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Introduction to topological string theory

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Abstract

These lecture notes give an elementary introduction to closed topological string theory intented for beginning PhD students. We discuss the central idea of localization, starting off with an easy example: zero dimensional QFT. We then describe $\mathcal{N} = 2$ and $\mathcal{N} = (2, 2)$ non-linear sigma-models and their topological twists. This results in two types of topological *field* theories, called the A and the B-model. Coupling these models to gravity allows to formulate the A- and B-model topological *string* theories. Finally, we briefly discuss three important asymptotic expansions of the topological string free energy.

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Chapter 1

Zero-dimensional supersymmetric QFT

1.1 A simple model

Addressing the question as to why topological theories are of interest, we first develop some intuition for the concepts of localization and deformation invariance. These two features underly the power of supersymmetric and especially topological field and string theories. We will discuss the simplest case, a sigma-model with real valued fields and then move on to a simple model with complex valued fields, allowing to introduce important concepts such as chiral and anti-chiral fields as well as their corresponding SUSY cohomology rings.

We start with the simplest case of a zero-dimensional supersymmetric QFT, where the path integral is just an ordinary integral and everything can in principle be calculated exactly. As a base manifold (or "world volume" manifold) we simply choose a point $\mathcal{W} = p$, and the target space will be the real line $X = \mathbb{R}$. The field content of this theory consists of a real bosonic 'scalar' field, X, which can be regarded as a map $X : p \mapsto \mathbb{R}$, and two real Grassmann variables ψ_1, ψ_2. The most general Grassmann-even Lagrangian, or action (the distinction is trivial in zero dimensions) we can write down for this field content is the following:

$$S(X, \psi_1, \psi_2) = S_0(X) - \psi_1\, \psi_2\, S_1(X)\,, \tag{1.1}$$

where S_0 and S_1 are functions of the boson X. The 'Euclidean' path integral is the following:

$$Z = \int dX\, d\psi_1\, d\psi_2 e^{-S(X,\psi_1,\psi_2)}\,. \tag{1.2}$$

We will use the convention that $\int d\psi_1\, d\psi_2\, \psi_1\, \psi_2 = 1$. This means that the path integral simplifies to

$$Z = \int dX\, S_1(X)\, e^{-S_0(X)}\,. \tag{1.3}$$

We will make a special choice for the action, such that it is 'supersymmetric' (in some sense). Let us define it as follows:

$$S(X, \psi_1, \psi_2) = \frac{1}{2}\,(\partial h)^2 - \psi_1\, \psi_2\, \partial h^2\,, \tag{1.4}$$

where $h = h(X)$ is some function of the scalar. This action is invariant under the following fermionic symmetry:

$$\begin{aligned} \delta X &= \epsilon\, \psi_1 + \epsilon\, \psi_2\,, \\ \delta\psi_1 &= \epsilon\, \partial h\,, \\ \delta\psi_2 &= -\epsilon\, \partial h\,, \end{aligned} \tag{1.5}$$

where ϵ is a Grassmann parameter. One can easily check that this path integral is exactly invariant under this symmetry.

1.2 Localization

In principle, we should compute the partition function (1.2) by carrying out an integration over one bosonic and two fermionic variables. However, in light of the supersymmetry exhibited by our action, one might wonder, whether one could accomplish this task in a simpler way by conveniently redefining the variables. Suppose, for instantce, that we had to compute the following ordinary integral on the real plane:

$$I = \int dx\, dy\, g(x,y)\,, \tag{1.6}$$

and we knew that the integrand f is invariant under rotations on the plane

$$\begin{aligned} x \to x_\theta &= x\,\cos(\theta) - y\,\sin(\theta)\,, & (1.7)\\ y \to y_\theta &= y\,\cos(\theta) + x\,\sin(\theta)\,. & (1.8)\end{aligned}$$

Then, we would switch to polar coordinates, and factor the angular integral out so that we would only have to integrate over the radius and then multiply by $2\,\pi$. Although this is overkill, let us do it through the Fadeev-Popov trick:
We define the function $\Delta(x,y)$ as follows:

$$\Delta(x,y)^{-1} = \int d\theta\delta[f(x_\theta,y_\theta)] = \int d\theta\delta[y_\theta]\,. \tag{1.9}$$

Here, the integrand is a Dirac delta-function, and $f(x,y) = y$ is our 'gauge fixing' condition. In this case, we see that $\Delta(x,y) = x$. Using the shift invariance of the $d\theta$ measure, one can show that $\Delta(x_\theta,y_\theta) = \Delta(x,y) = x$. Next, we insert '1' into our original integral:

$$I = \int dx\, dy\, g(x,y)\Delta(x,y) \int d\theta\, \delta[y_\theta]\,. \tag{1.10}$$

Using the θ-invariance of Δ, $g(x,y)$, and the measure $dx\,dy$, we can rewrite this as follows:

$$\begin{aligned} I &= \int d\theta \int dx_\theta\, dy_\theta\, g(x_\theta,y_\theta)\,\Delta(x_\theta,y_\theta)\,\delta[y_\theta] & (1.11)\\ &= \int d\theta \int d\tilde{x}\, d\tilde{y}\, g(\tilde{x},\tilde{y})\,\Delta(\tilde{x},\tilde{y})\,\delta[\tilde{y}]\,, & (1.12)\end{aligned}$$

where we have renamed x_θ and y_θ as dummy variables with tilde. Now, all we have to do is substitute Δ:

$$I = \int d\theta \int d\tilde{x}\,\tilde{x}\,g(\tilde{x},0)\,. \tag{1.13}$$

Here, one recognizes the Jacobian from going over to polar coordinates, if one thinks of $\tilde{x}$ as the radial coordinate.
The idea with a supersymmetric action is analogous to what we have just seen. One of the fermionic integrations can be traded by the supersymmetry parameter ϵ, and the full integral should factorize into a single fermionic integral times the 'volume' of the trivial fermionic integration. We expect something of the form:

$$Z = \int d\epsilon \int d\tilde{X}\, d\tilde{\psi}_2 e^{-S(\tilde{X},0,\tilde{\psi}_2)}\,\Delta\,, \tag{1.14}$$

and since $\int d\epsilon = 0$, the whole integral must vanish.
We start by defining the Fadeev-Popov determinant Δ:

$$\begin{aligned}\Delta(X,\psi_1,\psi_2)^{-1} &= \int d\epsilon\,\delta[f(X_\epsilon,\psi_{1\epsilon},\psi_{2\epsilon})] = \int d\epsilon\,\delta[\psi_{1\epsilon}] &(1.15)\\ &= \int d\epsilon\,\delta[\psi_1+\epsilon\,\partial h] = \int d\epsilon\,(\psi_1+\epsilon\,\partial h) = \partial h\,, &(1.16)\end{aligned}$$

where we have chosen a gauge-fixing function that sets ψ_1 to zero. Again, Δ can be shown to be invariant under supersymmetry. So, after repeating the exercise from before, we arrive at the following result:

$$Z = \int d\epsilon \int d\tilde{X}\,d\tilde{\psi}_2\,e^{-S(\tilde{X},0,\tilde{\psi}_2)}\frac{1}{\partial h}\,. \tag{1.17}$$

As long as h doesn't have any critical points, $Z=0$ due to the fact that $\int d\epsilon\cdot 1 = 0$. However, this is not valid if there are critical points. Where did we make the assumption that $\partial h \neq 0$? When we wrote

$$\Delta^{-1} = \int \epsilon\,\delta[\psi_1+\epsilon\,\partial h] = \partial h\,. \tag{1.18}$$

At the critical points of h, this 'Jacobian' breaks down. In other words, the coordinate transformation becomes singular. This shouldn't come as a surprise. Remember, that what we tried to do is to perform one of the ψ_1 fermionic integration by fixing the variable to zero, and then sweeping out its whole range of integration by acting on it with the supersymmetry transformations in (1.6). However, these transformations have fixed points, namely the critical points of h. At those points, ψ_1 is left invariant, so it makes sense that we can't 'sweep out' a whole range of integration with this trick.
This has two consequences: First, the partition function Z will not be zero. Second, all contributions to Z will come from the values of X that are critical points of h. This is known as *localization.*
In this case, we can carry out the integration exactly, so this will be a nice check for our intuition. First, we rewrite the partition function (1.2) by integrating out the fermions, i.e. in the form (1.3):

$$Z = \int dX\,d\psi_1\,d\psi_2\,e^{-\partial h^2/2+\psi_1\,\psi_2\,\partial^2 h} = \int dX e^{-\partial h^2/2}\,\partial^2 h\,. \tag{1.19}$$

In order for this integral to be finite, we need to assume that $\partial h^2 \to \infty$ as $X\to\pm\infty$. Now, we can make the change of variable $\partial h \to Y$:

$$Z = \int dY\,e^{-Y^2/2}\,dY\,. \tag{1.20}$$

However, if h has critical points, then this change of variables will not be injective. This means that the Y integral will retrace its steps a few times before going to $\pm\infty$. If h has an odd number of critical points, say three, then ∂h will start out at (without loss of generality) at $-\infty$ for $X=-\infty$, then cross its first zero, continue to a local maximum, then cross another zero, go down to a local minimum, and then go back up through its third zero, continuing to ∞. This means that the height between the local maximum and the local minimum will be covered twice in ascending direction, and once in descending direction, i.e. effectively once in ascending direction. So the integral is simply a Gaussian, giving $\sqrt{2\,\pi}$.
If h has instead an even number of critical points, say two, then the Y integration will retrace its steps, going from ∞ to ∞, effectively covering no height. Hence the integral will be zero. So Z is counting the critical points of h, weighing each by the sign of $\partial^2 h$ at that point. One can show that

$$Z = \sum_i \frac{\partial^2 h(x_i)}{|\partial^2 h(x_i)|}\,, \tag{1.21}$$

where the x_i are the critical points.
Another point of view on this localization phenomenon, which will be useful to us is known as *deformation invariance.* Redefine the function h as $h = t\,\tilde{h}$. Then, our partition function becomes

$$Z = \int dX e^{-t^2\partial\tilde{h}^2/2}\,t\,\partial^2\tilde{h}\,. \tag{1.22}$$

Differentiating this w.r.t. t, we get a vanishing surface term. This means that Z, is insensitive to rescalings of h. Hence, by taking $t \to +\infty$, we force the integrand to almost vanish everywhere except at the critical points. By regarding t^2 as $1/\hbar$, we recognize this as the zero-dimensional analogue of the semi-classical approximation in QFT. In this case, however, we have just shown that *the semi-classical approximation is exact*. This is a feature that comes up in all supersymmetric QFTs. Supersymmetric partition functions, and correlators of operators that are themselves invariant under supersymmetry can be computed exactly by means of the semi-classical approximation. This is a tremendous simplification, which one would love to have in non-SUSY QFTs such as QCD, where the strong coupling regime is hopelessly intractable.

Chapter 2

Universal properties for localization

The phenomenon of localization that we observed in the previous section permeates the arena of supersymmetric quantum field theories. In fact, it even takes place in non-supersymmetric models. The real property that makes this happen is not the supersymmetry *per se*, but the fact that the theory is *cohomological.* Let us now define a set of axioms that make a quantum field theory cohomological.
One starts with an action S that is invariant under some global symmetry $\delta_\epsilon S = 0$. Provided this symmetry also leaves the path integral measure invariant, one can construct an operator Q, based on this symmetry, that will act on the Hilbert space and on other operators as follows:

$$\delta_\epsilon\, \mathcal{O} = \begin{cases} i\,\epsilon[Q, \mathcal{O}] & \text{if at least one operator is bosonic,} \\ i\,\epsilon\{Q, \mathcal{O}\} & \text{if both operators are fermionic.} \end{cases} \tag{2.1}$$

In the cases we will study, Q will always be some supersymmetry generator. States that are symmetric are by definition annihilated by Q. If the vacuum state is annihilated, one says that the symmetry is not spontaneously broken.
The first three requirements we make to have a cohomological theory are the following:

- The symmetry must be nilpotent, i.e. $Q^2 = 0$.
- The vacuum must be symmetric, i.e. $Q\,|0\rangle$.
- Physical observables are defined by operators $\mathcal{O}$ such that
  $[Q, \mathcal{O}] = 0$ or $\{Q, \mathcal{O}\}$. (Without loss of generality, we will take the commutator).

As a consequence of these axioms, one can easily deduce that any correlator containing an operator of the form $\mathcal{O} = [Q, \mathcal{L}]$, i.e. *exact* operators, is automatically zero

$$\langle 0|\, \mathcal{O}_1 \ldots [Q, \mathcal{L}] \ldots \mathcal{O}_n |0\rangle = 0\,. \tag{2.2}$$

All one needs to do, is use the fact that Q will (anti-)commute through all other operators and reach the vacuum, which it will then annihilate. All this means that the physical operators of this theory are given by the cohomology ring of Q, i.e. all operators annihilated by Q, known as *closed* operators, modulo the equivalence

$$\mathcal{O} \sim \mathcal{O} + [Q, \mathcal{L}]\,. \tag{2.3}$$

Finally, we will require that the Lagrangian L be Q-exact

$$L = [Q, V] \quad \text{hence} \quad S = [Q, \int V]\,. \tag{2.4}$$

This requirement has two important properties: First, it means that the path integral is invariant under variations of the metric,

$$
\begin{aligned}
\frac{\delta}{\delta h^{\alpha\beta}} \langle \mathcal{O} \rangle &= \frac{\delta}{\delta h^{\alpha\beta}} \int d[\phi] \mathcal{O}\, e^{-[Q, \int V]} \\
&= -\int d[\phi] \mathcal{O}\, [Q, \frac{\delta}{\delta h^{\alpha\beta}} \int V] = \langle \mathcal{O}\, [Q, \frac{\delta}{\delta h^{\alpha\beta}} \int V] \rangle = 0\,.
\end{aligned}
\tag{2.5}
$$

This presupposes, of course, that the operator $\mathcal{O}$ is itself independent of the metric. Theories with this property are called *topological*, due to their independence of the metric.
The second important property that follows from the cohomological axioms, is the localization property. If we put Planck's constant back into our path integral

$$
\langle \mathcal{O} \rangle = \int \mathcal{O} \exp\Big(-\frac{S}{\hbar}\Big)\,,
\tag{2.6}
$$

then we realize that differentiating the correlation function w.r.t. $\hbar$ simply brings down a Q-exact operator, thereby yielding zero. Therefore, the theory is independent of $\hbar$, so we can just take the limit $\hbar \to 0$, and compute everything semiclassically. This is localization. To conclude, *for cohomological theories, the semiclassical approximation is exact!*.
Let us go back to our zero-dimensional example to see an easy implementation of this. First of all, notice that the supersymmetry transformations defined in (1.6) are not nilpotent, so we need to do some rewriting to make this happen. We will introduce a new bosonic auxiliary field H into the system and rewrite the action (1.4) as follows:

$$
S = \frac{1}{2} H^2 + i\,\partial h\, H - \psi_1\, \psi_2\, \partial h^2\,.
\tag{2.7}
$$

The new field H appears at most quadratically in this action, so we can eliminate it at will by completing the square and performing a Gaussian integration. This is equivalent to substituting the equations of motion of H into (2.7). In doing this, we retrieve the original system (1.4). The new supersymmetry rules now become the following:

$$
\begin{aligned}
\delta X &= \epsilon\,(\psi_1 + \psi_2)\,, \\
\delta \psi_1 &= \epsilon\, i\, H\,, \\
\delta \psi_2 &= -\epsilon\, i\, H\,, \\
\delta H &= 0\,.
\end{aligned}
\tag{2.8}
$$

By substituting the e.o.m. $H = -i\,\partial h$, we retrieve the old transformation rules (1.6), however, this new 'offshell' form of the rules is now nilpotent (i.e. $Q^2 = 0$.) By computing the *superdeterminant* or *Berezinian* of these transformations, one sees that the path integral measure is left invariant by this supersymmetry. Now, we can rewrite the action S as follows:

$$
S = \frac{1}{4}\, Q\Big[(\psi_1 - \psi_2)\,(2\,\partial h + i\,H)\Big]\,.
\tag{2.9}
$$

If $\mathcal{O}$ is an closed operator, i.e. $\delta\mathcal{O} = 0$, then the correlator of $\mathcal{O}$ with any exact operator must vanish

$$
\langle \mathcal{O}\, [Q, V] \rangle = \int \mathcal{O}\, \delta V e^{-S} = \delta\Big(\int \mathcal{O}\, V e^{-S}\Big)\,.
\tag{2.10}
$$

Similarly, if we put a coupling constant t next to the action, then differentiating w.r.t. t yields a zero correlator. This implies that the theory localizes around the minima of the action.

