3. Public Transportation: A Dynamic Model of Mode Choice and Self-Organization

Abstract

A model is presented to describe the dynamics of transportation mode choice in which the interaction between transportation users and a public transportation authority results in self-organization. The model illustrates that a sufficient number of connections between a central city and its suburbs are required for self-organization to occur whereby public transportation use and service will grow. It also demonstrates the existence of bifurcation points at certain times in a system's evolution. It is at these junctures that government policy action can have the most effect with the minimum input.

Introduction

This paper presents a new model of transportation mode choice which illustrates the use of non-linear differential equations exhibiting instabilities. This type of equation has been used to examine dissipative structures in chemical systems(1) and in the biological(2-5) and social sciences (6-10). Recently, this concept has been applied to transportation mode choice(11-12) which, in this paper, is given
a spatial dependence by considering a policy of establishing public transportation connections between a central city and the suburbs surrounding it. The question asked here is how this policy initiative may serve to enhance public transportation ridership and how large an investment is needed for the initiative to "catch on."

While the particular model developed here is not intended to give actual numerical values, it does provide a theoretical framework so that such questions may be realistically addressed with the proper elaboration of spatial detail and with an expansion of the factors which influence behavioral actions.

The model, like those in References 11 and 12, postulate certain basic behavioral actions on the part of the actors, namely, the users of transportation and the public transportation authority. The interaction among these elements, each with its own behavioral criteria for action, permits different evolutions of the system depending on the strength of the interaction. This, in turn, depends upon such factors as fares charged and costs of running the transportation system. Other factors will also be discussed in the next section. What is important in this description is that a particular evolutionary path may become significantly altered if the strength of any one of these factors becomes great enough (exceeds a critical threshold).

In this case, the system may become unstable to the "stimulus" being imposed, for example, the fares charged, and jump to another state which is stable. This phenomenon is a simple example of order through fluctuation(1), where the fluctuation here is a change in strength in the factors affecting the behavior of the actors. We shall see that in addition to finding very stable regimes of behavior where only a prohibitively large change will affect the system's stability, other regimes exist where even small changes will markedly alter the system's behavior (this happens when the system is near one of its bifurcation points).

The factor we shall examine in this paper is a policy initiative of adding or subtracting public transportation lines that connect a central city with the surrounding suburbs. Even in the very simple model offered here, we shall find the existence of a bifurcation point around which the addition of only one more connecting line can profoundly alter the system's response, resulting in a significant increase in public transportation usage over that which existed previously. It is around such points of bifurcation where policy initiatives will be most effective in influencing the response of the system.

Presentation of the Model

To help us develop a model of interaction between the users of transportation and policy implementations of a public transport mode, we imagine a central city surrounded by suburbs, shown schematically in Figure 1.

Within the central city there are automobile users, designated $x$ and public transportation users, designated $y$ (for specificity we shall take the public transportation mode to be the bus, though with a suitable change of scale this could be a subway system). The equations governing the time rate of change of these users of transportation has the general form (see our previous paper, Reference 11, for a justification of this form):
Public Transportation: A Dynamic Model

\[
\begin{align*}
\dot{x} &= \frac{DA_x}{A_x + A_y} - x; \quad \dot{y} = \frac{DA_y}{A_x + A_y} - y
\end{align*}
\]

(1)

Here, \( D \) is the estimated number of people who want to use transportation which is assumed to vary slowly compared with the changes of \( x \) and \( y \), so that it may be considered constant (no new users are brought into the system during the time of interest). \( A_x \) and \( A_y \) are the attractiveness for car and bus, respectively, and the terms containing them in Equations (1) express the fraction of the total demand that is attracted to the car or to the bus. The explicit form of these attractivities will be discussed after we introduce the other variables of the system.

