

Self-Organization and Dissipative Structures

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17. Fluctuations in Demand and Transportation Mode Choice

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INTRODUCTION

The problem this article addresses is the structuring of a system of transportation which is subject to fluctuations in human behavior. It is related to the article in this collection, by P. Allen, on dynamic urban growth models and employs the concept of order through fluctuation in nonlinear far-from-equilibrium systems and may be viewed as focusing on the transportation aspects of those models.

In order to treat the transportation aspects explicitly, several models covering different aspects of transportation were developed. These aspects included the spatial structuring of two similar transportation networks in competition for riders, the physical growth of a transportation system as a function of urban population density, and a model of choice between competing modes of transportation. This article will develop only the last in detail and summarize one of the others in an appendix. These models have the characteristics both of being dynamic and of allowing fluctuations in individual behavior to play a role in the system's evolution. The models, therefore, consider that a deterministic dynamic evolution of a system is always subject to change because of fluctuations.

The method used was to find solutions of the deterministic equations of the model and then to examine the stability of these solutions to fluctuations. For example, if a stationary state solution in one regime of demand for transportation is found to exist, the stability of this solution to fluctuations in demand must also be examined.

Some stationary states may be unstable even to small fluctuations in demand and others, though locally stable, may become unstable if a sufficiently large fluctuation were to occur. The system may then adopt a new solution

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which is stable to the fluctuation. In general, there exist threshold values of demand at which the system's evolutionary path is altered. These changes in a system's evolutionary path are an essential part of the process, and both the time evolution within regimes and the evolution that occurs as thresholds are exceeded and bifurcations occur must be considered.

TRANSPORTATION MODE CHOICE MODEL

Equations of the Model

In developing the equations of the model of transportation mode choice, we note that a number of kinetic equations are possible. We choose one that corresponds to the logistic equation, but one could quite plausibly arrive at this type of equation from a consideration of generalized transition probabilities, as will be shown in an appendix.

We suppose that the total number of transportation users evolves according to the logistic law:

$$\dot{x} + \dot{y} = D - (x + y), \quad (1)$$

where x represents the number of users of one mode of transportation, which we shall take to be the car, and y represents the number of users of a second mode of transportation, which we shall take to be the bus. D is the total number of transportation users.

We further suppose that a function $F_c(x,y)$ measures the attractivity for the car mode; and that $F_b(x,y)$ measures the attractivity for the bus mode. The relative attractivities are then $F_c(x,y)/(F_c+F_b)$ and $F_b(x,y)/(F_c+F_b)$. The equations of evolution for x and for y are then given by:

$$\begin{aligned} \dot{x} &= \frac{D F_c(x,y)}{F_c(x,y) + F_b(x,y)} - x \\ \dot{y} &= \frac{D F_b(x,y)}{F_c(x,y) + F_b(x,y)} - y, \end{aligned} \quad (2)$$

so that $DF_c(x,y)/(F_c(x,y)+F_b(x,y))$, for example, represents potential demand for the car mode.

Attractivity Functions

In order to illustrate the kinds of results this model will yield, we shall take some very simple forms for the attractivity functions. In a real situation, the

attractivities for particular modes will, of course, depend upon a number of factors, including, for example, perceived sociological status associated with a mode and the number of crimes occurring in a particular mode of transportation. Conceptually, it is not difficult to include such factors in the attractivity functions, rather the difficulties are empirical. If empirical data were available on the factors affecting mode choice, then we could, in principal, write attractivity functions which reflected these factors of mode choice.

In the first model to be presented here, we shall assume that the most important characteristic of a transportation mode that affects mode choice is its speed. For this first model we neglect all other factors determining mode choice. In a second model, to be presented in a later section, we shall consider some psychological factors as well.

Attractivity, a Function of Mode Speed Alone

The attractivity functions will be assumed to depend only on the mode speeds as follows:

$$F_c(x,y) = v_c^p, \quad F_b(x,y) = v_b^q, \quad (3)$$

where v_c and v_b are the car and bus speeds and p and q are powers. We further assume that the car and bus speeds can be represented as in fig. 17.1 and fig. 17.2.

The car-speed relation comes from empirical data that car speeds decrease with increasing congestion. The bus-speed curve assumes that, up to a certain density, the bus speed will increase with the number of users. This assumes that more buses will be put into service as the demand for buses increases, thus reducing the overall time of bus transit. The curve also shows the possibility for bus congestion.