Chapter 3

Topological field theory

In this section, we will study a quantum field theory in two Euclidean dimensions, namely a supersymmetric non-linear sigma model with a Kähler target space. After introducing a trick known as the *topological twist*, we will be able to show that this model also exhibits localization. However, instead of simply counting critical points of a function, like the zero-dimensional model, we will see that the topological sigma model counts more interesting geometric quantities.

3.1 $\mathcal{N} = 2, d = 2$ supersymmetric nonlinear sigma model

Let us begin by writing down a nonlinear sigma model in two dimensions. One starts with a Riemann surface Σ (the worldsheet), on which the theory is defined. One then considers maps Φ, of this surface into a target space X, $\Phi : \Sigma \to X$, where X is interpreted as the physical spacetime. Taking local coordinates x^I for the target space, on then defines a set of fields ϕ^I on the worldsheet by the composition $\phi^I = \Phi \circ x^I$. The can be regarded as embedding functions into a patch of the target space. The action of the theory is then defined so that its minimization corresponds to the minimization of the area of the worldsheet. In other words, the maps ϕ at their classical value are such that the area of $\Phi(\Sigma)$ is minimized. This leads to the well-known Polyakov action. Generalizing this theory to have $\mathcal{N} = 2$ supersymmetry (on the worldsheet, that is), leads to the following action:

$$\begin{aligned} S \;=\; & \int_\Sigma d^2z \, \big(\frac{1}{2} g^{IJ} \, \partial_z \phi^I \, \partial_{\bar z} \phi^J + \frac{i}{2} g_{IJ} \, \psi_-^I \, D_z \, \psi_-^J + \frac{i}{2} g_{IJ} \, \psi_+^I \, D_{\bar z} \, \psi_+^J \\ & + \frac{1}{4} R_{IJKL} \, \psi_+^I \, \psi_+^J \, \psi_-^K \, \psi_-^L \big) \, . \end{aligned} \tag{3.1}$$

Here, g^{IJ} is the target space metric, and R_{IJKL} is its Riemann tensor. The ψ's are taken to transform as spinors w.r.t. worldsheet rotations and as vectors w.r.t. target space Lorentz transformations. The $\pm$ subscript denotes the 'handedness' of the spinor as follows:

$$\begin{aligned} z &\mapsto e^{i\,\alpha} \, z \\ \psi_\pm &\mapsto \psi_\pm \, e^{\pm\, i\, \alpha/2} \, . \end{aligned} \tag{3.2}$$

To make contact with standard notation in the literature, we introduce the definition of the *canonical bundle*, K. For a complex n-dimensional manifold, it is defined as the bundle of $(n, 0)$-forms, i.e. forms of the form (no pun intended)

$$\omega = \omega_{z_1,\ldots,z_n} \, dz^1 \wedge \ldots \wedge dz^k \, . \tag{3.3}$$

Because it contains the maximal number of dz's, it is actually a complex one-dimensional vector space above each point of the manifold, i.e. it is a complex line bundle. In the case of a Riemann surface, it corresponds to the space of $(1,0)$ forms, which is simply the dual to the bundle of holomorphic vectors $T^{1,0}\Sigma$, i.e. vectors of the form

$$v = v^z \partial_z \,. \tag{3.4}$$

Hence, we will make the identification $K = \overline{T^{1,0}\Sigma} = T^{0,1}\Sigma$. Similarly, one defines the *anti-canonical bundle*, $\overline{K}$ as the bundle of $(0,n)$-forms. In our case, we can make the following identifications:

$$\begin{aligned} K &\equiv \Omega^{1,0} = \overline{T^{1,0}} = T^{0,1} \\ \overline{K} &\equiv \Omega^{0,1} = T^{1,0} = \overline{T^{0,1}} \end{aligned} \tag{3.5}$$

A $(1,0)$-form ω transforms[1] as follows under a phase rotation of the coordinate z:

$$\begin{aligned} z &\mapsto z' = z\, e^{i\,\alpha} \\ \omega = \omega_z\, dz &\mapsto \omega' = \omega_z\, dz' = \omega_z\, e^{i\,\alpha}\, dz \equiv \omega'_z\, dz \\ \text{therefore} \quad \omega_z &\mapsto \omega_{z'} = \omega_z\, e^{i\,\alpha} \end{aligned} \tag{3.6}$$

Looking at the spinor transformation rules (3.3), we notice that the $(+)$-spinors transform as 'square roots' of $(1,0)$-forms, and $(-)$-spinors as 'square roots' of $(0,1)$-forms. In bundle language, one could define new line bundles denoted as $K^{1/2}$ and $\overline{K^{1/2}}$, which can be thought of as square roots of the canonical and anti-canonical bundles, respectively. In terms of global data, this means that the transition functions defining them are elements of $U(1)$ that square to the transition functions defining K and $\overline{K}$. These are the line bundles in which these spinors live.

Taking into account the target space index of the spinors, we can now define the latter as follows:

$$\begin{aligned} \psi_+^I &\in \Gamma(\Sigma, K^{1/2} \otimes \Phi^*(TX))\,, \\ \psi_-^I &\in \Gamma(\Sigma, \overline{K^{1/2}} \otimes \Phi^*(TX))\,, \end{aligned} \tag{3.7}$$

where $\Gamma(\Sigma, E)$ means 'space of sections of the bundle E', and the Φ^* is the pullback map of the tangent bundle[2] TX onto the worldsheet. The covariant derivatives on the spinor are defined as

$$D_{\bar{z}}\psi_+^I = \partial_{\bar{z}}\psi_+^I + \partial_{\bar{z}}\phi^I\, \Gamma^I_{JK}\psi_+^K \,, \tag{3.8}$$

where we use the target space Christoffel symbol, and a similar definition applies to the negative spinor. This theory exhibits $\mathcal{N} = 2$ supersymmetry under the following transformations:

$$\begin{aligned} \delta\phi^I &= i\,\epsilon_-\, \psi_+^I + i\,\epsilon_+\, \psi_-^I\,, \\ \delta\psi_+^I &= -\epsilon_-\, \partial_z\phi^I - i\,\epsilon_+\, \psi_-^K\, \Gamma^I_{KM}\, \psi_+^M\,, \\ \delta\psi_-^I &= -\epsilon_+\, \partial_{\bar{z}}\phi^I - i\,\epsilon_-\, \psi_+^K\, \Gamma^I_{KM}\, \psi_-^M\,. \end{aligned} \tag{3.9}$$

Here, the tranformation parameters ϵ_+ is a holomorphic section of $K^{1/2}$ and ϵ_- and anti-holomorphic section of $\overline{K^{1/2}}$. They must be (anti-)holomorphic to be able to be pulled through the derivatives upon varying the Lagrangian.

3.2 $\mathcal{N} = (2,2), d = 2$ supersymmetric nonlinear sigma model

Having introduced the $\mathcal{N} = 2$ model, we can now make some more assumptions that will enhance the structure of this theory. First of all, we will assume that the target space X is a complex manifold. For a (real)

[1] Note that we are taking the 'active' point of view of transformations.

[2] The pullback is actually defined for dual vectors, but using the non-degenerate metric of X, one can define it for the tangent space.

$2\,n$-dimensional manifold, this means that the manifold can be described by a family of charts $\{U_\alpha, z^i_\alpha, z^{\bar{i}}_\alpha\}$, where $i = 1, \ldots, n$, such that transition functions don't mix the holomorphic and anti-holomorphic coordinates, i.e. on an intersection $U_\alpha \cap U_\beta$

$$z^i_\alpha = z^i_\alpha(z^1_\beta, \ldots, z^n_\beta), \quad \text{and} \quad z^{\bar{i}}_\alpha = z^{\bar{i}}_\alpha(z^{\bar{1}}_\beta, \ldots, z^{\bar{n}}_\beta)\,. \tag{3.10}$$

This means that we can rewrite the nonlinear sigma model in (3.1) by decomposing the fields as

$$\phi^I = \{\phi^i, \phi^{\bar{i}}\}, \quad \psi^I_\pm = \{\psi^i_\pm, \psi^{\bar{i}}_\pm\}\,, \quad \text{and} \quad g^{IJ} = \{g^{i\bar{j}}, g^{\bar{i}j}\}\,. \tag{3.11}$$

The statement of complex geometry is that this splitting of indices into barred and unbarred indices is consistent throughout all patches of X. Although transition functions won't mess up this splitting, the supersymmetry transformations (3.10) will. This is due to the presence of the Christoffel symbol in the rules. Just because the coordinate transformations preserve the (anti-)holomorphicity of a vector field does not mean that parallel transporting it will. As long as the Christoffel symbols have mixed (both barred and unbarred) indices, this will happen. Therefore, if we want our supersymmetry transformations to respect the complex structure of X, we must require that the metric on X be such that parallel transport of vectors preserves the decomposition $TX = T^{1,0}X \oplus T^{0,1}X$. Metrics that satisfy this condition are called *Kähler* metrics. They imply exactly what we want, i.e. that the only non-vanishing Christoffel symbols have pure indices:

$$\text{only} \quad \Gamma^I_{JK} \neq 0 \quad \text{and} \quad \Gamma^{\bar{I}}_{\bar{J}\bar{K}} \neq 0\,. \tag{3.12}$$

Now, we can write our model as follows:

$$\begin{aligned} S \;\; = \;\; & \int_\Sigma d^2z\,\Big(\frac{1}{2}\,g^{i\bar{j}}\,\partial_z\phi^i\,\partial_{\bar{z}}\phi^{\bar{j}} + \frac{1}{2}\,g^{\bar{i}j}\,\partial_z\phi^{\bar{i}}\,\partial_{\bar{z}}\phi^j \\ & + i\,g_{\bar{i}j}\,\psi^{\bar{i}}_-\,D_z\,\psi^j_- + i\,g_{\bar{i}j}\,\psi^{\bar{i}}_+\,D_{\bar{z}}\,\psi^j_+ + \frac{1}{4}\,R_{i\bar{j}k\bar{l}}\,\psi^i_+\,\psi^{\bar{j}}_+\,\psi^k_-\,\psi^{\bar{l}}_-\,\Big)\,. \end{aligned} \tag{3.13}$$

The supersymmetry rules are

$$\begin{aligned} \delta\phi^i &= i\,\alpha_-\,\psi^i_+ + i\,\alpha_+\,\psi^i_-\,, \\ \delta\phi^{\bar{i}} &= i\,\tilde{\alpha}_-\,\psi^{\bar{i}}_+ + i\,\tilde{\alpha}_+\,\psi^{\bar{i}}_-\,, \\ \delta\psi^i_+ &= -\tilde{\alpha}_-\,\partial_z\phi^i - i\,\alpha_+\,\psi^k_-\,\Gamma^i_{km}\,\psi^m_+\,, \\ \delta\psi^{\bar{i}}_+ &= -\alpha_-\,\partial_z\phi^{\bar{i}} - i\,\tilde{\alpha}_+\,\psi^{\bar{k}}_-\,\Gamma^{\bar{i}}_{\bar{k}\bar{m}}\,\psi^{\bar{m}}_+\,, \\ \delta\psi^i_- &= -\tilde{\alpha}_+\,\partial_{\bar{z}}\phi^i - i\,\alpha_-\,\psi^k_+\,\Gamma^i_{km}\,\psi^m_-\,, \\ \delta\psi^{\bar{i}}_- &= -\alpha_+\,\partial_{\bar{z}}\phi^{\bar{i}} - i\,\tilde{\alpha}_-\,\psi^{\bar{k}}_+\,\Gamma^{\bar{i}}_{\bar{k}\bar{m}}\,\psi^{\bar{m}}_-\,, \end{aligned} \tag{3.14}$$

This is referred to in the literature as $\mathcal{N} = (2,2)$ supersymmetry, because there are two holomorphic and two anti-holomorphic SUSY parameters. The spinors and parameters are sections of the following bundles:

$$\begin{aligned} \psi^i_+ &\in \Gamma(\Sigma, K^{1/2} \otimes \Phi^*(T^{1,0}X))\,, \quad \psi^{\bar{i}}_+ \in \Gamma(\Sigma, K^{1/2} \otimes \Phi^*(T^{0,1}X))\,, \\ \psi^i_- &\in \Gamma(\Sigma, \overline{K^{1/2}} \otimes \Phi^*(T^{1,0}X))\,, \quad \psi^{\bar{i}}_- \in \Gamma(\Sigma, \overline{K^{1/2}} \otimes \Phi^*(T^{0,1}X))\,, \\ \alpha_+\,, \tilde{\alpha}_+ &\in \Gamma(\Sigma, K^{1/2})\,, \\ \alpha_-\,, \tilde{\alpha}_- &\in \Gamma(\Sigma, \overline{K^{1/2}})\,. \end{aligned} \tag{3.15}$$