Also within the central city, the variable describing the quality of bus service is designated as \( L \), whose time rate of change is given by:

\[
\dot{L} = v_y + \sum_j vy_j - KL
\]

(2)

where \( v \) is the fare (\( v_y \) then is the revenue received); \( K \) is the cost per unit of service offered (\( KL \) then represents the costs of providing service); \( y_j \) is the number of residents of suburb \( j \) who use the bus in the central city (\( \sum_j vy_j \) then represents the revenue that the city bus system receives from the users \( y_j \). Clearly, other terms could be added to Equation (2) to express more realistically the dynamics of the public transportation mode, for example, a state subsidy term could be added. However, this will not be done here since Equation (2) is sufficient to demonstrate the feedback phenomena responsible for bifurcation as discussed in the Introduction.
The variable describing the commuter connection between the central city and the suburb \(j\) is designated \(C_j\) whose time rate of change is given by:

\[
\dot{C}_j = v'y_j - K'C_j
\]  

(3)

which is similar in form to Equation (2). The quality of service between the central city and the suburb \(j\) is assumed to be given simply by the net income: revenues received, \(v'y_j\), less maintenance costs, \(K'C_j\), where \(v'\) and \(K'\) are the fares and unit maintenance costs, respectively.

The equations for \(y_j\) and \(x_j\), representing people who live in suburb \(j\) and use the bus in the central city and people who live in \(j\) and commute by car, respectively, have the same form as Equations (1):

\[
\dot{x}_j = \frac{D_j x_j}{A'_x + A'_y} - x_j ; \quad \dot{y}_j = \frac{D_j y_j}{A'_x + A'_y} - y_j
\]  

(4)

where \(D_j\) is the demand for transportation between suburb \(j\) and central city. \(A'_x\) and \(A'_y\) are the attractiveness for car and bus commuting, respectively.

It is now necessary to give explicit representation to the attractiveness \(A_x\), \(A'_x\), \(A_y\), and \(A'_y\). We assume a constant attractiveness for car use within the central city, given by \(g\), and another constant attractiveness for car use between suburb and central city, given by \(g_j\):

\[
A_x = g \quad ; \quad A'_x = g_j
\]  

(5)

For bus usage, we assume that the attractiveness depend upon the quality of service offered. Hence, for people who live in the central city, we assume:

\[
A_y = L + C_j
\]  

(6)

and for people who live in a suburb \(j\) and who commute via public transportation to the city where they use the central city bus system, we have:

\[
A'_y = L + C_j
\]  

(7)

We note, in passing, that this form (7) is only one of several that may be used. For example, the product of \(C_j\) with \(L\) could be used to describe the situation where the absence of either \(C_j\) or \(L\) would preclude growth of \(y_j\).

Using these attractiveness in Equations (1) and (4) give for the evolution of car and bus users:

\[
\dot{x}_j = \frac{D_j x_j}{C_j + L + g_j} - x_j
\]  

\[
\dot{y}_j = \frac{D_j (C_j + L)}{C_j + L + g_j} - y_j
\]  

(8)

(9)

In words, these equations state that the growth in bus and car users depends upon the relative attraction of the
bus or the car mode, which in turn, depends upon the relative quality of service offered. In the case of central city residents this relative quality of service is \( \frac{L}{L+g} \) for the bus and \( \frac{g}{L+g} \) for the car, while for suburban residents it is \( \frac{L+C_j}{L+C_j+g_j} \) for the bus and \( \frac{g_j}{L+C_j+g_j} \) for the car.

Relative Growth Rate Assumption

The system of equations (2) and (3) for the growth of the central city bus system and the growth of the commuter lines, respectively, together with Equations (8) and (9) for the growth of transportation users, describe the model. However, we shall now assume that any central city bus or commuter line growth occurs much more slowly than does the reaction of people to mode choice. This assumption allows us to replace Equations (8) and (9) with their stationary state values, while retaining the time rate of change form for the bus and commuter line equations (that is, we assume that there is a time long enough so that the \( y \)'s and the \( x \)'s have attained a stationary state while the \( L \) and \( C_j \) still have not). With this assumption, Equations (8) and (9) become:

\[
y = \frac{vBL}{L+g} ; x = D-y
\]

(10)

\[
y_j = \frac{vD_j(C_j+L)}{C_j+L+g_j} ; x_j = D_j - y_j
\]

(11)

Substituting these expressions into Equations (2) and (3) then gives:

\[
\dot{L} = \sum_{j} \frac{vD_j(C_j+L)}{C_j+L+g_j} - KL
\]

(12)

\[
\dot{C}_j = \frac{vD_j(C_j+L)}{C_j+L+g_j} - K'C_j
\]

(13)

as the model equations for the growth of the central city bus service, \( L \), and the commuter lines, \( C_j \).