The curves for the car and bus speeds can be represented analytically by

$$v_c = \frac{1}{\beta + \mu (x + \nu y)^n}, \quad n > 0 \quad (4)$$

$$v_b = \frac{y^r}{\beta' + \rho (\nu' x + y)^m}, \quad \begin{matrix} m \geq 0 \\ r \geq 0 \\ m > r \end{matrix} \quad (5)$$

where the constants are attributes of the roadway and give the slope of the curves. In subsequent work, we shall be assuming special bus lanes so that there will be no car-bus interactions, ($\nu = \nu' = 0$).

Figure 17.1 illustrates the type of results that are obtained with this

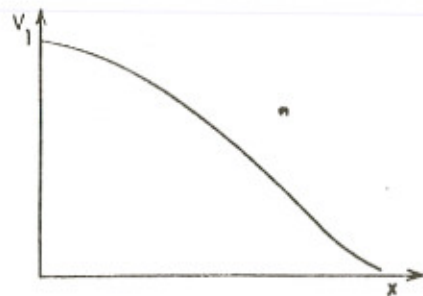


Fig. 17.1. Car speed, car user density relation.

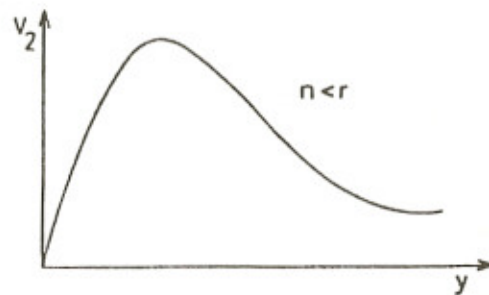


Fig. 17.2. Bus speed, bus user density relation.

model by taking some simple special cases. The first case assumes $p = q = 1$ and

$$v_x = 1/x \quad v_y = y. \tag{6}$$

With this assumption the equations of evolution of car and bus users become:

$$\dot{x} = \frac{D/x}{1/x + y} - x \tag{7}$$

$$\dot{y} = \frac{Dy}{1/x + y} - y.$$

The stationary states are defined as those for which $\dot{x} = 0, \dot{y} = 0$. We find the following stationary states for this system

$$x = D, y = 0 \tag{8}$$

$$x = 1, y = D - 1.$$

A stability analysis of equations (7) shows that the state for which $y = 0$ (no bus riders) becomes unstable if the demand for transportation D becomes sufficiently large $D > 1$. Taking D as a bifurcation parameter, the bifurcation diagram for this is shown in fig. 17.3.

For $D < 1$ a stable stationary state exists, namely, $(x = D, y = 0)$. The other stationary state $(x = 1, y = D - 1)$ is unstable in this regime. For higher values of D , such that $D > 1$, the state $(x = D, y = 0)$ is unstable and the other state is stable. There is thus a minimum value of transit demand before the bus mode becomes a viable possibility. In x, y space, the situation is depicted in fig. 17.4.

The car and bus speeds in these two regimes of transportation demand are shown in fig. 17.5 and fig. 17.6. The figures show that once bus service becomes an alternative, its speed will increase with increasing demand. The car speed will remain constant in this model.

Finite car speeds. If we had made the car speeds somewhat more realistic by limiting them at the low end, the above results would not change in any qualitative way. When a constant factor, a , is introduced in the first equation of (6), equations (6) become

$$v_x = \frac{1}{a + x} \quad v_y = y. \tag{9}$$

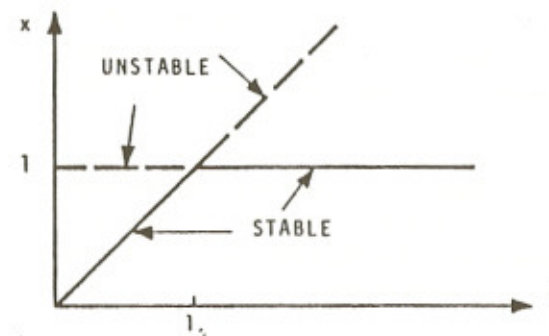


Fig. 17.3. Bifurcation diagram corresponding to equation (7).