3.3 R-symmetry

Now that we have established the supersymmetry of the sigma model, let us note two more symmetries it enjoys that go by the name of *R-symmetry*. More precisely, the *vector R-symmetry* and the *axial R-symmetry*, whose generators we will denote by F_V and F_A, respectively. These two symmetries act on the fermions only, and are defined as follows:

$$
\begin{aligned}
e^{i\,\alpha\,F_V}\,\{\psi^i_\pm,\psi^{\bar{i}}_\pm\} &\;\mapsto\; \{e^{-i\,\alpha}\,\psi^i_\pm, e^{i\,\alpha}\,\psi^{\bar{i}}_\pm\}\,,\\
e^{i\,\alpha\,F_A}\,\{\psi^i_\pm,\psi^{\bar{i}}_\pm\} &\;\mapsto\; \{e^{\mp i\,\alpha}\,\psi^i_\pm, e^{\pm i\,\alpha}\,\psi^{\bar{i}}_\pm\}\,.
\end{aligned}
\tag{3.16}
$$

3.4 The superalgebra

Now that we have defined all transformations explicitly on the fields, let us write down the corresponding superalgebra. First, we define the supersymmetry generators $Q_\pm$ and $\overline{Q}_\pm$ such that a total supersymmetry variation is expressed as follows:

$$
\delta = i\,\alpha_-\,Q_+ + i\,\alpha_+\,Q_- + i\,\tilde{\alpha}_-\,\overline{Q}_+ + i\,\tilde{\alpha}_+\,\overline{Q}_- \tag{3.17}
$$

These generators obey the following anti-commutation relations[3]:

$$
\{Q_\pm,\overline{Q}_\pm\} = P \pm H\,, \tag{3.18}
$$

where P and H are the Euclidean versions of the generators of space and time translations.
Let M, be the generator of Euclidean 'Lorentz boosts', i.e. $SO(2)$ rotations. Then it satisfies the following commutation relations:

$$
[M,Q_\pm] = \mp Q_\pm \quad \text{and} \quad [M,\overline{Q}_\pm] = \mp\overline{Q}_\pm\,. \tag{3.19}
$$

Finally, we write the commutation relations with the R-symmetry generators:

$$
\begin{aligned}
[F_V,Q_\pm] &= Q_\pm\\
[F_V,\overline{Q}_\pm] &= -\overline{Q}_\pm\\
[F_A,Q_\pm] &= \pm Q_\pm\\
[F_A,\overline{Q}_\pm] &= \mp\overline{Q}_\pm\,.
\end{aligned}
\tag{3.20}
$$

3.5 Anomalies

Having defined the R-symmetries (3.16) of the action (3.13), we would like to know whether they are symmetries of the full quantum theory. We will answer this question by carefully defining the measure of the path integral. To do this we begin by changing the basis of the fermions to 'momentum' space.

Concretely, we write the fermions in an eigenbasis of the operators that act on them in the action. For simplicity, let us suppress spacetime indices and write the fermions as $\psi_\pm$ and $\overline{\psi}_\pm$. The ψ_- will be decomposed into eigenspinors of the $D_{\bar{z}}D_z$ operator. If we denote the eigenvalues as λ_n, then we can write the spinor as follows:

$$
\psi_- = \sum_{n=0}^{\infty}\sum_{\alpha} c_{n,\alpha}\psi_-{}^{(n,\alpha)}\,, \tag{3.21}
$$

where the n labels the eigenvalue and the α labels the degeneracy of that eigenvalue. We can split up this sum into zero and non-zero modes

$$
\psi_- = \sum_{\alpha} c_{0,\alpha}\psi_-{}^{(0,\alpha)}\,, + \sum_{n=1}^{\infty}\sum_{\alpha} c_{n,\alpha}\psi_-{}^{(n,\alpha)}\,, \tag{3.22}
$$

where $\lambda_0 = 0$. Notice that a zero mode of $D_{\bar{z}}D_z$ is necessarily annihilated by D_z. The argument to prove this goes as follows: Suppose there is a ψ^i_- such that $D_{\bar{z}}D_z\psi_- = 0$. Then, we can take the complex conjugate

[3] We only write the non-vanishing relations.

of that solution, $\overline{\psi}_+$, and compute the following

$$\begin{aligned} 0 &= -\int \overline{\psi}_+ D_{\bar{z}} D_z \psi_-^i = +\int D_{\bar{z}}\overline{\psi}_+ D_z \psi_-^i \\ &= +\int \Big(D_{\bar{z}}\psi_-\Big)^* D_z\psi_- \geq 0\,, \end{aligned} \tag{3.23}$$

where the last inequality is saturated if and only if $D_z\psi_- = 0$.
We can similarly make a decomposition of $\overline{\psi}_-$ into eigenspinors of the same operator, and $\overline{\psi}_+$ and ψ_+ into eigenvectors of $D_z D_{\bar{z}}$. Having made this change of basis, we can rewrite the fermionic measure of the path integral roughly as follows:

$$\Big(\prod_\alpha d\psi_+{}^{(0,\alpha)}\Big)\Big(\prod_\alpha d\psi_-{}^{(0,\alpha)}\Big)\Big(\prod_\alpha d\overline{\psi}_+{}^{(0,\alpha)}\Big)\big(\prod_\alpha d\overline{\psi}_-{}^{(0,\alpha)}\big)\times(\text{nonzero modes})\,. \tag{3.24}$$

In this basis, the action will look like this

$$S \sim \sum_{n,m\neq 0}\sum_{\alpha,\beta}^{\infty}\Big(\overline{\psi}_+^{(n,\alpha)}\, D_{\bar{z}}\,\psi_+^{(m,\beta)} + \overline{\psi}_-^{(n,\alpha)}\, D_z\,\psi_-^{(m,\beta)}\Big)\,. \tag{3.25}$$

The point now is that the zero modes don't appear in the action, because they have been killed by the corresponding operator. This means that the Grassmann integration over these modes will go unsaturated, i.e. we will have integrals of the form $\int d\psi\cdot 1 = 0$. So the whole path integral will vanish unless we add some fermions to the integrand. The question is, how many fermions of each kind do we have to add. If it turns out that we have to add the same amount of fermions for each kind, say ℓ, then non-vanishing correlators for this theory will look like this

$$\langle \psi_+^1\ldots\psi_+^l\,\psi_-^1\ldots\psi_-^l\,\overline{\psi}_+^1\ldots\overline{\psi}_+^l\overline{\psi}_-^1\ldots\overline{\psi}_-^l\rangle\,. \tag{3.26}$$

Such a correlator has neutral R_V and R_A charge, so no anomaly is present. If, however, the number of zero modes of the different kinds of operators differ, then there will be an anomaly. So we need to count the (differences) in zero modes. First of all, we note that the number ℓ_+ of ψ_+ zero modes is equal to the number of $\overline{\psi}_-$ zero modes. This follows from the fact that

$$\Big(D_{\bar{z}}\psi_+\Big)^* = D_z\overline{\psi}_-\,. \tag{3.27}$$

Similarly, the number ℓ_- of ψ_- zero modes equals the number of $\overline{\psi_+}$ zero modes. This means that the non-vanishing correlator will have neutral R_V charge, because there is an equal number of barred and unbarred spinors.
Now, we would like to know the difference $\ell_+ - \ell_-$. The space of sections of $K^{1/2}\otimes\Phi^*(T^{(1,0)})$ (i.e. ψ_+ spinors) that are annihilated by D_z is denoted in the language of sheaf cohomology as $H^0(K^{1/2}\otimes\Phi^*(T^{(1,0)}))$. There is a powerfull theorem that can give us information about this space, namely the Riemann-Roch theorem. It states that, given a bundle E on a complex n-dimensional space Y, the following relation holds:

$$\sum_{i=0}^{n}(-1)^n\,\mathrm{dim}H^i(E) = \int_Y \mathrm{ch}(E)\,\mathrm{td}(Y)\,. \tag{3.28}$$

On the left hand side, $H^i(E)$ means $(0,i)$-forms taking values in E that are in the kernel of a covariantized $D_{\bar{z}}$ operator, modulo elements in the image of this operator. On the right hand side, ch and td stand for the Chern character and Todd class of the bundles in their argument. In our case, the alternating sum terminates at the first cohomology

$$\mathrm{dim}H^0 - \mathrm{dim}H^1\,. \tag{3.29}$$

We were after just H^0, and now we're stuck with its difference with H^1. What can we make of that second term? One can use of what is known as *Serre duality*, which states the following:

$$H^i(E) = H^{n-1}(K \otimes \overline{E})^* \,, \tag{3.30}$$

where $\overline{E}$ is the dual bundle, and the * means 'dual vector space'. In our case, this means

$$H^1(K^{1/2} \otimes \Phi^*(T^{(1,0)}X)) = H^0(K \otimes \overline{K^{1/2}} \otimes \Phi^*(T^{(0,1)}X))^* = H^0(K^{1/2} \otimes \Phi^*(T^{(0,1)}X))^* \,. \tag{3.31}$$

This is the dual to the space of $\overline{\psi}_+$'s, so its dimension is just ℓ_-. In other words, the Riemann-Roch theorem (3.28) already gives us the difference $\ell_+ - \ell_-$. So let us compute the right hand side. First, we need $\mathrm{ch}(K^{1/2} \otimes \Phi^*(T^{(1,0)}X))$. We can simplify this by using the rule that the Chern character of a tensor product bundle equals the product of the Chern characters of the individual bundles. Using the fact that $K = \overline{T^{(1,0)}\Sigma} = T^{(0,1)}\Sigma$ we can rewrite this as follows:

$$\begin{aligned}
\mathrm{ch}(K^{1/2} \otimes \Phi^*(T^{(1,0)}X)) &= \mathrm{ch}(K^{1/2})\mathrm{ch}(\Phi^*(T^{(1,0)}X)) = \sqrt{\mathrm{ch}(K^{1/2})}\Phi^*(\mathrm{ch}(T^{(1,0)}X)) \\
&= \left(1 - \frac{1}{2}\, c_1(T^{(1,0)}\Sigma)\right)\left(d + \Phi^*(c_1(T^{(1,0)}X))\right) \qquad (3.32) \\
&= 1 + \Phi^*(c_1(T^{(1,0)}X)) - \frac{d}{2}\, c_1(T^{(1,0)}\Sigma) \,, \qquad (3.33)
\end{aligned}$$

where d is the complex dimension of the target space. For simplicity we will denote $T^{(1,0)}$ by T and $T^{(0,1)}$ by $\overline{T}$. In the last line, we have thrown away terms that are higher than two-forms, since we are integrating over a surface. For the Todd class, we write

$$\mathrm{td}(T\Sigma) = 1 + \frac{1}{2}\, c_1(T\Sigma) \,. \tag{3.34}$$

Multiplying it all together, and keeping only the two-forms, we get

$$\ell_+ - \ell_- = \int_\Sigma \Phi^*(c_1(TX)) \,. \tag{3.35}$$

Therefore, a correlation function must be of the form

$$\langle (\psi_+)^{\ell_+}, (\overline{\psi}_-)^{\ell_+}, (\psi_-)^{\ell_-}, (\overline{\psi}_+)^{\ell_-} \rangle \,. \tag{3.36}$$

such that the difference is given by (3.35). If $c_1(TX) \neq 0$, this means that an operator with non-zero R_A charge has a vev (vacuum expectation value), which means that the symmetry is spontaneously broken. This is an anomaly in the symmetry.
This can also be seen by looking at how the path integral measure (3.24) transforms. Note, that the $D_z D_{\bar{z}}$ and $D_{\bar{z}} D_z$ operators provide an isomorphism between the spaces of ψ_+ and ψ_- non-zero modes, so we know that the non-zero mode part of the measure is R-charge neutral (under both R-symmetries). However, the Riemann-Roch theorem tells us that the zero modes part of the measure will contain a different number of ψ_+'s and ψ_-'s. Therefore, the measure itself has non-zero R_A-charge, which means it is not invariant under the symmetry. The condition for this anomaly to vanish $c_1(TX) = 0$. This means that the target space has to be a Calabi-Yau manifold.

To summarize: The R_V-charge is always preserved for any Kähler target space. The R_A-charge is preserved if and only if the target space is Calabi-Yau.

Chapter 4

Twisting: A/ B model topological field theory

Now that we have built a supersymmetric sigma model we would like to exploit the power of localization to make quantum mechanically exact calculations. However, we are not quite ready to do this because we have overlooked one obstruction. Usually, when studying *globally* supersymmetric models on a classical level, one takes spacetime to be a product of space and time, $Y \times \mathbb{R}$. In two dimensions, this would amount to taking the worldsheet to be $S^1 \times \mathbb{R}$. Then, one proceeds to define supersymmetry transformations with covariantly constant infinitesimal parameters ϵ, which would just need to be constant on a circle.
Here, we are doing something different. Because we are treating the system in a Euclidean path integral, and are taking the worldsheet to be an arbitrary Riemann surface with an arbitrary metric, we want supersymmetry parameters ϵ that are well defined and covariantly constant throughout the whole worldsheet, not just a spacelike slice. This means, that our epsilons need to be covariantly constant sections of $K^{1/2}$ and $\overline{K^{1/2}}$. If these bundles are not trivial, then they will only admit sections with at least one vanishing point. One can show that if an object that is everywhere covariantly constant has a vanishing point, then it must be zero everywhere. Hence, it will not be possible in general to use the supersymmetry techniques for our models, unless we change something about the theory.
The modification we need to make, is to replace the spinor fields by fields that are sections of trivial bundles. This trick, which we will explain in detail in what follows, is known as the *topological twist*, and it was invented by Witten.