Analytical Solutions of the Model

A Simplified Version of the Model

The equations for the growth of central city bus service (12) and for the growth of suburban-central city commuter lines (13), may be easily solved analytically for its stationary state values if we make the simplifying assumptions that each suburb has the same transportation demand

\[
D_1 = D_2 = \ldots = D_n = D
\]

(14)

and that each suburban-central city automobile attractivity term, \( g_j \), is the same

\[
g_1 = g_2 = \ldots = g_n = g_c
\]

(15)

For completeness we may note that we have implicitly assumed that there is a reaction rate \( \gamma \) which multiplies the right hand side of Equation (12). This reaction rate \( \gamma = \gamma ', \gamma '' \), where \( \gamma \) would be the corresponding reaction rate to mode choice in Equations (8)(9). This idea is shown by use of the stationary state values as given in (10) and (11).
We further simplify the calculations by taking the fares and maintenance costs to be the same

\[ v = v' \quad K = K' \]  \hspace{1cm} (16)

We point out that these simplifying assumptions do not allow us to distinguish between suburbs, nor allow us to investigate the effect of the installation of different levels of commuter service to the different suburbs. We are thus restricted to the investigation of the effect of adding or subtracting commuter line connections between "equal" suburbs and the central city. That is, we shall be concerned here only with the effect of the number of lines on transportation usage and growth. \(^2\)

Rewriting Equations (12) and (13) to reflect the above assumptions gives:

\[ L = \frac{vDL}{L + g} \cdot \sum_j \frac{vD_C(C_j + L)}{C_j + L + g} - KL \]  \hspace{1cm} (17)

\[ C_j = \frac{vD_C(C_j + L)}{C_j + L + g} - KC_j \]  \hspace{1cm} (18)

\(^2\)We also note that while this paper considers only one central city, it is also of interest to consider and compare a spatial structure with more than one central city (one or more central business districts) serving a number of suburban locations.

---

**Analysis of the Model**

**Introductory Analysis.** Before studying the model Equations (17) and (18) we introduce in this section an analysis of a much simpler case in order to show more clearly how the suburban and central city systems influence each other in producing growth of public transportation. We shall begin with a central city only, adding first one commuter line and then another to see how their introduction influences public transportation in the central city. Following this simple case we will then return to the more complicated, general set of coupled central city-suburban equations (17) and (18).

Beginning with only a central city and no connection to the suburbs, the equation which governs the time evolution of central city bus service is (from (17))

\[ L = \frac{vDL}{L + g} - KL \]  \hspace{1cm} (17a)

We are interested in the conditions for growth of central city bus service. From (17a) we find that there will be growth if \( vD/g > K \). When this condition is met, any perturbation \( eL \) around the zero stationary state solution of (17a), \( L = 0 \), will grow in time. Thus \( L = 0 \) is an unstable solution whenever \( K < vD/g \), and bus service will grow. Physically, this is clear: \( L \) will grow whenever the costs \( K \) of providing service are sufficiently small (or the demand \( D \) for transportation is sufficiently great).

We now ask, similarly, what the conditions for commuter line growth may be if central city bus service is non-
We are interested in the conditions for growth of commuter line service. From (18a) we find that there will be growth if \( \frac{v_D s_c}{g_o} \geq K \). When this condition is met, any perturbation \( C\) around the stationary state solution \( C=0 \) of (18a) will grow in time. Thus, \( C=0 \) is unstable whenever \( K \geq \frac{v_D s_c}{g_o} \): when costs are low (or demand high) the system will grow.

We now wish to see the effect of adding a commuter link on central city bus service. The effect of such a link is to enhance central city bus service as can be seen by noting that the non-trivial stationary state solution for commuter service is given by (from Equation 18a):

\[
C = \frac{v_D}{K} - g_o \tag{18b}
\]

which, as a physical solution, is positive and thereby produces growth \( \lambda > 0 \) in the central city bus system. And, similarly, if the central city bus service has reached its non-trivial stationary state

\[
L = \frac{v_D}{K} - g \tag{17b}
\]

this positive term will produce growth \( \dot{C} > 0 \) in the commuter line system as is evident from Equation (18).

We illustrate this phenomenon of mutual reinforcement in Figure 2 which shows the time history of the system when two commuter lines are successively introduced.