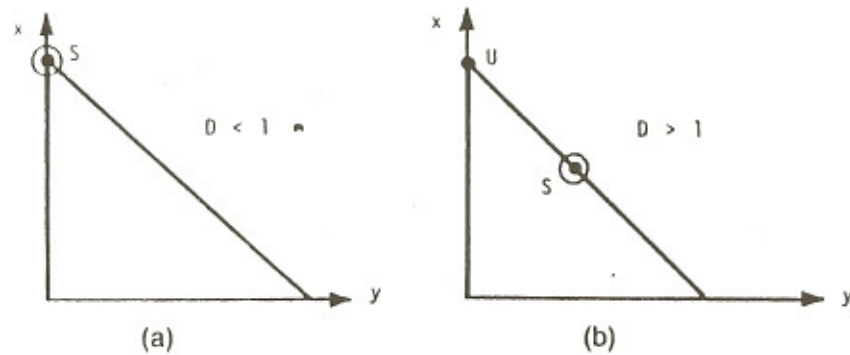


Fig. 17.4. Car user density versus bus user density; the stable state is represented by S , the unstable state by U .

This form for v_x limits the car speeds at low densities. The stationary states are now given by

$$x = D, y = 0 \quad (10)$$

$$x^* = (-a \pm \sqrt{a^2 + 4})/2.$$

The state $(x = D, y = 0)$ is unstable and (x^*, y^*) is stable for $D > D^*$ where

$$D^* = (-a + \sqrt{a^2 + 4})/2. \quad (11)$$

The results are qualitatively the same as in the first case, here the critical density for stability is D^* where before it had the numerical value of 1. We note that this stability condition is identical to the condition for existence of the solution x .

Finite car speeds and bus congestion effects. The purpose of this section is to show that, if we also make the bus-speed term somewhat more realistic, the previous results do not change qualitatively. Putting congestion effects into the bus-speed expression we obtain in place of equation (9)

$$v_x = \frac{1}{a + x} \quad v_y = \frac{\theta y}{\gamma + y}, \quad (12)$$

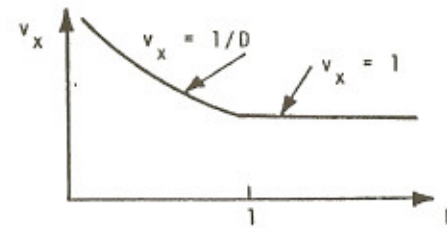


Fig. 17.5. Car speed as a function of demand.

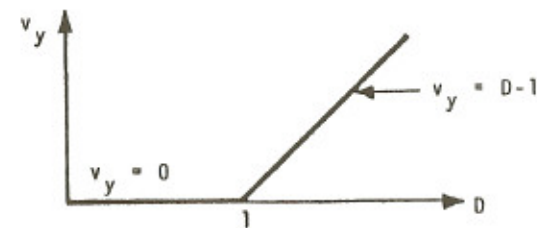


Fig. 17.6. Bus speed as a function of demand.

Again, one stationary state of the system is given by $(y = 0, x = D)$, which becomes unstable when the demand exceeds a critical value D^* , where

$$D^* = (-a + \sqrt{a^2 + 4\gamma/\theta})/2. \quad (13)$$

The other stationary state is x^* given by:

$$x^* = \frac{-a\theta - 1 + \sqrt{(a\theta + 1)^2 + 4\theta(D + \gamma)}}{2\theta} \quad (14)$$

The bifurcation diagrams for this case are shown in fig. 17.7 and fig. 17.8 and are seen to be qualitatively similar to the previous, using simpler forms for the car- and bus-speed relations.

Internal perturbations near a stationary state are such that $\delta y \ll y_s$ and since $y_s = 0$, the perturbations δy must be introduced as an external factor corresponding to the introduction of a new mode of transportation. Thus the examples here give the conditions under which the system becomes unstable

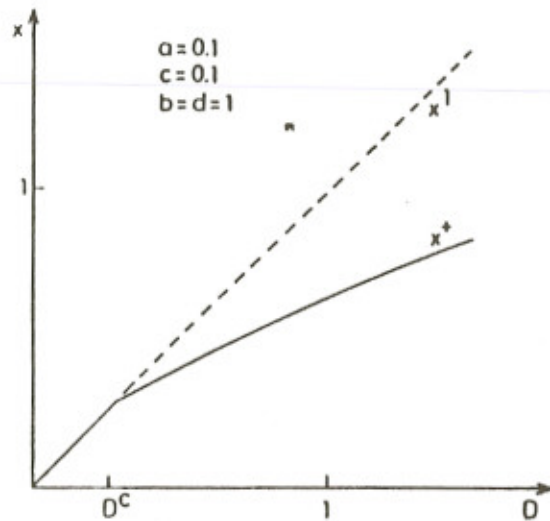


Fig. 17.7. Bifurcation diagram, x versus D .