4.1 The twists

There are many ways to define and interpret the topological twist, however, we will present the two that are most pragmatic. When one defines the $\mathcal{N} = (2,2)$ model (3.13), one first determines what kind of fields the theory should have (by superspace methods, for instance), and then one writes a Lagrangian such that it is Lorentz invariant (rotation invariant in the Euclidean case). What we will do now is the opposite. We will start with the form of a Lorentz invariant Lagrangian

$$L \sim \quad \text{bosons} \quad + \overline{\psi_+}\, D_{\bar{z}}\, \psi_+ + \overline{\psi_-}\, D_z \psi_- \,, \tag{4.1}$$

and reverse-engineer what the transformation properties of the ψ's should be. Let us start with the first fermionic term. Clearly, the derivative transforms as a dual anti-holomorphic vector[1] , i.e. as a section of $\overline{K}$

$$\begin{aligned} z &\mapsto e^{i\,\alpha}\, z \\ D_{\bar{z}} &\mapsto e^{-i\,\alpha}\, D_{\bar{z}} \,, \end{aligned} \tag{4.2}$$

[1] Remember, we are taking the 'active' point of view.

and there is nothing we can do about that. So, in order to have a Lorentz invariant term in the Lagrangian, it is clear that the product of the two ψ's must transform oppositely to $D_{\bar{z}}$

$$\begin{aligned} z &\mapsto e^{i\,\alpha}\, z \\ \psi_+\, \overline{\psi}_+ &\mapsto e^{+i\,\alpha}\psi_+\, \overline{\psi}_+\,, \end{aligned} \tag{4.3}$$

i.e. their product must be a section of K. How should the spinors transform individually? The first obvious choice is to take them each to be a section of $K^{1/2}$. However, as we argued before, this bundle may be non-trivial. The other two inequivalent choices [2] are the following: We can take ψ_+ to be a section of the trivial line bundle $\mathbb{I}$, (i.e. a scalar), and $\overline{\psi}_+$ to be a section of K (i.e. a holomorphic one-form). This is called the (+) twist. We can also do the opposite, i.e. let ψ_+ be a section of K and $\overline{\psi}_+$ a section of the trivial bundle. This is obviously called the (−) twist. To summarize

$$(\psi_+, \overline{\psi}_+) \in \Big(\Gamma(K^{1/2}), \Gamma(K^{1/2})\Big) \to \begin{cases} \Big(\Gamma(\mathbb{I}), \Gamma(K)\Big) & (+)\text{ twist} \\ \Big(\Gamma(K), \Gamma(\mathbb{I})\Big) & (-)\text{ twist}\,. \end{cases} \tag{4.4}$$

Similarly, we define the (±) twists for the other two spinors as follows:

$$(\psi_-, \overline{\psi}_-) \in \Big(\Gamma(\overline{K^{1/2}}), \overline{\Gamma(K^{1/2})}\Big) \to \begin{cases} \Big(\Gamma(\mathbb{I}), \Gamma(\overline{K})\Big) & (+)\text{ twist} \\ \Big(\Gamma(\overline{K}), \Gamma(\mathbb{I})\Big) & (-)\text{ twist}\,. \end{cases} \tag{4.5}$$

Up to inversion of the worldsheet complex structure, i.e. switching the definitions of z and $\bar{z}$, the are only two inequivalent choices we can make for whole model:

- The 'A-model': Taking the (−) twist for ψ_+ and the (+) twist for ψ_-.
- The 'B-model': Taking the (−) twist for both ψ_+ and ψ_-.

To define the supersymmetry transformation rules for these new models, we must take into account that expressions like

$$\delta\phi \sim \epsilon\psi \quad \text{and} \quad \delta\psi \sim \epsilon\partial\phi \tag{4.6}$$

will only make sense if we also redefine the Lorentz properties of the ϵ's. The important result of these twisted models, is that now half of the supersymmetry parameters have become scalars (even though they are still Grassmann valued). This is what we needed. It is always possible to define globally constant non-zero scalars. Now, we can use the full power of supersymmetry, without the obstructions of the non-trivial worldsheet topology.

There is one more point of view for defining the twists that will be useful to us. We can do it via the algebra of symmetry generators. One can restate the A and B twists by saying that the Lorentz transformation properties of the new fields should be related to the old transformation properties as follows:

$$M_A = M_{\text{old}} - F_V \quad \text{for the A} - \text{model}\,, \tag{4.7}$$

$$M_B = M_{\text{old}} - F_A \quad \text{for the B} - \text{model}\,. \tag{4.8}$$

Defining $Q_A = \bar{Q}_+ + Q_-$ and $Q_B = \bar{Q}_+ + \bar{Q}_-$, one can check (using (3.20)) that $[M_A, Q_A] = 0$ and $[M_B, Q_B] = 0$, making Q_A (Q_B) a scalar under the new Lorentz group for the A-twist (B-twist). This means that one can define the A- and B- Lorentz group on arbitrary curved worldsheets.

In the next subsections, we will study both models in detail.

[2] up to a redefinition of the complex structure, $z \to \bar{z}$.

4.2 The A-model

4.2.1 The model

Now that the spinors are actually no longer spinors, let us rewrite them in a notation that makes their transformation properties more obvious:

$$\begin{aligned} \psi^i_+ &\mapsto \psi^i_z\,, \quad \psi^{\bar{i}}_+ \mapsto \chi^{\bar{i}} \\ \psi^i_- &\mapsto \chi^i\,, \quad \psi^{\bar{i}}_- \mapsto \psi^{\bar{i}}_{\bar{z}}\,. \end{aligned} \tag{4.9}$$

The A-model action now reads

$$S_A = 2t \int_\Sigma (g_{i\bar{j}} \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} + i g_{\bar{i}j} \psi^i_z D_{\bar{z}} \chi^{\bar{j}} + i g_{i\bar{j}} \psi^{\bar{j}}_{\bar{z}} D_z \chi^i + \frac{1}{2} R_{i\bar{j}k\bar{l}} \psi^i_z \psi^{\bar{j}}_{\bar{z}} \chi^k \chi^{\bar{l}})\,, \tag{4.10}$$

where we have introduced a coupling constant t up front. Looking at the old supersymmetry transformations (3.15), we see that α_- and $\tilde{\alpha}_+$ are now scalars, and α_+ and $\tilde{\alpha}_-$ are sections of the canonical and anti-canonical line bundle, respectively. Now we can have a globally supersymmetric system, by throwing away the two latter, and keeping the scalar SUSY parameters. Denoting the two parameters by α and $\tilde{\alpha}$, the new transformations read

$$\begin{aligned} \delta\phi^i &= i\,\alpha\chi \\ \delta\phi^{\bar{i}} &= i\tilde{\alpha}\,\chi^{\bar{i}} \\ \delta\psi^{\bar{i}}_z &= -\alpha\,\partial_z\phi^{\bar{i}} - i\,\tilde{\alpha}\,\chi^{\bar{k}}\,\Gamma^{\bar{i}}_{\bar{k}\bar{m}}\,\psi^{\bar{m}}_z \\ \delta\psi^i_{\bar{z}} &= -\tilde{\alpha}\,\partial_{\bar{z}}\phi^i - i\,\alpha\,\chi^k\,\Gamma^i_{km}\,\psi^m_{\bar{z}} \\ \delta\chi^i &= \delta\chi^{\bar{i}} = 0\,. \end{aligned} \tag{4.11}$$

Notice the similarity with the zero-dimensional model in (1.6). We can simplify this model by taking $\alpha = \tilde{\alpha}$. This corresponds to defining an operator, which we will call the A-SUSY operator Q_A, as $Q_A = \bar{Q}_+ + Q_-$. Using this new operator, we can now express the action (4.10) as

$$S_A = i\,t \int_\Sigma \{Q_A, V\} + t\int_\Sigma \Phi^*(K)\,, \tag{4.12}$$

where $V = g_{i\bar{j}}(\psi^i_{\bar{z}}\partial_{\bar{z}}\bar{\phi}^j + \partial_z\phi^i\psi^{\bar{z}}_{\bar{j}})$, and K is the Kähler form of the target space. Hence, one can almost express the Lagrangian as Q_A-exact, which would make the theory topological. The failure of this action to be a purely Q_A-exact action, is an integral over the worldsheet of the pullback of the spacetime Kähler form (to the worldsheet), i.e. the second term. However, this is not so bad. This term only depends on the homology class of the image $\Phi(\Sigma)$, of the worldsheet under the embedding map. This means that we can split up the path integral into different sectors according to this homology class and factor this Kähler term out, as follows:

$$Z = \sum_{\beta\in H_2(X,\mathbb{Z})} e^{-t\,K\cdot\beta} \int_{[\Phi(\Sigma)]\in\beta} d[\Phi]d[\chi]d[\psi]e^{-i\,t\int\{Q_A,V\}}\,. \tag{4.13}$$

It can be shown, that this theory is independent of the complex structures of Σ and X, and depends only on the Kähler class of the target space through the $\exp(-t\,K\cdot\beta)$. Otherwise, the model is *half topological* in the sense that it is independent of half of the moduli of the worldsheet and target space metrics.
Now that we have the action (almost) in 'cohomological form', we can argue as in the zero-dimensional case, that the path integral localizes to the minima of the action. For the bosonic part of the path integral, these are configurations satisfying

$$\partial_{\bar{z}}\phi^i = \partial_z\phi^{\bar{i}} = 0\,, \tag{4.14}$$

i.e. the holomorphic maps $\Phi : \Sigma \mapsto X$.

4.2.2 Anomalies

We will now repeat an analysis of the zero modes analogous to the one done for the untwisted model. We begin by noting that the number ℓ_χ of χ zero modes equals that of $\overline{\chi}$ zero modes (by simple complex conjugation of the zero mode equation), and similarly the number ℓ_ψ of $\overline{\psi}_z$ zero modes equals that of $\psi_{\bar{z}}$ zero modes. Therefore, there will be no R_V anomaly. There will be an R_A anomaly if $\ell_\chi \neq \ell_\psi$. The χ zero modes are elements of $H^0(\Phi^*(TX))$. So the Riemann-Roch theorem should tell us something about it

$$\int_\Sigma \mathrm{ch}(\Phi^*(TX))\,\mathrm{td}(T\Sigma) = \mathrm{dim}H^0(\Phi^*(TX)) - \mathrm{dim}H^1(\Phi^*(TX))\,. \tag{4.15}$$

By Serre duality, we can write $H^1(\Phi^*(TX)) = H^0(K \otimes \Phi^*(\overline{TX}))^*$. We can recognize this as the dual to the space of ψ zero modes. Hence, the Riemann-Roch theorem gives us exactly the difference we need

$$\begin{aligned}\ell_\chi - \ell_\psi &= 2\int_\Sigma (d + \Phi^* c_1(TX))\,(1 + \frac{1}{2}\,c_1(T\Sigma)) \\ &= 2\int_\Sigma \Phi^* c_1(TX) + d\,(1-g) \equiv 2\,k\,,\end{aligned} \tag{4.16}$$

where d is the complex dimension of the target space. The factor of two comes from the fact that the Riemann-Roch theorem computes complex dimensions. Here, we have used

$$\int_\Sigma c_1(T\Sigma) = \chi(\Sigma) = 2 - 2\,g\,. \tag{4.17}$$

This result tells us that a non-vanishing correlator must have $2\,k$ more insertions of χ operators than of ψ operators. These must come in equal numbers of holomorphic and anti-holomorphic versions of the operators.

4.2.3 Observables

Now, we are ready to discuss the physical observables of the A-model. The first ground rule is, these must be defined by correlators of operators that are closed under the Q_A operation. The second rule is that they must be topological, in the sense that the worldsheet and target metrics must not be involved in their construction. This is to ensure that the theory can later be coupled to gravity by integrating over the worldsheet metric, and to ensure that the correlators will only depend on the Kähler class of the target space. This means we cannot use the ψ's, since they contain a worldsheet Lorentz index that needs to be contracted with the metric. This leaves us with local operators of the form

$$\mathcal{O}(x) = C_{i_1\ldots i_p\bar{j}_1\ldots\bar{j}_q}(\phi(x))\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q}\,, \tag{4.18}$$

where C is a function of the ϕ's, and is antisymmetric in its indices due to the fact that it is contracted with Grassmann variables. Here, x is the worldsheet position of the operator insertion. By using the transformation rules in (4.12), one easily sees that the variation of such an operator is the following:

$$\{Q_A, \mathcal{O}(x)\} \simeq \frac{\partial C_{i_1\ldots i_p\bar{j}_1\ldots\bar{j}_q}(\phi(x))}{\partial\phi^{\bar{k}}}\,\chi^{\bar{k}}\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q} + \frac{\partial C_{i_1\ldots i_p\bar{j}_1\ldots\bar{j}_q}(\phi(x))}{\partial\phi^{k}}\,\chi^{k}\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q}\,. \tag{4.19}$$

In other words, if we view $C(\phi(x))$ as a (p,q) form on X, then

$$\{Q_A, \mathcal{O}_C\} \simeq \mathcal{O}_{dC}\,, \tag{4.20}$$

where dC is the de Rham exterior derivative on C. This means that we can identify (by a group isomorphism) the Q_A cohomology of physical operators with the de Rham cohomology of X, by viewing the χ^i as holomorphic differentials $d\phi^i$ and the $\chi^{\bar{i}}$ as the anti-holomorphic ones $d\phi^{\bar{i}}$:

$$\mathcal{H}(\mathcal{Q_A}) = \mathcal{H}_{\lceil\mathcal{R}}(\mathcal{X})\,. \tag{4.21}$$

The result in the previous subsection prescribes that a correlator is only non-vanishing if one inserts a certain number of χ's and ψ's, whereby the difference in the numbers is given by

$$2\,k = 2\int_{\Sigma} \Phi^* c_1(TX) + 2\,d\,(1-g)\,. \tag{4.22}$$

The theorem only gives us a difference, it does not tell us how many ψ's we need. We will restrict to the what is called the 'generic' case, where this number equals zero since, by our earlier statements, we have excluded considering ψ-insertions,

$$\dim H^1(\Phi^*(TX)) = 0\,. \tag{4.23}$$

However, the non-generic case can also be treated by introducing certain generalizations.
One is studying correlators of the form $\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle = \int \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\psi e^{-S} \mathcal{O}_1 ... \mathcal{O}_n$ (n-point functions) with $\mathcal{O}_i \in H^{(p_i,q_i)}(X)$. Our anomaly analysis gave us selection rules that tell us, which correlators are not trivially zero and thus contain the interesting information. The vector anomaly cancellation rule gave us $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i$, and this combined with the axial anomaly cancellation tells us $k = \sum_{i=1}^n p_i = \sum_{i=1}^n q_i$ with k given by our Riemann-Roch calculation. For the sake of clarity, let us summarize the vector and axial charges of Grassmann worldsheet scalar insertions:

Operator insertion	$U(1)_V$ vector charge	$U(1)_A$axial charge
$\chi^{i_1} ... \chi^{i_p}$	$-p$	p
$\chi^{j_1} ... \chi^{j_q}$	q	q

So, what kind of information does the topological A-model compute? The answer is, that it counts the number of holomorphic maps (corresponding to worldsheet instantons) from the worldsheet Σ to the target space X. By the property of localization, our correlators reduce to