The figure shows the system starting off with no commuter links \( C=0 \) when a perturbation \( dL=0 \) is introduced. \( L \) grows since the costs \( K \) have been made sufficiently small \((KvD/g)\). This growth approaches a steady state (given by Equation 17b). At some later time, \( t=9 \) units, a commuter line is introduced into the system. We observe the growth of that commuter line for costs \( K\dot{v}_D/g_o \) (sufficiently small to allow growth to occur) and observe its rather strong effect on the subsequent growth of central city bus service \( L \). The growth in \( L \) and \( C \) continues until new steady states are reached for these quantities. Finally, the Figure shows the introduction of a second commuter line at a still later time. We observe its growth to a steady state as well as the growth of central city bus service \( L \) which it helped stimulate.

For completeness, Figure 3 illustrates the coexisting time history of bus users, as central city service and commuter service grow. We observe the increase in bus ridership corresponding to growth in \( L \) until a steady state is reached and the ridership saturates. However, the introduction of a commuter line causes a further increase in bus ridership until that too saturates. The figure shows two such commuter lines and the consequent increase in ridership.

Stability Analysis. Returning now to the model equations (17) and (18) for the growth of the central city bus system and commuter lines, we find immediately the stationary state solutions:
which expresses the absence of any central city bus service and the absence of any commuter service. From Equations (10) and (11), we see that it implies all car usage, that is:

\[
\begin{align*}
    y = 0 & \quad y_j = 0 \\
    x = 0 & \quad x_j = 0
\end{align*}
\]

Though this is a solution of the model equations, its existence is not enough to ensure that it will persist for all (or even for any) of the values of the parameters. If the system is unstable to perturbations around this state and a perturbation here would be equivalent to the introduction of a bus line into the system, then the state will not persist. On the other hand, if (19) is stable to the perturbations, then the zero state will persist even though the bus line was introduced. It is therefore, important to know the range of values for which the stationary state is stable, and from the point of view of the decision maker, it is important to know the values of the parameters at which the system is most sensitive to change, so that a timely policy initiative may be made to effect a desired change and also to be aware when an unwanted change may be likely to occur. In other words, for which values of the parameters of the system (here, for example, level of fares, costs of maintaining the service, demand, number of commuter lines, etc.) should the decision maker be alerted to a likely significant change of system performance (change of state) and in what direction is this change going to occur? Also, what "perturbation" must be imposed upon the system to affect its performance in a desired way? The model equa-
tion's solutions, their stability properties and numerical example should illustrate the above points.

The stability analysis is done by subjecting the model equations (17) and (18) to perturbations $dL$ and $dC_j$ around its stationary states. The first stationary state found was the trivial one given in Equation (10) in which the system permitted only automobile transportation. We want to investigate the stability of this "busless" state to perturbations in the quality of central city bus service, $dL$, and to perturbations in the quality of the service between central city and the suburbs $dC_j$.

Assuming that the time dependence of the variables $L$ and $C_j$ vary as exp $(wt)$, the model equations (17) and (18) become, when subjected to perturbations around the trivial stationary state,

$$\omega + K - \frac{v_D}{g} - \frac{nv_D}{g_c} \sigma L - \frac{nv_D}{g_c} \sigma C_j = 0 \quad (20)$$

$$\omega + K - \frac{v_D}{g_c} \sigma C_j - \frac{v_D}{g_c} \sigma L = 0 \quad (21)$$

Solving this system gives the following equation for $w$:

$$w^2 + (A+F)w + (AF-nB)^2 = 0 \quad (22)$$

with solutions:

$$w = \frac{-A+F \pm \sqrt{(A+F)^2 - 4(AF-nB)^2}}{2} \quad (22a)$$

where:

$$A = K - \frac{v_D}{g} - \frac{nv_D}{g_c} \quad (23)$$

$$B = \frac{v_D}{g_c} \quad (24)$$

$$F = K - \frac{v_D}{g_c} \quad (25)$$

It is easily seen that there will be a positive root $w>0$ whenever:

$$A + F < 0 \quad (26)$$

or whenever:

$$A + F > 0 \quad (27)$$

These conditions of stability may be summarized as in the schematic below.