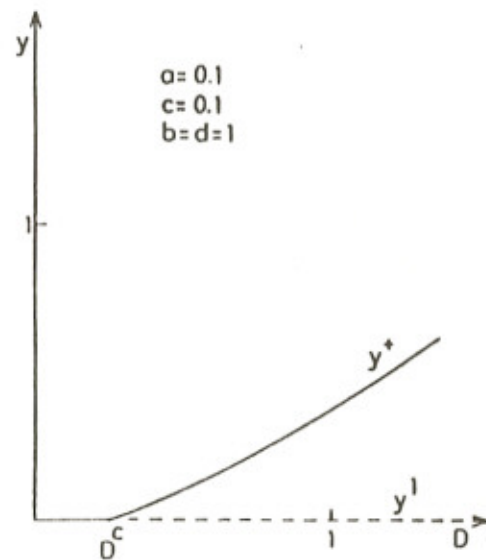


Fig. 17.8. Bifurcation diagram, y versus D .

tion that the system accept this new mode, $D > D^*$, depends upon the characteristics of the old, a , and those of the new, γ and θ .

Non-linear dependence upon speed. The previous attractivity functions were linearly dependent upon the mode speeds. It is interesting to ask how the solutions might change if a stronger-than-linear dependence of mode choice on mode speed were assumed. For example, let us take

$$F_x = v_x^2 \quad F_y = v_y^2 \quad (15)$$

with the simplest form for the speed-density relation

$$v_x = 1/x \quad v_y = y. \quad (16)$$

We then obtain for the equations of evolution the following:

$$\begin{aligned} \dot{x} &= \frac{D/x^2}{1/x^2 + y^2} - x \\ \dot{y} &= \frac{D y^2}{1/x^2 + y^2} - y. \end{aligned} \quad (17)$$

The stationary states in this case are given by

$$\begin{aligned} x &= D, y = 0 \\ (D - y)^3 &= 1/y. \end{aligned} \quad (18)$$

Taking the density D as the bifurcation parameter, we obtain a bifurcation diagram for the second stationary state as shown in fig. 17.9. The figure shows the existence of a minimum value of D , namely, D^* before bus transit is possible; however, here two solutions are possible. One solution gives the number of bus users increasing with increasing demand and the other, decreasing with increasing demand. Examining the stability of these two solutions, we find the upper branch (see fig. 17.9) to be stable and the lower one to be unstable to fluctuations. Thus, once a critical density for bus ridership is reached, an increasing number of bus riders will be attracted to the bus mode, while a decreasing number of car users will be attracted to the car mode. Beyond the critical density D^* any further increase in the number of cars leads to an unstable situation. This model thus predicts that fluctuations will drive the system to the stable branch in which the number of car users decrease with a further increase in demand. Fig. 17.10 shows the situation in (x, y) space. For a num-

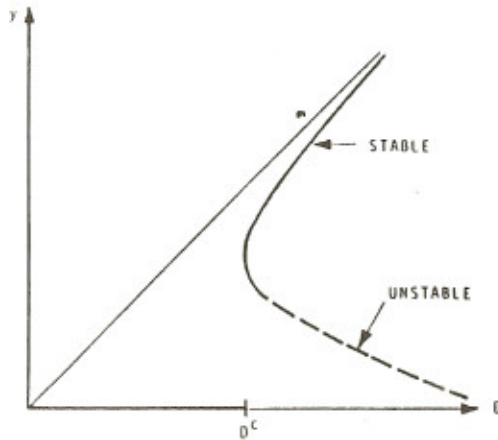


Fig. 17.9. Bifurcation diagram for the system of equation (17), y versus D .

ber of cars greater than a critical value numerically equal to $3^{1/4}$ and a number of bus users less than $9^{1/4}$ there is a solution U which is unstable. For a number of car users less than the critical value and a number of bus users greater than $9^{1/4}$ there is a stable stationary state labelled S in the figure. The figure also shows the curve which separates space into two regions, each region being under the influence of a stable stationary state.