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle \sum_{\beta \in H_2(X,\mathbb{Z})} e^{-t(K\cdot\beta)} \cdot N_g^\beta, \tag{4.24}$$

where N_g^β denotes the number of maps from our genus g Riemann surface (the worldsheet) into curve class β. Let us denote the moduli space of these holomorphic maps with $\mathcal{M}_{\Sigma_g}(X,\beta)$,

$$\mathcal{M}_{\Sigma_g}(X,\beta) := \{\phi : \Sigma_g \to X \quad | \quad \phi_*[\Sigma_g] = \beta \in H_2(X,\mathbb{Z}) \quad \text{and} \quad \bar{\partial}\phi = 0\}. \tag{4.25}$$

Note that a point in this moduli space is a map, ϕ. To give a more formal definition of what the topological A-model computes, one would like to integrate over this moduli space of maps. The χ-zero modes provide us with a measure on $\mathcal{M}_{\Sigma_g}(X,\beta)$. Let us therefore develop some intuition for this.
First of all, given a certain map ϕ, think of it's image $\mathrm{Im}(\phi)$ (a curve in X). If one deforms this map, it is intuitively clear that one is moving in a tangential direction. At the same time the map should remain holomorphic, so one is moving in $T^{(1,0)}X$ direction (or, more restrictive, in a direction in the holomorphic tangent space restricted to our Riemann surface). Remember that $\chi \in \phi^*(T^{(1,0)}(X))$. This is a first hint that tangential directions can be associated to χ-directions. Now, let us add a stronger argument. Taking a supersymmetry variation of our map (our bosonic worldsheet scalar), one gets something proportional to χ: $\delta\phi^i \propto \chi^i$. If one requires this deformation to be holomorphic, one gets the zero mode equation $\bar{\partial}\chi^i$ – the reader may consult [1] on this point. Therefore, we conclude that the χ-zero mode directions can be identified with directions in $T\mathcal{M}_{\Sigma_g}(X,\beta)$, providing a volume measure on our moduli space of interest.
By inserting our operators $\mathcal{O}_i$ at positions $x_i \in \Sigma_g$ one arrives at the notion of a punctured Riemann surface (with the n punctures at the points of the n operator insertions). Let us now define n evaluation maps ev_i, $(i = 1, ..., n)$, one associated to each operator one inserts:

$$\mathrm{ev}_i : \mathcal{M}_{\Sigma_g}(X,\mathbb{Z}) \to X, \quad \phi \to \phi(x_i). \tag{4.26}$$

The evaluation map sends a map ϕ to its evaluation at the point of insertion. An operator $\mathcal{O}_i$ inserted at x_i can now be seen as a pullback of $\omega_i \in H^*(X)$: $\mathcal{O}_i(x_i) \leftrightarrow \mathrm{ev}_i^*(\omega_i)$. Now, one can formally write out our correlator as follows:

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle = \sum_\beta e^{-t\,K\beta} \int_{\mathcal{M}_{\Sigma_{g,n}}(X,\beta)} \mathrm{ev}_1^*(\omega_1) ... \mathrm{ev}_n^*(\omega_n), \tag{4.27}$$

The latter expression can be seen as a definition of the Gromow-Witten invariant $N^{\beta}_{g,n}$, counting the number of holomorphic maps from the genus g Riemann surface with n punctures into curve class β in our target space (which we choose to be a CY).
Since the theory we are considering is cohomological, differentiating the path integral w.r.t. the Kähler parameter t yields zero[3]. Hence, we can safely take the $t \to \infty$ limit. In this limit, the dominant contributions come from the maps whose image belongs to the trivial class $\beta = 0$. In order words, those for which the image of the worldsheet is homologous to a point. Then, one has $\mathcal{M}_{\Sigma_g}(X, 0) \simeq X$ and one can has $\mathrm{ev}_i = \mathrm{id}_i$. In this case, one thus gets an integral over our target space CY. To finish this discussion, let us mention that for a CY 3-fold as a choice of target space, the selection rules require the insertion of a $(3, 3)$ total form degree, one hence gets an integral over the CY of the volume form. At genus one, no insertion is needed to find a non-zero result, and we are thus left with a partition function. At higher genus, there are no insertions possible. To summarize, in an A-model TFT we find the following (non-vanishing) GW invariants: $N^0_{0,3}$ by calculating a 3-point function, and $N^0_{1,0}$ by calculating the partition function.

4.2.4 An easy example: GW invariants of $X = \mathbb{CP}^1$rom TFTs

Let us start by noting the general formula

$$c(\mathbb{CP}^n) = (1 + K)^{n+1}$$

for the total Chern class of the complex projective n-space [4], where K denotes the Kähler form on $\mathbb{CP}^n$. Furthermore, $\int_{\mathbb{CP}^n} K^n = 1$.
When considering $X = \mathbb{CP}^1$, we get

$$c(\mathbb{CP}^1) = c_0 + c_1 = 1 + 2K, \tag{4.28}$$

and therefore $\chi(\mathbb{CP}^1) = \int_{\mathbb{CP}^1} c_1 = 2$. Let us also remind the reader that $\mathbb{CP}^1 \equiv S^2 \equiv \mathbb{C} + \{\infty\}$, from which one can recall the following cohomology groups:

$$\begin{aligned} H^0(\mathbb{CP}^1) &\cong \mathbb{Z}, \\ H^1(\mathbb{CP}^1) &= \emptyset, \\ H^2(\mathbb{CP}^1) &\cong \mathbb{Z}. \end{aligned} \tag{4.29}$$

Our selection rule(s) (4.16) for the insertions of (p, q)-forms into correlators for this model reads

$$\begin{aligned} \sum_i p_i = \sum_i q_i = k &= \int_{\Sigma_g} \Phi^*(c_1(X)) + \dim_{\mathbb{C}}(\mathbb{CP}^1)(1 - g), \\ &= \int_{\Phi_*(\Sigma_g)} 2K + (1 - g), \\ &= 2n + (1 - g), \end{aligned} \tag{4.30}$$

where in the last step, we used the fact that the images of the worldsheet $\Phi_*(\Sigma_g)$ are classified by $\mathbb{Z}$, so one can denote them by $\beta = n \cdot \mathrm{PD}[K]$, with $\mathrm{PD}[K]$ denoting the Poincaré dual cycle of K, which means that the integral produces an integer n.
Let us have a look at a TFT (one is not summing over worldsheet geometries, yet) with $\mathbb{CP}^1$ as a target space, and a genus zero (i.e. a sphere) worldsheet.
A-model TFT with $\Sigma_0 = S^2$ and $X = \mathbb{CP}^1$:
One has $\sum_i p_i = \sum_i q_i = 2n + 1$. For (worldsheet image) curve class $\beta = 0$, corresponds to $n = 0$, one has to insert a $(1, 1)$-form. The only choice is the Kähler form, so one gets a one-point function,

$$\langle \mathcal{O} \rangle = \int_{\mathbb{CP}^1} K = 1. \tag{4.31}$$

[3]Only the cohomological part is t-independent. The part that is factored out certainly depends on it.
[4]This formula can be obtained by considering an exact sequence, the Euler sequence (from there follows $c(\mathbb{CP}^n = (c(\mathcal{O}(1)))^n)$, check [2] for more detail.

At $n = 1$ one has to insert forms whose degrees add up to $(3, 3)$. Since the target space is a sphere, and hence the highest form on it is a two-form, one can only accomplish this by means of a 3-point function,

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = e^{-t} \int_{\mathcal{M}_{S^2}(\mathbb{CP}^1,1)} \mathrm{ev}_1^*(\omega_1)\mathrm{ev}_2^*(\omega_2)\mathrm{ev}_3^*(\omega_3). \tag{4.32}$$

Obviously, this integral counts holomorphic maps from three points to three points. Holomorphic maps from S^2 to S^2 are given by the Möbius transformations, $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$. There are four visible parameters, but one can rescale according to $((a, b), (c, d)) \to (\lambda(a, b), \lambda(c, d))$ without changing the transformation (the parameters are homogenous coordinates on the moduli space of maps). This means one is left with effectively three parameters characterizing such a map. Therefore, by specifying the image of three points, one uniquely determines map, and our final result for the 3-point function is e^{-t}.
For $\beta \geq 2 = n$ one similarly has an $(2n + 1)$-point function, mapping $2n + 1$ points to $2n + 1$ points. These kinds of maps are not one-to-one maps, but n-to-one maps. They are written as $f(z) = \frac{a_n z^n + \ldots a_1 z + b}{c_n z^n + \ldots c_1 z + d}$ and, therefore, (taking the scaling argument into account) are specified by $2n + 2 - 1$ parameters. $2n - 1$ is exactly the number of points we are fixing with the correlator, so the maps are uniquely determined, and one always has one map. We conclude, that we have completely understood the genus zero part of the A-model, and one has found the following GW invariants

$$N^{\beta}_{0,2\beta+1} = 1, \tag{4.33}$$

for $\mathbb{CP}^1$. Of course, one can now go on and have a look at other TFT's on $c\mathbb{P}^1$, such as the genus one (or higher genus) worldsheet theory. For example, at $g = 1$ one finds a partition function at $\beta = 0$, a 2-point function at $\beta = 1$ and a 4-point function (i.e. no insertions) at $\beta = 2$,... Note the difference between this case and a CY target space. For a CY, the number of insertions is defined by the genus and is independent (at a fixed genus) of the curve class β. Therefore, for a CY, one will never get any non-vanishing correlators (and GW invariants) at $g \geq 2$, the TFT are trivial. However, there will be remedy for this when considering A-model topological *string* theory (i.e. coupled to 2-dimensional gravity.)

4.3 The B-model

4.3.1 The model

Similarly to the treatment of the A-model, let us again rewrite (and for convenience recombine) the spinors in a notation which reflects their transformation properties:

$$\begin{array}{rcl} \psi^i_+ & \mapsto & \psi^i_z\,, \quad g_{i\bar{j}}(\psi^{\bar{j}}_+ - \psi^{\bar{j}}_-) \mapsto \theta_i\,, \\ \psi^i_- & \mapsto & \psi^i_{\bar{z}}\,, \qquad\quad \psi^{\bar{i}}_+ + \psi^{\bar{i}}_- \mapsto \eta^{\bar{i}}\,. \end{array} \tag{4.34}$$

The action for the B-model then reads

$$L = t \int_{\Sigma} (g_{i\bar{j}} \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{i}} + i g_{i\bar{j}} \eta^{\bar{i}} (D_{\bar{z}} \psi^i_z + D_z \psi^i_{\bar{z}}) + i\theta_i (D_{\bar{z}} \psi^i_z - D_z \psi^i_{\bar{z}}) + \frac{1}{2} R_{i\bar{j}k}{}^{l} \psi^i_z \psi^k_{\bar{z}} \eta^{\bar{j}} \theta_l). \tag{4.35}$$

The action is invariant under the supersymmetry transformations (for simplicity, we set $\alpha = \tilde{\alpha}$)

$$\begin{array}{rcl} \delta\phi^i & = & 0 \\ \delta\phi^{\bar{i}} & = & i\alpha\eta^{\bar{i}} \\ \delta\psi^{\bar{i}}_z & = & -\alpha \\ \delta\psi^i_{\bar{z}} & = & -\alpha d\phi^i \\ \delta\eta^{\bar{i}} & = & \delta\theta_i = 0\,. \end{array}$$

This time, we define the B-SUSY operator $Q_B = Q + Q$, which again allows us to express the B-model action as almost exact:

$$S_B = it \int_\Sigma \{Q_B, V\} + t \int_\Sigma (i\eta_i(D_{\bar{z}}\psi^i_z - D_z\psi^i_{\bar{z}}) + \frac{1}{2} R_{i\bar{j}k}{}^l \psi^i_z \psi^k_{\bar{z}} \eta^{\bar{j}} \theta_l) \tag{4.36}$$

with $V = -g_{i\bar{j}}(\psi^i_z \partial_{\bar{z}} \bar{\phi}^j + \partial_z \bar{\phi}^j \psi^i_{\bar{z}})$.
This Lagrangian is exact up to a term which does not depend on the Kähler metric of the target space (this is not obvious). Again, we have the action in (almost) cohomological form, and one can argue that the path integral localizes to the minima of the action. This time – note the contrast to the A-model case – the bosonic fields have to satisfy

$$\partial_z \phi^i = \partial_{\bar{z}} \phi^i = \partial_z \phi^{\bar{i}} = \partial_{\bar{z}} \phi^{\bar{i}} = 0, \tag{4.37}$$

so the path integral localizes to constant maps. (This can be found by analyzing the classical equations of motion of the cohomological part of the action.)

4.3.2 Anomalies

Following the treatment of the A-model, one can use the Riemann-Roch theorem to calculate the difference of needed χ and ψ insertions (although we will not consider ψ insertions) for an anomaly-free correlator. One again finds

$$\begin{aligned} \ell_{\eta,\theta} - \ell_\psi &= 2 \int_\Sigma (d + \Phi^* c_1(TX))\,(1 + \frac{1}{2}\,c_1(T\Sigma)) \\ &= 2 \int_\Sigma \Phi^* c_1(TX) + d\,(1-g) \equiv 2\,k\,, \end{aligned} \tag{4.38}$$

with d the complex dimension of the target space X. Note that (in analogy to what we have seen for the A-model) the number of χ_i zero modes equals the number of $\chi_{\bar{i}}$ zero modes (and the analogous statement for the ψ's holds too, of course). Again, one has no R_V anomaly. This means that one is studying insertions of $(0,d)$ forms with values in $\bigwedge^q T^{(1,0)}X$ (compare with the following discussion on observables of the B-model) at genus zero and a partition function at genus one. Without allowing for insertions of operators with opposite $U(1)_A$-charge (such as the ψ's) one cannot write down any correlators at higher genus [5].

4.3.3 Observables

What kind of physical operators can one write down in this case? Due to the same reasons as for the A-model, we will restrict to building operators with the worldsheet scalars, θ_i and $\eta^{\bar{i}}$, in order to ensure that we have Q_B-closed operators with no metric dependence. The The local operators read

$$\mathcal{O}(x) = C^{i_1 \ldots i_p}{}_{\bar{j}_1 \ldots \bar{j}_q}(\phi(x))\theta_{i_1}\ldots\theta_{i_p}\eta^{\bar{j}_1}\ldots\eta^{\bar{j}_q}\,, \tag{4.39}$$

where again C is a antisymmetric. Studying the variation of such an operator using (4.36), one finds

$$\{Q_B, \mathcal{O}(x)\} = -\frac{\partial C^{i_1 \ldots i_p}{}_{\bar{j}_1 \ldots \bar{j}_q}(\phi(x))}{\partial \phi^{\bar{k}}} \eta^{\bar{k}} \theta_{i_1}\ldots\theta_{i_p}\eta^{\bar{j}_1}\ldots\eta^{\bar{j}_q}. \tag{4.40}$$

This time our notation suggests a different identification. If one thinks of $C(\phi(x))$ as a $(0,q)$ form with values in the antisymmetrized product of p holomorphic tangent spaces of X (one denotes this space a $\bigwedge^q(T^{(1,0)}X)$, one can write

$$\{Q_B, \mathcal{O}_C\} = \mathcal{O}_{\bar{\partial} C}. \tag{4.41}$$

[5] This may be seen as one of the motivations to formulate TST: There one can construct anomaly-free correlators at higher genera, as new operators with the desired opposite $U(1)_A$-charge become available for insertion.

