car more quickly than suburban residents, \( x_j \), as has been observed.

It is also instructive to investigate the effect of changes of either fares or unit costs on the growth of bus service and ridership. We choose, therefore, a second example, shown in Figs. 9–11, which plots the stationary state solutions of the system as a function of the unit costs of
Fig 7. Central city bus ridership, $y/D$ (from central city residents) and $y/D_s$ (from suburban residents) vs number of central city-suburban links, $n$. Solid lines represent stable states. Dashed lines represent unstable states. Parameter values and critical values same as in Fig. 6.
Fig. 8. Central city car ridership, $x/D$ (from central city residents) and $x'/D_s$ (from suburban residents) vs number of central city-suburban links, $n$. Solid lines represent stable states. Dashed lines represent unstable states. Parameter values and critical values same as in Fig. 6.
Fig. 9. Central city bus service, L, vs unit costs of maintaining service, K. Solid lines represent stable states. Dashed lines represent unstable states. Parameter values: \( uD = 4500, vD = 450, g = 2, \sigma = 5, n = 10 \). Critical values: \( K^*_c = 3176, K^*_c = 63.75 \).

Fig. 10. Bus commuter service \( C_j \) vs unit costs of maintaining service, K. Solid lines represent stable states. Dashed lines represent unstable states. Parameter values and critical values same as in Fig. 9.
providing bus service. In the example, the fares are held constant as well as the number of connections, \( n \).

Figures 9 and 10 show that for very high unit costs, only one stationary state of the system exists, namely, the all car state, which may be interpreted to mean that there is not enough return on investment to initiate bus service. Of course, if we had put state subsidies (or other considerations) into the model, then some given level of service would exist. That would merely change the scale of service at which the jump would occur. The system would then have begun from a finite level of service rather than from a zero level.

Continuing with Figs. 9 and 10, we see that unit costs must decrease to below a critical value \( K^* \) before the growth of central city and commuter bus service can begin. In other words, the stationary state, \( (L \neq 0, C \neq 0) \) becomes a physical solution to the model equations only for values of \( K \) below \( K^* \). This state is stable to perturbations. The stationary state \( (L = 0, C = 0) \) continues to exist but becomes unstable to perturbations for costs below \( K^* \), and hence would not be an acceptable solution. Bus service (and ridership, see Fig. 11) continue to grow, once initiated, with the continued decrease in costs while car usage, correspondingly, declines.

**Dynamical simulation**

Figure 12 shows the dynamic approach of \( L(t) \) and \( C(t) \) to their respective stationary states. The figure is obtained from integrating the model eqns (12) and (13) for given numerical values of the parameters (shown in the Fig. Legend). The corresponding changes in ridership, \( y \) are shown in Fig. 13.

The dynamic simulation is started off by initiating bus service at an initial time \( t = 0 \) and allowing the system to grow. As bus service \( L \) begins to increase, it stimulates the growth of commuter line connections \( C \) according to the coupled set of eqns (12) and (13). Growth in each of the connections, \( C \), further stimulates the growth of \( L \) and so on until \( L \) and each of the \( C \)'s reach their stationary state values.
Fig. 12. Dynamic evolution of $L/g_c$ and each $C_j/g_c$ showing their coupled growth for the parameter values:

$K = 1.5$, $a = 4$, $b = 2$ and

- $n = 4$ solid line
- $n = 2$ dashed line
Fig. 13. Growth of bus ridership corresponding to growth in bus service $H_R$ and $C_R$ of Fig. 12 for the $n = 2$ case.
CONCLUSIONS

Using even a highly skeletal model of the interaction between transportation users and a public transportation authority, we find a discontinuous response of the system to changes in the parameters describing the system, for example, the costs and the number of links connecting the central city to its surrounding suburbs. We find that the system will persist at a low level of usage even when costs decrease and the number of service links increase unless the magnitude of such changes are sufficiently large. If not, such as investment will be "wasted". This is an example of the behavior of real systems, which instead of exhibiting monotonic growth (or decline) with changing stimuli, show very non-linear, discontinuous response to such stimuli. This is also an illustration of the concept of dissipative structures where a system maintained far from equilibrium will re-structure only if there is an amplification of the non-linear interactions, resulting in an instability in the original system. This phenomenon was exhibited in our model which discontinuously structured to a new regime of operation, the mixed mode state, only when a critical number of links was attained or a critical level of cost was reached to cause an instability in the all car state of the system.

REFERENCES


Transportation mode choice and city-suburban public transportation service

\[ A = K - nDg(L + g)^2 \]  
\[ B = nDgJ[(C_L + L + g)]^2 \]

where

\[ F = K - nDgJ[(C_L + L + g)]^2. \]

We note that eqn (A1) has the form

\[ w^2 + Sw + \Delta = 0 \]

where, using eqns (A5)–(A7):

\[ S = 2K - nDgJ(L + g)^2 - (n + 1)nDgJ[(C_L + L + g)]^2 \]

and,

\[ \Delta = [K - nDgJ(L + g)^2][K - DgJ[(C_L + L + g)]^2] - nKgJ[DgJ[(C_L + L + g)]^2]. \]

The stability criteria, in terms of \( S \) and \( \Delta \) are shown in the following schematic.

<table>
<thead>
<tr>
<th>Regions of stability, general case</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S = (K + A - (n + 1)B) &lt; 0 )</td>
</tr>
<tr>
<td>( \Delta = AK - AB - nBK &gt; 0 )</td>
</tr>
<tr>
<td>Unstable Node</td>
</tr>
<tr>
<td>( w_1 &gt; 0 )</td>
</tr>
<tr>
<td>( w_2 &gt; 0 )</td>
</tr>
</tbody>
</table>

For completeness, we may note that if the two roots for \( w \) are not real we would have the following additional conditions:

1. If \( w_1 \) and \( w_2 \) are complex conjugates, then the state is either a stable or unstable focus (producing either an oscillatory approach or an oscillatory departure) depending upon whether \( S > 0 \) or \( S < 0 \), respectively.
2. If the roots \( w \) are pure imaginary, \( S = 0 \) but \( \Delta \neq 0 \), then the system is only marginally stable.