General discussion. It is worthwhile to point out that even with the very simple forms used for the attractivity functions, bifurcations appear and thresholds of transportation demand exist before bus service becomes viable.

If we were to develop more realistic forms for the attractivity functions, which like human behavior can be expected to be quite nonlinear, we could also expect an even greater richness in the evolutionary branchings of the system. As each solution is subject to fluctuations the evolutionary path is also subject to new branchings and hence new paths of evolution will emerge. Fig. 17.11 schematically illustrates a general bifurcation diagram in which we use λ as a bifurcation parameter. The stationary state is represented by the variable x while λ is a parameter which measures the feedback effects in the system. We see from the figure that if the feedback parameter is sufficiently small, $\lambda < \lambda_0$, the system will have only one stationary state, x^0 . If, however, λ is sufficiently large, $\lambda > \lambda_0$, new stationary states may appear in the system. As λ increases, the number of possible stationary states of the system increases. If the system accepts the branch x^+ , then, as time increases, the prob-

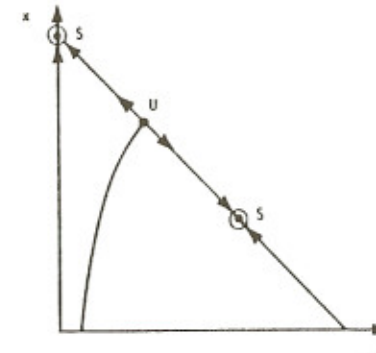


Fig. 17.10. Car user density versus bus user density for the system of equations (17); the critical numerical values are $x_c = 3^{1/4}$ and $y_c = 9^{1/4}$; unstable (U) for $x > x_c$ and $y < y_c$; stable (S) for $x < x_c$ and $y > y_c$.

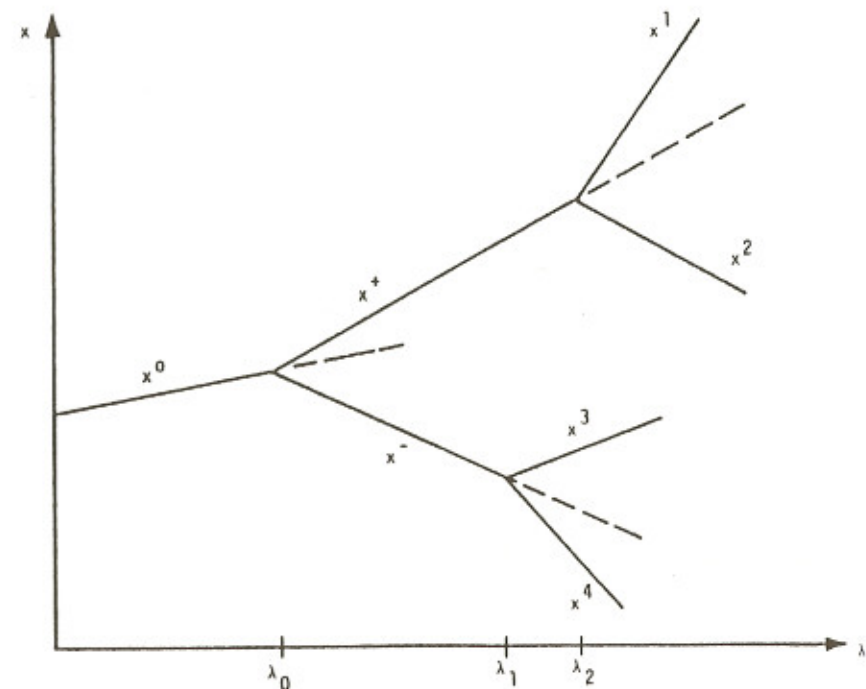


Fig. 17.11. General bifurcation diagram with λ as the bifurcation parameter.

of branches, x^+ or x^- made by the system near a bifurcation point λ_0 , determines the system's further evolution. This phenomenon underscores the importance of history in the evolution of systems.