**Regions of Stability of L=0, C_j=0 Solution**

<table>
<thead>
<tr>
<th>$A+F &lt; 0$</th>
<th>$A+F &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unstable Node</td>
<td>Saddle Point</td>
</tr>
<tr>
<td>Stable Node</td>
<td>Saddle Point</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AF - nb^2 &lt; 0</th>
<th>AF - nb^2 = 0</th>
<th>AF - nb^2 &gt; 0</th>
<th>AF - nb^2 &gt; 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1 &lt; 0$</td>
<td>$w_1 &gt; 0$</td>
<td>$w_1 &lt; 0$</td>
<td>$w_1 &gt; 0$</td>
</tr>
<tr>
<td>$w_2 &lt; 0$</td>
<td>$w_2 &gt; 0$</td>
<td>$w_2 &lt; 0$</td>
<td>$w_2 &gt; 0$</td>
</tr>
<tr>
<td>(2)</td>
<td>(3)</td>
<td>(1)</td>
<td>(4)</td>
</tr>
</tbody>
</table>
A positive root, \( a > 0 \), producing an unstable node or a saddle point, means that the system is unstable to the perturbation (the perturbation is amplified with time). Negative roots \( w_1 < 0 \) and \( w_2 < 0 \), producing a stable node, result in the damping of the perturbation with time.\(^3\)

The above four cases will now be studied in more detail and a diagram mapping out the regions of stability will be constructed.

**Case 1: Stable Node**

Let us call the stable node case 1. The condition that the trivial solution be stable is:

\[
\begin{align*}
A + F &> 0 \\
AF - nB^2 &> 0
\end{align*}
\]

Taking into account the effect of the second of these two on the first requires that \( A \) and \( F \) each be positive, and

\[
\begin{align*}
A > 0 &\quad \Rightarrow \quad K > a + nb \\
F > 0 &\quad \Rightarrow \quad K > b
\end{align*}
\]

where we have defined:

\[
a = vD/g \\
b = vD/g_c (=B)
\]

Thus, we must have:

\[K > a + nb\]

in order that the trivial solution be stable.

The other condition \( AF - nB^2 > 0 \) leads to the following inequality on \( n \) for the trivial solution to be stable:

\[
n < (K-a)(K-b)/Kb = n^0
\]

Both of these conditions (31) and (32) make sense physically: the business, all car state will be stable if public transportation costs \( K \) are sufficiently high, \( K > a + nb \), and if there are not enough public transportation accesses \( n \) to the central city.

**Case 2: Unstable Node**

The condition that the trivial solution be an unstable node is:

\[
\begin{align*}
A + F &< 0 \\
AF - nB^2 &> 0
\end{align*}
\]

which requires that \( A < 0 \), \( F < 0 \) and

\[
\begin{align*}
A < 0 &\quad \Rightarrow \quad K < a + nb \\
F < 0 &\quad \Rightarrow \quad K < b
\end{align*}
\]

\(^3\) The difference between an unstable node and a saddle point is that for the former all trajectories issue and diverge from the singular point, whereas for a saddle point only two asymptotes pass through the singular point (the saddle point); all trajectories except for one of the asymptotes diverge from the saddle point, which therefore is always unstable.
Thus, in order that the zero solution be an unstable node we must have:

\[ Kc_b \]  

as well as (from the second condition of (33))

\[ n < (K-a)(K-b)/K_b = n^c \]

Thus, if bus service costs \( K \) are sufficiently low \( Kc_b \), even if the number of commuter lines \( n \) \( > n^c \) bus service will grow (the busless state is unstable to perturbations). As we shall see in cases 3 and 4, when costs are higher, it will be necessary to create a number of commuter lines such that \( n \) \( > n^c \) in order to have bus service grow.

**Case 3 and 4: Saddle Points**

In these cases, the conditions that the trivial solution be a saddle point are:

**Case 3**

\[ A + F > 0 \]

\[ AF - nB^2 < 0 \]

**Case 4**

\[ A + F < 0 \]

\[ AF - nB^2 < 0 \]

In an analysis similar to that done for the first two cases, we find the following conditions for the busless state to be unstable:

\[ Kc_a + nb > n^c \]  

Even though costs are high, there is a sufficiently large number of connections to make the busless state unstable.

**Case 4**

Large number of connections and a sufficiently low cost

Costs still sufficiently low

A sufficiently large number of connections even though costs may be high

These four cases are shown in Figure 11 which maps out the regions of stability/instability of the busless, all car state in \((n, K)\) space.