By viewing the θ_i as vector fields $\partial_i = \frac{\partial}{\partial \phi^i}$ and the $\eta^{\bar{i}}$ as anti-holomorphic one forms $d\phi^{\bar{i}}$, acting with Q_B amounts to acting with the Dolbeault cohomology operator $\bar{\partial}$ (up to a sign). Thus, for the B-model, we can identify (again through a group isomorphism) (minus) the Q_B cohomology of physical operators with the Dolbeault cohomology of X:

$$\mathcal{H}(Q_B) = H_{\text{Dolb}}(X, T^{1,0}X). \tag{4.42}$$

A priori this seems to be a rather 'boring' result, because one does not need topological string theory to compute these cohomologies. However, the observables of the B-model on a Calabi-Yau manifold X can be shown to be related to the observables of the A-model on a different manifold $\tilde{X}$, which is known as the *mirror manifold*. This is (part of) the statement of *mirror symmetry*. Often, there are practical ways to construct such a mirror manifold $\tilde{X}$ given some Calabi-Yau X. Then, one can compute Gromov-Witten invariants of $\tilde{X}$, which are rather sophisticated topological invariants, by computing the 'easy' invariants of X. Unfortunately, we will not go into this topic in these notes.

Chapter 5

Topological string theory

5.1 A-model correlators and coupling to gravity

Let us go back to the A-model TFT (on a genus g Riemann surface Σ_g) for a moment. By analyzing the potential for anomalous R-symmetries, we found selection rules for correlators. We found (restricting to the generic case and respecting our allowed operator insertions) that the sum of the holomorphic degrees of our operator insertions as well as the sum of the anti-holomorphic degrees must be equal to an expression depending on the (complex) target space dimension, the first Chern class and the genus of our worldsheet:

$$\sum_k \deg_{p_k} \mathcal{O}_k^{(p_k,q_k)} = \int \phi^*(c_1(\mathcal{M})) + d(1-g),$$
$$\sum_k \deg_{q_k} \mathcal{O}_k^{(p_k,q_k)} = \int \phi^*(c_1(\mathcal{M})) + d(1-g). \quad (5.1)$$

Note that for a CY manifold, this means that one will only be able to find a non-vanishing correlator at genus zero and at genus one. The reason for this was that the degree of the insertion (as seen as an element of the target space de Rham cohomology $H^*(X)$) corresponds to the axial charge of the worldsheet operator, hence one cannot make any 'negative' insertions. As we are about to see, this situation will change when allowing a path-integral over the worldsheet metric field. This is what is meant by coupling the (twisted) topological sigma model to gravity.

The intuitive reasoning for why the higher genus correlators on a CY vanish goes as follows: When considering a TFT, one works with a fixed genus g Riemann surface (with a fixed complex structure class). In general, there are just no holomorphic maps to X.

However, if one would allow different complex structure classes for a genus g Riemann surface, one should find such maps, at least at isolated points in the (complex structure) moduli space $\mathcal{M}_g$ of these surfaces. Again, put differently, for a given genus Riemann surface and n insertions, the conditions for the n points to be mapped to the n points in X representing our cycles, overdetermine the map. In general, such a map won't exist. Let this serve as a motivation to include a path-integral over all possible metrics on Σ_g (which will split up into a sum over each complex structure class and an integral over all conformally equivalent metrics in each such class). We therefore expect to be able to get interesting results at every genus from a TST on a CY manifold.

Let us begin by discussing the integration over all possible metrics on a genus g Riemann surface. Here, one is confronted with a problem known from ordinary string theory. Schematically, we are dealing with partition function of the form

$$Z = \sum_{\Sigma_g} \int \mathcal{D}[g_{\Sigma_g}] \int \mathcal{D}[\phi]\mathcal{D}[\xi]\mathcal{D}\psi e^{-\int_{\Sigma_g} L[g,\phi,\xi,\psi]}, \quad (5.2)$$

which we expect to localize at the constant map contributions. The integral over all metrics is interpreted as a sum over all genus g Riemann surfaces with all possible metrics. Topological string theory, much like ordinary string theory, is a 2d conformal field theory. This means that this path-integral is overcounting metrics, as one has the conformal gauge symmetry. One can fix a conformal gauge on a Riemann surface - which requires the introduction of a ghost and an anti-ghost, according to the FP trick. As known from the bosonic string, a different gauge has to be chosen for each complex structure class. The volume of the conformal group is then factored out, and one is left with an integral over the moduli space of complex structures $\mathcal{M}_g$ on every genus g Riemann surface.

Recall that an almost complex structure (in this case on a Riemann surface) is an endomorphism $J \in \mathrm{End}(T\Sigma_g)$ (i.e. $J \in T\Sigma_g \otimes T^*\Sigma_g$, or — put differently – J is a $(1,1)$-tensor) squaring to minus the identity, $J^2 = -1$. Defining projections for vector fields onto their holomorphic and anti-holomorphic parts (with respect to the almost complex structure J) according to

$$P = \mathbf{1} - \frac{iJ}{2},$$
$$\bar{P} = \mathbf{1} + \frac{iJ}{2},$$

one can nicely formulate the condition for J to be a complex structure. For two vector fields X, Y on Σ_g, (using the Lie bracket for vector fields) the (integrability) condition reads

$$\bar{P}[PX, PY] = 0.$$

This can be shown to be equivalent to the vanishing of the Nijenhuis tensor

$$N[X,Y] = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = 0. \tag{5.3}$$

To define an integral over the moduli space of complex structures, one has to define a measure. To achieve this, one is interested in the directions tangent to $\mathcal{M}_g$ Therefore, study an (infinitesimal) deformation of the complex structure, $J \to J + \epsilon$. One can choose coordinates $(z, \bar{z})$ on Σ_g such that J can be written as $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ as a map from $T\Sigma_g$ to $T\Sigma_g$. For an infinitesimal deformation, the equation $(J+\epsilon)^2 = -\mathbf{1}$ tells us that $\epsilon = \begin{pmatrix} 0 & \epsilon_1 \\ \epsilon_2 & 0 \end{pmatrix}$.

The requirement that the Nijenhuis tensor vanish for $J + \epsilon$ (for an infinitesimal variation) leads to the restrictions $\bar{\partial}\epsilon_1 = 0$ and $\partial\epsilon_2 = 0$. In other words, ϵ_1 is a $(0,1)$ form with values in $T^{(1,0)}\Sigma_g$; remember, it only has off-diagonal components, so to be valued in the holomorphic tangent bundle, it has to be of type $(0,1)$. This means $\epsilon_1 \in \Lambda^{(0,1)}(T^{(1,0)}\Sigma_g)$, and, accordingly $\epsilon_2 \in \Lambda^{(1,0)}(T^{(0,1)}\Sigma_g)$. These tangent bundle valued forms are known as 'Beltrami differentials' in the literature and are written as

$$\begin{aligned} \mu &= \mu^z{}_{\bar{z}}(z)\partial_z d\bar{z}, \\ \bar{\mu} &= \mu^{\bar{z}}{}_z(\bar{z})\partial_{\bar{z}} dz. \end{aligned} \tag{5.4}$$

As $\bar{\partial}\epsilon_1 = 0$ and $\partial\epsilon_2 = 0$, we have found that deformations of the complex structure are classified by $H^{(0,1)}(\Sigma_g, T_{\mathrm{hol}}\Sigma_g)$ and $H^{(1,0)}(\Sigma_g, T_{\text{anti-hol}}\Sigma_g)$. The dimensions of these spaces is of course the same, so in order to calculate the dimension of our moduli space, we can restrict to the holomorphic one. We can write $H^{(0,1)}(T^{(1,0)}\Sigma_g) = H^1_{\bar{\partial}}(T^{(1,0)}\Sigma_g) = H^1(T\Sigma_g)$, where the first equality is mere notation and the second equality follows from the Čech-Dolbeault isomorphism.

We have found $T\mathcal{M}_g = H^1(T\Sigma_g)$. This allows us to calculate the dimension of the moduli space $\dim(\mathcal{M}_g) =$

$\dim T\mathcal{M}_g = \dim H^1(T\Sigma_g)$ with the Riemann-Roch formula. Note

$$\begin{aligned}
\chi(\Sigma_g, T\Sigma_g) &= \dim H^0(T\Sigma_g) - \dim H^1(T\Sigma_g), \\
&= \int_{\Sigma_g} \mathrm{ch}(T\Sigma_g)\,\mathrm{td}(T\Sigma_g), \\
&= \int_{\Sigma_g} (\dim_{\mathbb{C}} T\Sigma_g + c_1(T\Sigma_g))(1 + \frac{c_1(T\Sigma_g)}{2}), \\
&= \frac{3}{2}\int_{\Sigma_g} c_1(\Sigma_g) = \frac{3}{2}\chi(\Sigma_g) = 3(1-g).
\end{aligned} \tag{5.5}$$

In the generic setting when $\dim H^0(T\Sigma_g) = 0$ – which is true for higher genus Riemann surfaces, $g \geq 2$, one finds $\dim(\mathcal{M}_g) = 3(g-1)$. Let us quickly discuss the two special cases, the sphere and the torus. For $g = 0$, one finds $\dim H^0(T\Sigma_0) = 3$, hence $\dim(\mathcal{M}_0) = 0$ (and not 3 as one might expect). For the torus, one finds $\dim H^0(T\Sigma_1) = 1$, which yields $\dim(\mathcal{M}_1) = 1$ (this result will be known to the reader from an introduction to string theory; of course this is the modular (complex structure) parameter τ of a torus (regulating the corresponding rectangle's shape, whereas the size of the torus is regulated by the Kähler parameter).
The fact that we have found the dimension of our moduli space to generally be $3(g-1)$ will have an important implication. Namely, it will select the 'relevant' dimension for TST on a CY to be three complex dimensions[1], which of course seems extremely convenient.
For a genus g Riemann surface Σ_g, let us denote the basis for the tangent space to the moduli space $\mathcal{M}_g$ by $(\mu_i)^z_{\bar{z}}$ and $(\bar{\mu}_{\bar{j}})^{\bar{z}}_z$ $(i, \bar{j} = 1, ..., 3(1-g))$. These are also familiar from bosonic ST and are called Beltrami differentials. The crucial point now is that each of these Beltrami differentials can be contracted with an operator of axial ghost number minus one – a fact familiar from the quantization of the bosonic string.
This is the worldsheet partner of the energy-momentum tensor,

$$T = \{Q, G\}, \tag{5.6}$$

so (as the energy-momentum tensor is traceless, one can write $G = \begin{pmatrix} G_{zz} & 0 \\ 0 & G_{\bar{z}\bar{z}} \end{pmatrix}$). One can (for a genus g surface Σ_g) form

$$\begin{aligned}
\beta_i &= \int d^2z\, G_{zz} (\mu_i)_{\bar{z}}{}^z, \\
\beta_{\bar{j}} &= \int d^2z\, G_{\bar{z}\bar{z}} (\mu_{\bar{j}})_z{}^{\bar{z}},
\end{aligned} \tag{5.7}$$

for $i, \bar{j} = 1, ..., 3g-3$. These are used to define a measure with the desired charge on the moduli space $\mathcal{M}_g$. Note that the product of all the constructed expressions has axial charge $6(1-g)$). As each of the μ's can be seen as a basis vector in the tangent space, it is clear that a measure will be formed with their dual one-forms, which will be denoted by dm_i and $dm_{\bar{j}}$.
Now we are ready to define the TS free energy (one gets a partition function for $g \geq 2$):

$$F_g = \int_{\mathcal{M}_g} \prod_{i,\bar{j}=1}^{3g-3} dm_i dm_{\bar{j}} \langle \beta_i \beta_{\bar{j}} \rangle. \tag{5.8}$$

Remember, that on a torus, we only have one Beltrami differential of each type and can make an insertion of axial ghost charge $-1 - 1 = -2$ (one minus coming from the β and one minus coming from the $\bar{\beta}$.) Our selection rule tells us that we still need to make an insertion of total form-type $(1, 1)$. On the sphere, we have no available Beltrami differentials, and we need to make an insertion of type $(3, 3)$.

[1]There is no critical dimension in the sense of ordinary string theory as one is not troubled with the issue of a conformal anomaly when dealing with our 'twisted' Virasoro algebra.

This allows us to formally write the topological string partition sum as a sum over the genus (in analogy to ordinary ST, one can weigh them by an appropriate coupling constant, called the topological string coupling λ),

$$F_{\text{top}} = \sum_g \lambda^{2g-2} F_g, \tag{5.9}$$

to which we will come back to in a minute.
As we have seen, the twisted topological σ model has a very similar structure to the bosonic string, and in fact, one can apply BRST quantization to the topological sigma model coupled to gravity in analogous fashion [2]. The field χ plays the role of the ghost c, G plays the role of the anti-ghost b, and the supersymmetry operator plays the role of the BRST operator. We will refrain from copying material on these topics from other sources, where the topic is well explained. The reader interested in studying the formulation of TST as a twisted $c = 0$ CFT can have a look at [3].