Attractivity, a Function of Mode Speed and Psychological Factors

We now wish to look at the transportation mode choice problem with an added psychological factor. We shall assume that mode choice depends not only on the mode speed, but also on the publicity given to that mode and on imitation. By *imitation* we mean that people tend to copy the behavior of others that gives rise to a positive reinforcement of mode choice.

We take for the attractivity functions

$$F_1(x, y) = v_1^s F_1 \quad (19)$$

$$F_2(x, y) = v_2^s F_2,$$

where F_1 and F_2 are given by

$$F_1 = \theta + \alpha x \quad (20)$$

$$F_2 = \theta' + \alpha' y.$$

The constant terms θ and θ' are publicity terms for the modes, and the terms depending upon the density of users are the imitative terms proportional to the number of people already using these modes.

When equation (19) and equation (20) are used, the equations of evolution become

$$\dot{x} = \frac{D(\theta/x + \alpha)}{\theta/x + \alpha + \theta'y + \alpha'y^2} - x \quad (21)$$

$$\dot{y} = \frac{D(\theta'y + \alpha'y^2)}{\theta/x + \alpha + \theta'y + \alpha'y^2} - y,$$

where we have used equation (6) for the part of the attractivity function that depends upon mode speed.

A stationary state of the system is $(x = D, y = 0)$, while the other stationary states are given by the roots of the cubic equation:

$$x^3 - x^2 \left(\frac{\theta'}{\alpha'} + D \right) + \frac{\alpha}{\alpha'} x + \frac{\theta}{\alpha'} = 0. \quad (22)$$

We shall discuss the case for which there is no publicity for car use ($\theta = 0$). In this case the other stationary states of the system given by equations (21) are obtained as the roots of the quadratic equation:

$$\alpha' y^2 + (\theta' - \alpha' D)y + (\alpha - D\theta') = 0. \quad (23)$$

Two critical densities of demand emerge from the analysis:

$$D_1^c = \alpha/\theta' \quad (24)$$

$$D^c = (\sqrt{4\alpha\alpha'} - \theta')/\alpha'. \quad (25)$$

The demand D_1^c is the critical density for the $(x = D, y = 0)$ stationary state. For sufficiently high demand, $D > D_1^c$ the $y = 0$ state is unstable to fluctuations. The other critical density D^c given by equation (25) gives the condition for the other stationary states to exist.

There are two interesting situations, one for which there is great publicity for bus use, $\theta' > \alpha\alpha'$, depicted in fig. 17.12, and one for which there is little publicity, $\theta' < \alpha\alpha'$ shown in fig. 17.13.

Fig. 17.12 shows that for $D < D_1^c$ there is only one stable stationary state, $(y = 0, x = D)$, labeled y^1 in the figure. When the density becomes high enough, $D > D_1^c$, this state y^1 becomes unstable, but another one appears, that is labeled y^* and is stable and increases with increasing demand for transportation. This result is similar to the results of the previous models. For the case of little publicity for the bus, fig. 17.13 shows that there is a region of intermediate transportation demand in which the system will accept the stationary states y^* , y^- , and y^1 . The y^- state is unstable and, hence, any perturbation will immediately move the system away from this state. In this region of transportation demand, a given size perturbation Δy , is needed to bring the system from the state y^1 (where $y = 0$) to the state y^* (which increases with increasing demand). The value of the perturbation Δy , needed for this transformation decreases as the transportation demand D , increases. That is, the size of the fluctuation necessary to cause the system to jump from the state y^1 to the state y^* decreases as the demand for transportation increases, as one would intuitively expect.

For larger densities, $D > D_1^c$, whatever the value of the perturbation, Δy , the system will spontaneously go over to the stationary state (y^*, x^*) .

Fig. 17.14 shows how publicity for the bus is related to the total demand for transportation. The figure also shows the conditions under which coexistence between the bus and car modes is possible. The figure illustrates that coexistence is more easily accomplished if the publicity for bus use is high, (in order for there to be both car and bus users the demand for transportation

APPENDIX A

Transition Probabilities and the Logistic Equation

In this appendix we show that the equations of evolution used in this paper follow quite plausibly from some general considerations of transition probabilities and utility functions.