The figure shows the busless state to be always unstable (either an unstable node or a saddle point) when costs \( K \) are sufficiently low (in the figure, for \( Kc_a \)). Bus service in this regime, if initiated, will grow as will be seen later when we discuss the non-trivial solutions of the model.

The effect of adding commuter service is most evident for costs \( Kc_a \). Here we see that even for high maintenance costs \( Kc_a \), if there is a sufficiently large number of connections \( > n^c \), the busless state remains unstable to fluctuations such as the initiation of bus service. Not until both the costs \( K \) exceed a critical threshold, \( a + nb \), and the
number of connections falls below a critical value $n^c$ is the busless all car state a stable solution of the model.

Non-Trivial Solutions. In addition to the trivial stationary state solution (19), there are other stationary state solutions of the model (Equations (17) and (18)), which, after some straightforward algebra, turn out to be given by the cubic equation:

$$c_3^3 + a_2 c_3^2 + a_1 c_3 + a_0 = 0$$  \( (41) \)

with

$$a_2 = \frac{1}{n+1} \left\{ (n+2)g_c + \frac{vD}{K} - (n+1) \left( \frac{2vD_s}{K} + g \right) \right\}$$  \( (42) \)

$$a_1 = \frac{1}{n+1} \left\{ \frac{vD_s}{K} \left[ (n+1) \left( \frac{vD_s}{K} + 2g \right) - (n+2)g_c \right] - \frac{2vD}{K} \right\} + g_c \left( g_c + \frac{vD}{K} - g \right)$$  \( (43) \)

$$a_0 = \frac{vD_s}{(n+1)K} \left\{ \frac{vD_s}{K} \left[ \frac{vD}{K} - (n+1)g \right] + g_c \left( g - \frac{vD}{K} \right) \right\}$$  \( (44) \)

where $n$ is the number of links between the central city and its surrounding suburbs (one link per suburb). The corresponding stationary state solution for $L$ is given by
We note here that for a given $L$, Equation (45) is a quadratic equation in $C$ which has only one positive root, and to each value of $L$ corresponds a unique value of $C$; vice versa, a solution of the cubic equation (47) will be acceptable only if $L > 0$.

Inspection of the cubic equation (47) shows that there will be at least one positive (and hence physically acceptable) root if the coefficient $a_0 < 0$. This condition on $a_0$ is equivalent to the condition

$$K^2 - K(a + (n+1)b) + ab < 0 \quad (46)$$

In terms of the number of connections, $n$, this condition becomes

$$n > \frac{(K-a)(K-b)}{Kb} \quad (47)$$

The right hand side of (47) is precisely $n^0$, the critical number of links that was found to determine the stability of the trivial state. Thus, just at the point $n = n^0$ where the trivial solution becomes unstable a new solution emerges for positive central city and suburban bus service.

In fact, if we were to substitute $n = n^0$ into the cubic equation, we would find that $a_0 = 0$ and hence that $C = 0$ was a solution. As $n$ increased, the solution $C$ would also increase from that zero value to the stationary state value corresponding to the given $n > n^0$.

---

Figure 5 depicts this situation: Below a critical number of connections $n < n^0$ and above a critical cost $K^0$, which is obtained as a solution of Equation (46), where the trivial solution is stable, there is no physically acceptable root to the cubic equation. Thus, the only stationary state in this regime is the busless one.

For a larger number of connections, $n > n^0$, Figure 5 shows the existence of one stable stationary state (the trivial solution is unstable in this regime). This stable stationary state grows in magnitude as the costs $K$ continue to decrease.

We now point out that not all positive solutions $C$ of the cubic are allowed since $L$ must too be a positive quantity. In fact, the condition that $L > 0$ is (from Equation (45)) that

$$C > \frac{vd_s}{K} - \frac{a_0}{K} \quad (48)$$

In terms of the reduced variable $C_R$, actually plotted in Figure 5, the condition is:

$$C_R > 1 - h \quad (49)$$

All values of $C_R$ that fall below the 1-h line in Figure 5 give physically unacceptable negative values for $L$. We also point out that there are no positive values of $C_R$ that lie between the 1-h line and the given positive $C^0$ root. When we substitute 1-h into the cubic equation we find that its value is negative. And there, of course, is no intersection because no point on the 1-h line is a solution of the cubic.
Summarizing Figure 5, which is meant to provide a numerical example which illustrates the discussion of this section, we have plotted the reduced variable \( \frac{C_h}{\gamma_c} \) versus \( K \), and find that there is no physically acceptable solution of the cubic for \( K > K_c^\ast \). However, for all values of the costs \( K < K_c^\ast \) (where the trivial solution is unstable), there is a physically acceptable solution which grows as costs decrease. This solution is stable. Other positive \( \lambda_{1,2} \) roots fall below the 1-h line \( (K_c^\ast \) always less than \( b \) \) which are therefore unacceptable as previously discussed \( (L < 0 \) there).

In the next section we provide additional simulations of the model.

### Simulations

#### Stationary States Simulation

The analysis in the preceding sections is now further illustrated with the aid of several numerical examples.

In the first example, the stationary state solutions of Equations (41) were obtained, their stability to perturbations determined, and plotted as a function of an increasing number of central city-suburban public transportation links. The results are shown in Figures 6 through 8. Also shown on the figures are the all car stationary state solutions.

Figure 6 shows that the only possible stationary state solution of the system is a stable all car solution \( (L = 0, C_j = 0) \) for values of \( n \alpha n^0 \). The stationary state solutions of the cubic, Equation (41) are non-physical in this regime, and hence do not exist. However, beyond the critical number \( n^c \), the all car solution, while still a physical and hence
Figure 6. Central city bus service, L, and bus commuter service, C_j, versus number of central city-suburban links, n.

Solid lines represent stable states.
Dashed lines represent unstable states.
Parameter values: \( \nu D/K = 1.8, \nu D_j/K = 0.18, g = 2, g_c = 5. \)
Critical values: \( n^c = 2.68. \)

Figure 7. Central city bus ridership, \( y/D \) (from central city residents) and \( y_j/D \) (from suburban residents) versus number of central city-suburban links, n.

Solid lines represent stable states.
Dashed lines represent unstable states.
Parameter values and critical values same as in figure 6.
possible solution of the system, becomes unstable to perturbations. Also, at this critical value, one of the solutions to the cubic becomes realizable and is stable to perturbations. This means that for values of \( n^0 \) any perturbation, no matter how slight, will cause the all car solution to jump to the stable branch. This stable stationary state grows with increasing \( n \). Specifically, as can be seen from Figure 6, once \( n^0 \), central city bus service \( L \) grows quite rapidly with increasing \( n \) as does, though less rapidly, the quality of commuter service between central city and the suburbs, \( C_j \).

The corresponding increase in bus ridership is shown in Figure 7. We note that continued investment in links \( n \), though increasingly improving bus service \( L \), does not result in as rapid an increase in bus ridership, which begins to flatten out. In other words, there is diminishing return on investment.

Also as this bus ridership increases, there is a corresponding decrease in car usage as can be seen in Figure 8. This figure shows, as well, that central city residents \( x \), give up their car more quickly than suburban residents, \( x_j \), as has been observed.

It is also instructive to investigate the effect of changes of either fares or unit costs on the growth of bus service and ridership. We choose, therefore, a second example, shown in Figures 9-11, which plots the stationary state solutions of the system as a function of the unit costs of providing bus service. In this example, the fares are held constant as well as the number of connections, \( n \).

Figures 9 and 10 show that for very high unit costs,
Figure 9. Central city bus service, L, versus unit costs of maintaining service, K.

Solid lines represent stable states.
Dashed lines represent unstable states.
Parameter values: \( v_0=4500, v_0s=4500, g=2, g_c=5, n=10 \).
Critical values: \( K^+_c = 3176, K^-_c = 63.75 \).

Figure 10. Bus commuter service \( C_1 \) versus unit costs of maintaining service, K.

Solid lines represent stable states.
Dashed lines represent unstable states.
Parameter values and critical values same as in figure 9.
only one stationary state of the system exists, namely, the all car state, which may be interpreted to mean that there is not enough return on investment to initiate bus service. Of course, if we had put state subsidies (or other considerations) into the model, then some given level of service would exist. That would merely change the scale of service at which the jump would occur. The system would then have begun from a finite level of service rather than from a zero level.