5.2 Asymptotic expansions of the topological string

In the large radius limit (A-model), or in the large complex structure limit (B-model), the topological string free energy on a Calabi-Yau X can be written as a perturbative power expansion in the topological string coupling λ as $F_{\text{top}} = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$. The contributions F_g are understood to arise from different worldsheet genera in analogy to ordinary string theory. As discussed earlier, the correlation functions for the B-model are related to constant maps (as worldsheet embeddings), whereas in the A-model case the embedding maps are required to be holomorphic; such maps are called worldsheet instantons.
For the A-model, one can write the topological string free energy as

$$F_{\text{top}} = \frac{1}{6\lambda^2} D_{ABC} t^A t^B t^C - \frac{1}{24} c_{2A} t^A + F_{\text{non-pert}}. \tag{5.10}$$

The D_{ABC}s are the triple intersection numbers $\int_X J_A J_B J_C$, $c_{2A} = \int_X J_A \cdot c_2(T^{(1,0)}X)$, t_A denote the complexified Kähler moduli of X, λ is the topological string coupling constant. The non-perturbative (in t) part $F_{\text{non-pol}}$ is either given in the Gromov-Witten form, F_{GW}, the Gopakumar-Vafa form, F_{GV}, or the Donaldson-Thomas form, F_{DT}. Note that it is non-perturbative in α and also, that the contributions depend on the CYs (complexified) Kähler moduli (in the exponent), but not on the complex structure moduli of X. The first term in F_{top} comes from the genus zero, constant map ($\beta = 0$) contribution and can be interpreted as the zero-instanton sector, determined by classical intersection numbers on X, which are proportional to the volume of the CY. The topological string partition function reads $\mathcal{Z}_{\text{top}} = e^{F_{\text{top}}}$. In the following, we briefly introduce the reader to these three asymptotic expansions and the different corresponding sets of topological invariants. We start off by familiarizing the reader with the notion of asymptotic series
It is common in theoretical physics that exact solutions to a problem cannot be found. The promiment method of perturbation theory often offers an option to approximate the solution of interest in given background limits b close to b_0 (weak/strong coupling, near a saddle point etc.). However, it may well occur, that the perturbation series approximating the solution $A(b)$ (e.g. value of an integral or solution to a non-linear differential equation) in some small perturbation order parameter ϵ such as α in QED,

$$A(b) = A(b_0 + \epsilon) \equiv A_0(b_0) + \epsilon A_1(b_0) + \epsilon^2 A_2(b_0) + ... \tag{5.11}$$

is in fact divergent, $\lim_{N\to\infty} \sum_{k=0}^{N} \epsilon^k A_k \to \infty$, almost everywhere, even close to the background b_0. Still, an asymptotic series can be an extremely useful tool, as decades of quantum physics have shown. The divergence of the series might be attributed to the fact that the perturbative expansion misses certain non-perturbative physical effects. It is sometimes possible that one finds a 'non-perturbative completion' of the series that renders it finite.
This leads to the notion of asymptotic series as opposed to convergent series. Let us compare the two different definitions – we will briefly discuss them in the simplest setting: on the space of real functions $C(\mathbb{R})$

[2] One can state that the twisted $\mathcal{N} = (2, 2)$ algebra is isomorphic to the (BRST enlarged) algebra of the bosonoic string.

in one variable.
First, the **convergent series**: Recall that near a point $x = a$ within a radius of convergence r, one calls a power series $\sum_{n=1} c_n(x-a)^n$ a uniformly convergent approximation to the function $f(x)$, if (for $|x-a| < r$)

$$\lim_{N\to\infty}[f(x) - \sum_{n=0}^{N} c_n(x-a)^n] = 0. \tag{5.12}$$

One can express this (by Taylor's theorem) as

$$f(x) = f(a) + \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x), \tag{5.13}$$

where the rest R_N satisfies $\lim_{N\to\infty} R_N(x) = 0$.
On the other hand, an **asymptotic series** relies only on the behaviour of the rest R_N as one approaches the point $x = a$. This means that one calls $\sum_{n=0}^{N} c_n(x-a)^n$ an asymptotic approximation to the function $f(x)$ near $x = a$ if for *any fixed* N, the rest R_N dissapears faster than $(x-a)^N$ in the limit $x \to a$:

$$\lim_{x\to a}\frac{1}{(x-a)^N}[f(x) - \sum_{n=0}^{N} c_n(x-a)^n] = 0. \tag{5.14}$$

This can alternatively be expressed as $R_N(x) = \mathcal{O}(((x-a)^{N+1}))$, where the $\mathcal{O}$ is the big 'order symbol'. One writes

$$f(x) = \mathcal{O}(g(x)), \tag{5.15}$$

which means that $f(x)$ is *of order* $g(x)$: There exists some constant C such that $|f(x)| \le C|g(x)|$.
Now let us discuss the utility of an asymptotic series. Choosing a certain valueof interest, $x = b$ (somewhere 'near' $x = a$), one might like to find a good approximation of $f(b)$. What one will find is that the partial sums $\sum_{n=0}^{N} c_n(b-a)^n$ will seem to converge towards $f(b)$, successively as one includes more and more power terms up to some 'optimal' upper limit N_{max}, and then start to diverge again (thus an important point in this context is to find out where one should optimally cut off the series to have as good an approximation as possible near the point b). The crux of the matter is, that one can easily find that $\sum_{n=0}^{\infty} c_n(b-a)^n \to \infty$, but for any b one will be able to find an N_{max}. Typically, the closer one gets to a the higher it will be. Or, put differently, no matter how many terms one chooses to include, one will always be able to get close enough to a such that the series becomes a good approximation for values within that bound. For the sake of illustration, let us include one example to illustrate these concepts and hopefully make them completely clear.
Let us set $f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}dt$ and check that

$$\sum_{n=0}^{\infty}(-1)^n n! x^n \tag{5.16}$$

is an asymptotic series of $f(x)$ as $x \to 0^+$. The plus next to the zero indicates that this statement holds when approaching zero from the positive side. As a side remark: It is worthwile to spend some fun time experimenting with the partial sums of this series numerically with a program such as Mathematica!
Using the integral representation of the factorial $n! = \Gamma(n+1) = \int_0^\infty e^{-t}t^n dt$ one can write the partial sums S_N as

$$\int_0^\infty e^{-t}(\sum_{n=0}^{N}(-tx)^n)dt = f(x) - (-x)^{N+1}\int_0^\infty \frac{e^{-t}t^{N+1}}{1+xt}dt. \tag{5.17}$$

As $|\int_0^\infty \frac{e^{-t}t^{N+1}}{1+xt}dt| < |\int_0^\infty e^{-t}t^{N+1}dt| = |(N+1)!|$, one can read off

$$|f(x) - S_N(x)| = |R_N| < |x^{N+1}||\int_0^\infty e^{-t}t^{N+1}dt| = |x|^{N+1}(N+1)!, \tag{5.18}$$

which indeed shows that $R_N(x) = \mathcal{O}(x^{N+1})$, or

$$\lim_{x\to 0}(\frac{f(x) - S_N(x)}{x^N}) = 0. \tag{5.19}$$

Note that, for a given asymptotic series, the function which it approximates is not uniquely defined, as there are functions whose asymptotic series equal zero. For example, the function e^{-1/x^2} is asymptotic to zero as $x \to 0$:

$$\lim_{x\to 0}\frac{e^{-1/x^2} - 0}{x^N} = 0. \tag{5.20}$$

Had we been given the asymptotic series $\sum_{n=0}^{\infty}(-1)^n n! x^n$, we wouldn't have known whether we were approximating $f(x) = \int_0^\infty e^{-t}t^n dt$ or $f(x) + e^{-1/x^2}$!
Let us now turn back to topological strings. Our function $f(x)$ is assumed to be the full non-perturbatively complete free energy F_{top}, and we are going to discuss asymptotic expansions of it. Note that as one only knows asymptotic expansions (and maybe the expansion points in the CY moduli space), one still does not know which function one is approximating – this function would be the complete free energy of the topological string!

5.3 Gromov-Witten invariants

The first asymptotic form of the topological string is the Gromov-Witten expansion. It looks like

$$F_{GW}(\lambda, q) = \sum_{g\geq 0}\sum_{\beta\in H_2^+(X)} N_g^\beta q^\beta \lambda^{2g-2} \tag{5.21}$$

and is interpreted as an asymptotic expansion of the free energy in the string coupling λ (a good approximation in the large radius limit); a sum over every worldsheet genus, in analogy to ordinary string theory, and a sum over 'positive' elements of the second cohomology $H_+^2(X)$ (cycles with a positive coefficient). It is a sum over worldsheet instantons with amplitudes

$$q^\beta := e^{2\pi i \beta_A t^A}. \tag{5.22}$$

Roughly, the Gromov-Witten expansion of F_{top} receives contributions from Riemann surfaces of genus g embedded in the CY X, each of them weighed by its complex spacetime image area $e^A = q$, and multiplied with the number of existing maps ϕ from a given worldsheet Σ_g of genus g to a curve $\phi(\Sigma_g) \subset X$ in the (integral) spacetime homology class β, counted by the Gromov invariants N_g^β.
These have conjectured to be rational (in general). These are an example of a set of topological invariants of the target space X, which are calculated by the A-model topological string theory. They are the observables of a mathematical version of TST, Gromov-Witten theory, have been heavily studied within the framework of Algebraic Geometry, and have been rigorously defined.
For now, let us content ourselves by noting that they are defined via virtual fundamental classes [3] on the moduli space $\mathcal{M}_{g,n}(X, \beta)$. It is a generalization of the famous Deligne-Mumford space $\mathcal{M}_{g,n}$ of n-punctured stable maps [4]. The space $\mathcal{M}_{g,n}(X, \beta)$ is the moduli space of holomorphic maps ϕ from a worldsheet Σ_g of genus g (Riemann surface) with n punctures and a choice of complex structure (metric) such that $\phi_*[\Sigma_g] = \beta$ – it has to map to worldsheet into a curve in X of homology class β. Let us call this the moduli space of TST. Note that – in line with our earlier discussions – its virtual dimension can be found to be

$$\dim(\mathcal{M}_{g,n}(X, \beta)) = d(1-g) - 3(1-g) + n + \int_{\Sigma_g}\phi^*(c_1(X).) \tag{5.23}$$

[3] A fundamental class of an (orientable) manifold is a homology class in the top integral homology which is given by a specfic orientation, giving meaning to integration over the manifold.

[4] To be more precise, one has to consider a compactification of this space, to render it irreducible, compact, connected and non-singular: this construction is called the Deligne-Mumford stack $\bar{\mathcal{M}}_{g,n}$

As for TFT, let us define the evaluation maps for each insertion point x_i on this moduli space into the target space (which we choose to be the CY X) as

$$\mathrm{ev}_i : \mathcal{M}_{g,n}(X,\beta) \to X, \quad \mathrm{ev}_i(\phi) := \phi(x_i), \tag{5.24}$$

meaning it evaluates the worldsheet embedding at the insertion point x_i. Again, imagine that for every insertion point, one has an operator corresponding to a form ω_i in $H^*(X)$. One again needs to respect the selection rule, for the total form degree to match the dimension of our moduli space, and then one define an integration on the moduli space, which will give rise to general Gromov-Witten invariants for any genus,

$$N^{\beta}_{g,n} = \int_{\mathcal{M}_{g,n}(X,\beta)} \mathrm{ev}^*(\omega_1 \otimes ... \otimes \omega_n), \tag{5.25}$$

counting the number of maps from a genus g surface (including allowing its metric / complex structure to vary), n punctures into curve class β on X. Note that when studying a topological sigma-model on a CY X, one already found the Gromov-Witten invariants $N^{\beta}_{0,3}$. For genus $g \geq 2$, the selection rule does not require any specific insertion (because one has compensated the positive axial $U(1)$-charge by inserting the negatively charged G-operators), one can also set $n = 0$ (in fact, for a CY, considering our local operators, this is the only thing one can do to obtain something non-vanishing), and one gets a whole series of Gromov-Witten invariants $N^{\beta}_{g,0} = N^{\beta}_{g}$. These, together with $N^{\beta}_{0,3} = N^{\beta}_{0}$ and $N^{\beta}_{1,1} = N^{\beta}_{1}$ are the ones appearing in our first asymptotic expansion and are often referred to as *the* Gromov-Witten invariants of X at genus g and of class β.
Let us structure the Gromov-Witten series into contributions from constant maps ($\beta = 0$) at genus zero, genus one and at higher genus, and a non-constant map part ($\beta \neq 0$ – sometimes also called the reduced Gromov-Witten potential):

$$F_{GW} = \frac{1}{\lambda^2} F^{\beta=0}_0 + F^{\beta=0}_1 + \sum_{g \geq 2} \lambda^{2g-2} F^{\beta=0}_g + \sum_{g \geq 0} \sum_{\beta \neq 0} \lambda^{2g-2} F^{\beta}_g. \tag{5.26}$$

The constant map contributions – the ones where the whole genus g Riemann surface is mapped to a point in the CY – are dominant when studying a CY at large volume. We have already found N^0_0 - it is determined by classical intersection theory on X and is just given (up to a constant) by the intersection number $D_{ABC} = \int_X J_A \wedge J_B \wedge J_C$.
At genus one, the contribution is obtained from a virtual class calculation. Furthermore, at every genus $g \geq 2$, the constant map contributions are completely known (check [4]):

$$F^{\beta=0}_g = N^0_g = \frac{(-1)^g \chi(X)}{2} \int_{\mathcal{M}_g} c^3_{g-1}, \tag{5.27}$$

where $\chi(X)$ is the Euler character of the CY, $\mathcal{M}_g$ is the moduli space of Riemann surfaces (complex curves) of genus g and c^3_{g-1} is the $(g-1)$th Chern class of the Hodge bundle over $\mathcal{M}_g$. These this type of Hodge integrals appearing have been solved (check [5]) and can be expressed as $\frac{|B_{2g}||B_{2g-2}|}{2g(2g-2)(2g-2)!}$ (the B_gs are Bernoulli numbers); The constant map contributions are therefore completely understood.

5.4 Gopakumar-Vafa invariants

Next, we would like to develop some intuition for another type of topological invariants that topological string theory calculates. These are the coefficients of another type of asymptotic expansion of the topological string free energy. They are a result of a reinterpretation of what TST computes. In order to explain this, we need to talk about one of the physical applications of TST.

When considering a compactification of type II string theory on a CY 3-fold, one obtains an $\mathcal{N}=2$ supergravity as an effective theory. For a IIA theory one has $h^{1,1}(X)$ vector- and $h^{2,1}(X)+1$ hyper-multiplets (the role of $h^{1,1}(X)$ and $h^{2,1}(X)$ is exchanged for type IIB) as well as one gravity multiplet. Each one of these multiplets contains a complex scalar. Due to the highly restrictive nature of $\mathcal{N}=2$ supersymmetry, it turns out that these two types of scalars (vector- and hyper-multiplets) decouple from in other in the so-called *F-terms*.

The (super)gravity multiplet contains a $U(1)$ gauge field known as the *graviphoton*, which will play an important role in this discussion.

The action of IIA(B) supergravity contains, of course, the Einstein-Hilbert term plus kinetic terms for the aforementioned scalars, but it also has corrections in α' which have higher powers of the Riemann tensor. These are know as *higher curvature corrections.* Due to the supersymmetry of the system, one can split up these terms into F-terms and D-terms. We will focus on the F-terms of the vector-multiplets, which are of the form

$$\int d^4x F_g(t^A) R_+^2 F_+^{2g-2}, \tag{5.28}$$

with R_+ and F_+ denoting the self-dual (i.e. $*F_+ = F_+$) parts of the 4d Riemann curvature and the graviphoton field strength. The t^A's correspond to the vector-multiplet complex scalars, which can be shown to parametrize the Kähler moduli space of the CY manifold on which the supergravity was compactified. The $F_g(t^A)$'s are the genus g amplitudes of the A-model topological string theory. These terms exist for every genus and can be interpreted as giving rise to the interaction between two gravitons and $2g-2$ graviphotons.