We consider a system having N different possible states, N^2 possible transitions. We define $g(i, j)$ as the utility function of the transition i to j . This utility function will be a function of $g(i)$, $g(j)$ (utility functions associated with each state) and the distance (or a generalization of the distance) between the state i and the state j . In general, individuals will try to make transitions which increase their utility function. The probability per unit time to make a transition from the state i to the state j will be

$$P(i, j) = g(i, j) / \sum_{k=1}^N g(i, k) . \quad (\text{A-1})$$

Assuming that the transitional probabilities $P(i, j)$ exist, we can describe the evolution of the population $x(i)$ as the probability of a transition from state j to state i multiplied by the number of people at j , minus the opposite transition:

$$\dot{x}(i) = \sum_{j=1}^N P(j, i) x(j) - \sum_{j=1}^N P(i, j) x(i) . \quad (\text{A-2})$$

If we replace the transition probabilities $P(i, j)$ with the generalized utility functions $g(i, j)$ we obtain:

$$\dot{x}(i) = \sum_{j=1}^N \frac{g(j, i) x(j)}{\sum_k g(j, k)} - x(i) , \quad (\text{A-3})$$

which has the same form as the equations of evolution that we have been using. For example, consider the problem of transportation mode choice and assume that the utility function is given by

$$g(i, j) = g(i) g(j) . \quad (\text{A-4})$$

This assumption leads to an equation of evolution of the form:

$$\dot{x}(i) = x_i P(i) - x(i) , \quad (\text{A-5})$$

where x_i is the total population and

$$P(i) = g(i) / \sum_{k=1}^N g(k) . \quad (\text{A-6})$$

If x is the number of car users the equations of evolution (A-5) become

$$\begin{aligned} \dot{x} &= D P(x) - x \\ \dot{y} &= D P(y) - y , \end{aligned} \quad (\text{A-7})$$

where D has replaced x_i . The probabilities $P(x)$ and $P(y)$ are given by:

$$\begin{aligned} P(x) &= g(x) / (g(x) + g(y)) \\ P(y) &= g(y) / (g(x) + g(y)) . \end{aligned} \quad (\text{A-8})$$

With these definitions of the transitional probabilities and an identification of the utility functions with the attractivity functions used in the paper, the evolution equations (A-7) are of the same form as the equations (2) of this article. This analysis, however, proceeded from a probabilistic model to a deterministic one involving average values for the variables. There is thus an implicit assumption that we are dealing with a sufficiently large number of individuals. The model may thus be said to be a phenomenological one for the frequency of transition.

APPENDIX B

Transportation System Growth and Density of Users

In this appendix we briefly indicate how one may treat the interrelation between the growth of a transportation mode and the density of users of transportation.

For one mode of transportation L , we have the evolution equations

$$\begin{aligned} \dot{y} &= \frac{D y}{y + p} - y \\ \dot{L} &= G(L, y) , \end{aligned} \quad (\text{B-1})$$

where y is the number of users of the transportation mode L , D is the total demand for the mode, ρ is the attractivity of other transportation modes, and L is the number of miles of highway or railroad track or the like.

As an example, if we assume that the attractivity of a transportation mode is only a function of the infrastructure L that it offers, then,

$$\dot{y} = \frac{D L}{\rho + L} - y \quad (\text{B-2})$$

where ρ represents competing modes. If we assume that the physical growth of the transportation mode depends only on the influx of money and the cost of running the system, K , we have

$$\dot{L} = (\phi - K) L \quad (\text{B-3})$$

If we represent the money flux ϕ as $\alpha(t) + \nu y$ where $\alpha(t)$ are state subsidies or capital grant monies, and νy is the revenue received from the users of the mode, then the equations of evolution become, assuming $\alpha = 0$,

$$\dot{y} = \frac{D L}{\rho + L} - y \quad (\text{B-4})$$

$$\dot{L} = \nu y - K L .$$

When we study the stationary states of this system we find that one stationary state is $(L = 0, y = 0)$. This is stable when maintenance costs are too high, $K > D\nu/\rho$. The other stationary state (L^*, y^*) becomes viable when the influx of money exceeds the maintenance costs, $K < D\nu/\rho$. We note that the condition of stability has the density of users in it, and, hence, the viability of a transport mode is a function of the number of users of that mode (recalling that the state subsidy and capital grant money-terms were assumed to be zero). The condition for stability would then indicate that a high-cost transportation system is justified for central cities where urban population densities are high.