Continuing with Figures 9 and 10, we see that unit costs must decrease to below a critical value $K^*_c$ before the growth of central city and commuter bus service can begin. In other words, the stationary state, $(L=0, C^*_j=0)$ becomes a physical solution to the model equations only for values of $K$ below $K^*_c$. This state is stable to perturbations. The stationary state $(L=0, C^*_j=0)$ continues to exist but becomes unstable to perturbations for costs below $K^*_c$ and hence would not persist. Bus service (and ridership, see Figure 11) continue to grow, once initiated, with the continued decrease in costs while car usage, correspondingly, declines.

**Dynamical Simulation**

Figure 12 shows the dynamic approach of $L(t)$ and $C(t)$ to their respective stationary states. The figure is obtained from integrating the model equations (17) and (18) for given numerical values of the parameters (shown in the Figure Legend). The corresponding changes in ridership, $y$ and $y_j$, are shown in Figure 13.

The dynamic simulation is started off by initiating bus service at an initial time $t=0$ and allowing the system to
Figure 12. Dynamic evolution of $L/x_C$ and each $C_j/x_C$ showing their coupled growth for the parameter values:

$K=1.5$, $a=4$, $b=2$ and

- $n=4$ solid line
- $n=2$ dashed line

Figure 13. Growth of bus ridership corresponding to growth in bus service $L/x_C$ and $C_j/x_C$ of figure 12 for the $n=2$ case.
grow. As bus service L begins to increase, it stimulates the growth of commuter line connections C according to the coupled set of Equations (17) and (18). Growth in each of the connections, C, further stimulates the growth of L and so on until L and each of the C's reach their stationary state values.

Conclusions

Using even a highly skeletal model of the interaction between transportation users and a public transportation authority, we find a discontinuous response of the system to changes in the parameters describing the system, for example, the costs and the number of links connecting the central city to its surrounding suburbs. We find that the system will persist at a low level of usage even when costs decrease and the number of service links increase unless the magnitude of such changes are sufficiently large. If not, such as investment will be "wasted." This is an example of the behavior of real systems, which instead of exhibiting monotonic growth (or decline) with changing stimuli, show very non-linear, discontinuous response to such stimuli. This is also an illustration of the concept of dissipative structures where a system maintained far from equilibrium will re-structure only if there is an amplification of the non-linear interactions resulting in an instability in the original system. This phenomenon was exhibited in our model, which discontinuously structured to a new regime of operation, the mixed mode state, only when a critical number of links was attained or a critical level of cost was reached to cause an instability in the all car state of the system.

In summary then, this chapter provided an example of a system having real world dynamic non-linear properties strongly coupled to decisions being made by the actors in the system. The actors here were the decision makers involved in developing new suburban transit links to the central city and the users of transportation. Their interaction with the intrinsic dynamics of the system illustrated the self organization and structuring of the transportation system. Whenever the required combination of transit links and low costs were allowed to occur, or were initiated by the decisions makers, bus transit operation emerged as a stable growing component of the transportation network.

References


4. Adaptive Economics

Abstract

Urban and regional planning and specific government policy at all levels require an improved understanding of the complex interplay of forces that influence the form and quality of urban life and govern the evolving relationships among various economic regions. Many of these influences are economic in nature, but since their illumination must clearly involve basic considerations of human behavior and organization, it would appear that the comparative static, equilibrium methodology of orthodox economics needs to be augmented by a broader framework that incorporates dynamic structure and disequilibrium behavior. Such a framework is provided by adaptive economic theory and models.

Adaptive economics involves the extension of economic theory to incorporate the point of view that economic decision-makers are "boundedly rational" (have imperfect information, limited foresight, finite cognitive powers, and changing preferences) and consequently can make plans that are only suboptimal or temporarily optimal. It stresses the implication that economic decisions are imperfectly coordinated so that transactions and behavior must evolve out of equilibrium. It represents economizing behavior as governed by various adaptive processes such as feedback control, behavioral rules, trial and error search, sub-optimization with feedback and other sequential decision procedures. It emphasizes that the basic economic entities (firms, households, banks, government agencies) evolve: their activities, numbers, rules of behavior and organization development in a complex process displaying various modes of change (growth, oscillations, decay), and that exhibit changing phases which represent stages or epochs with distinct structural characteristics.

Support for this study was provided by the U.S. Department of Transportation under Contract No. DTRI 57-81-F-0144.