In 1998 Gopakumar and Vafa gave a reinterpretation of F_g inspired by these F-terms, which we will sketch. The reader is referred to the original papers [6] and [7] for more detail.

We begin by making the following crucial observation: The dilaton is a scalar field that belongs to a hyper-multiplet. As we noted before, hyper-multiplets and vector-multiplets are forbidden to couple through F-terms due to the $\mathcal{N}=2$ supersymmetry. Therefore, the terms in (5.28) must be independent of the string coupling g_s, which is dictated by the value of the dilation at spatial infinity. This is a non-renormalization theorem.

In our description of the topological string in the previous chapter, we worked in the limit $g_s \to 0$. The approach of Gopakumar and Vafa was to exploit the g_s-independence of the F-terms and compute the F_g in the strong string coupling regime. At weak coupling, the F_g's are interpreted as topological string correlation functions. At strong coupling, however, the fundamental degrees of freedom are no longer strings, but D-branes. This is not hard to see; the tension of a D-brane is inversely proportional to the string coupling, e.g. for the D0-brane, $T_{D0} = \frac{1}{g_s l_s}$, whereas for the fundamental string $T_{F1} = \frac{1}{2\pi l_s^2}$.

How can the F-terms be reinterpreted as effective terms for a theory of D-branes? If one considers D2-branes wrapped on two cycles of CY and D0-branes, both objects give rise to massive particles in four dimensions that couple to the graviton and the graviphoton fields. Since these particles are massive, one can compute a Wilsonian-like effective action where these are ‘integrated out’ and all that is left are new coupling terms for the graviphoton and the graviton. The calculations of Gopakumar and Vafa show that the F-terms in (5.28) are precisely the effective terms one gets after integrating out the effects of wrapped D-branes.

This idea was inspired by an old QED calculation by Schwinger, where he considered the theory of a charged scalar field ϕ of charge e in the presence of a background $U(1)$ field-strength, and integrated out the scalar:

$$e^{-S_{\text{eff}}[F]} = \int d[\phi] \exp\left(-\int |D\phi|^2 + m^2 |\phi|^2\right), \tag{5.29}$$

where D is the $U(1)$-covariant derivative, F is the field strength, and $S_{\text{eff}}[F]$ is the effective action for the gauge field. The result of Schwinger's calculation is non-perturbative in the mass of the scalar, and is of the following form:

$$S_{\text{eff}} = \frac{1}{4}\int_\epsilon^\infty \frac{ds}{s} \frac{e^{-sm^2}}{2\sinh(\frac{seF}{2})}, \tag{5.30}$$

where ϵ is a cutoff that can be interpreted as the cutoff for the effective action. This effective action takes into account effects due to internal propagators and loops of the scalar field in correlators with photon external legs.

D-branes whose spatial worldvolume is entirely wrapped in the CY manifold look like particles in four dimensions, but not necessarily scalars. Depending on how the branes are wrapped, they can give rise to different representations of the rotation group in the three external spatial dimensions. Gopakumar and Vafa repeated the Schwinger calculation for all possible particles arising form wrapped D-branes. However, to simplify their calculation and unify the contributions from different possible D-branes, they used the M-theory picture, which we now briefly describe.

Imagine that one starts out with a type IIA string theory on $\mathcal{M}^4 \times X$ with X denoting our CY 3-fold. This setting is dual to M-Theory on $\mathcal{M} \times X \times S^1$ at weak string coupling. Note that the radius of the eleventh M-Theory dimension S^1 is given by the string coupling, $R = l_s g_s$ (l_s denotes the string length). Fundamental strings of type IIA ST (in our case completely located in the Calabi-Yau manifold X) can be seen as M2-branes wrapped on the 11th M-Theory circle S^1 (effectively 'losing' one of their dimensions). M2-branes that are wrapped along the circle can have Kaluza-Klein excitations along the circle, which correspond to D0-branes in the IIA picture. M2-branes that are not wrapped along the S^1 correspond to D2-branes. The D0-branes in 4d spacetime as well as D2-branes wrapped on 2-cycles in $X \times S^1$ give rise to particles (and moreover, to bound state of particles, e.g. N D0-branes, corresponding to the Nth excitation) in 4d spacetime.

The contributions to an effective action obtained from integrating out the particles corresponding to D0- and D2-branes contain some changes and generalizations due to the (dimension and the) fact that one sums over particles in different representations, depending on the corresponding BPS multiplets. For the D0-branes (as particles in 4d spacetime in a BPS multiplet of central charge Z) in a constant self-dual graviphoton field F_+, GV found

$$\frac{1}{4}\int_\epsilon^\infty \frac{ds}{s\sinh^2(\frac{s}{2})} e^{-\frac{sZ}{F_+}}. \tag{5.31}$$

These sorts of terms appear with generalizations when the particles carry spin. D2 branes wrapped on various cycles give rise to particles in 4d with multiplets living in various representations of $SO(4) = SU(2)_L \times SU(2)_R$. As also the offshell particles contribute in a loop, it turns out to be useful to use an M-Theory perspective, as M2-brane on-shell amplitudes appear to include D2-brane off-shell amplitudes. All of these BPS particle contributes summed up and nicely arranged, GV end up with a formula

$$\sum_{h\geq 0}\sum_{\beta\in H_2^+(X)}\sum_{d\geq 1} n_{h,\beta}\frac{1}{d}[2\sin(\frac{d\lambda}{2})]^{2h-2} e^{2\pi i d\beta_A t^A}, \tag{5.32}$$

where they have identified a new type of BPS invariant.
Roughly, one might imagine numbers $n^\beta_{(j_L,j_R)}$ appearing in this formula, where $n^\beta_{(j_L,j_R)}$ counts the number of BPS states of M2-branes wrapping a curve (two cycle) of class β in the Calabi-Yau X corresponding to particles in a (j_L, j_R)-representation of $SO(4)$. It turns out that these numbers are sensitive to changes in the theories parameters, e.g. the complex structure moduli – they will not serve as BPS invariants.
On the other hand, the numbers

$$n^\beta_{j_L} = \sum_{j_R}(-1)^{2j_R}(2j_R+1)n^\beta_{(j_L,j_R)} \tag{5.33}$$

are invariant. These are related to the Gopakumar-Vafa invariants by

$$n^\beta_{j_L} = \sum_{h\geq|2j_L|}\binom{2\beta}{h+2j_L} n_{h,\beta}. \tag{5.34}$$

The proposal of Gopakumar and Vafa is to resum the non-perturbative part of the topological string free energy as follows

$$F_{GV} = \sum_{h\geq 0}\sum_{\beta\in H_2^+(X)}\sum_{d\geq 1} n_{h,\beta}\frac{1}{d}[2\sin(\frac{d\lambda}{2})]^{2h-2} e^{2\pi i d\beta_A t^A}. \tag{5.35}$$

The GV invariants $n_{h,\beta}$ have been conjectured to be integers, whereas the GW invariants are known not to be integral. Hence, this is more than the mere rewriting of the partition function, the GV invariants have, in some sense, enhanced the geometric interpretation of topological string theory.
Note that the summation index h does not correspond to the genus of Riemann surfaces, but that a given GV invariant can contribute to all orders in the topological string expansion (quite unlike a GW invariant). One can see this nicely when looking at the $\beta = 0$ contribution

$$F_{GV}^{\beta=0} = n_{0,0} \sum_{d \geq 1} \frac{1}{d(2\sin(\frac{d\lambda}{2}))^2}. \tag{5.36}$$

Note that the GV invariants vanish for $h > \beta$.
One can rewrite the $\beta = 0$ contribution and one sees that it converges:

$$F_{GV}^{\beta=0} = n_{0,0} \ln \prod_{k=1}^{\infty} (1 - e^{\pm i\lambda k}) \tag{5.37}$$

where the plus sign in the exponent corresponds to $\mathrm{Im}(\lambda) > 0$ whereas the minus sign applies for $\mathrm{Im}(\lambda) < 0$. One calls this the McMahon form as it can be nicely written using the McMahon function $M(q) = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}$: $F_{GV}^{\beta=0} = -n_{0,0} \cdot \ln M(e^{\pm i\lambda k})$. Using $n_{0,0} = -\frac{1}{2}\chi(X)$ it can be neatly written as

$$F_{GV}^{\beta=0} = (\ln[M(e^{\pm i\lambda k})])^{\chi(X)/2}. \tag{5.38}$$

It can be seen as a non-perturbative (in the topological string coupling) completion of the $\beta = 0$ (constant map) sum in the GW form (which diverges!).
It can also be shown that the $h = 0$ as well as the $h = 1$ part (with arbitrary curve classes β allowed) converge at least within certain regions of the topological string coupling, but there are strong indications that the $h > 1$ part of the partition function diverges.
One can rewrite the $\beta \neq 0$ in a similar manner, so, for convenience, let us write down the GV partition function in its new form, split up into three contributions:

$$\mathcal{Z}_{\mathrm{GV}} = e^{F_{GV}} = \mathcal{Z}_{\mathrm{GV}}^{\beta=0} \cdot \mathcal{Z}_{GV}^{',h=0} \cdot \mathcal{Z}_{GV}^{',h>0}, \tag{5.39}$$

where

$$\begin{aligned}
\mathcal{Z}_{\mathrm{GV}}^{\beta=0} &= \prod_{k=1}^{\infty} (1 - e^{\pm \lambda k})^{k \cdot n_{0,0}}, \\
\mathcal{Z}_{GV}^{',h=0} &= \prod_{k \geq 1, \beta > 0} (1 - e^{\pm i\lambda k} \cdot q^{\beta})^{k \cdot n_{0,\beta}} \qquad (5.40) \\
\mathcal{Z}_{GV}^{',h>0} &= \prod_{h \geq 1, \beta > 0} \prod_{l=0}^{2\beta-2} (1 - e^{\pm i\lambda(h-l-1)} \cdot q^{\beta})^{(-1)^{h+l}\binom{2h-2}{l} n_{h,\beta}},
\end{aligned}$$

(5.41)

with $q^{\beta} = e^{2\pi i \beta_A t^A}$.
To summarize, let us note that the Gopakumar-Vafa resummation of the topological string free energy highlights that the topological string counts degeneracies of certain BPS states, namely a gas of spinning M2-branes (they have angular momemtum) wrapping 2-cycles in the CY. Therefore, they can be used to count 5-d black hole degeneracies.

5.5 Donaldson-Thomas invariants

This brings us to the discussion of the third asymptotic expansion of the (A-model) topological free energy. It is called the Donaldson-Thomas form, as it shows that the topological string can be interpreted as computing yet another set of topological invariants of the Calabi-Yau X, namely the Donaldson-Thomas invariants. These were originally discovered in a mathematical setting by Donaldson and Thomas, and were at first conjectured to be related to the Gromov-Witten invariants, [8] and [9]: $F_{\text{GW}} = F_{\text{DT}}$. Mathematically, they can be rigorously formulated as invariants counting ideal sheafs, however we will be more interested in their physical interpretation. It is the relation between the GV and the DT form which we will discuss in the following, discovered in [10].

For a CY manifold X one can interpret a Donaldson-Thomas invariant $N_{\text{DT}}(\beta, n)$ as counting the number of D6-D2-D0 BPS bound states of a D6 wrapping the CY, with the D2-brane wrapped on a cycle of homology class $\beta \in H_2(X, \mathbb{Z})$ and with n D0-branes added. The 6+1 dimensional world-volume gauge theory of this D6-brane reduces to a 6d theory, when the brane is wrapped on $X \times S^1$ with the S^1 interpreted as a Euclidean time circle, in the limit that the radius of this circle goes to zero.

The DT invariants arise as Euler characters counting the number of BPS states of D2-D0-branes in the 6d (topologically twisted) $U(N)$ worldvolume gauge theory on the D6 brane wrapping X. Note that the N in $U(N)$ is the number of D6-branes present.

Keeping this in mind, one can view the GW-DT 'correspondence' as an open-closed duality, relating a theory of closed strings, given by the GW interpretation, to a theory of open strings on a D-brane, at low-energy described by a worldvolume gauge theory leading to the DT interpretation of the topological string. However, as mentioned before, we will now focus on the GV-DT correspondence.

We start off with a system of D-branes on $\mathbb{R}^3 \times S^1_\tau \times X$, where the S^1_τ is the Euclidean time circle. Then, we uplift this system to M-theory in eleven dimensions. This brings us naïvely to $S^1_M \times \mathbb{R}^3 \times S^1_\tau \times X$, where S^1_M is the M-theory circle. This lift has been used to connect BPS black holes in 4 and 5 dimension in [11]. One finds that the D2- and D0-branes get mapped to a system of M2 branes with KK momentum along S^1_M. The effect of uplifting the D6-brane is to 'twist' the geometry by turning the $S^1_M \times \mathbb{R}^3$ factor into a non-trivial fibration called *Taub-NUT space*. This means that now the eleven-dimensional geometry is $\text{TN}_4 \times S^1_\tau \times X$, where the TN_4 is a space that locally looks like $S^1_M \times \mathbb{R}^3$, but is globally different from a product space (and is also not an S^1-bundle over $\mathbb{R}^3$). There is an appropriate generalization of this background, called Multi-Taub-NUT, if one lifts several D6-branes. While we will not get into the issue of adding D4-branes (or M5-branes in the M-Theory picture), the reader can rest assured that this will not fundamentally affect the arguments that will follow and the resulting modifications can be incorporated. The Taub-NUT metric looks like

$$ds^2_{\text{TN}} = R^2[(1 + \frac{1}{r})^{-1}(d\chi + \vec{A}d\vec{r})^2 + (1 + \frac{1}{r})dr^2] \tag{5.42}$$

with $\chi \in S^1$ and $\vec{r} \in \mathbb{R}^3$, so it can be seen as a fibration of a circle S^1 over $\mathbb{R}^3$ with the radius of the S^1 disapearing at the center $r = 0$ of $\mathbb{R}^3$ and attaining a finite value R at infinity ($r \to \infty$).

Note that there are now *two* different circles in our eleven-dimensional geometry. We can write the full 11-d space as follows:

$$\text{TN}_4 \times S^1_\tau \times X = \mathbb{R}^3 \bowtie S^1_{\text{M}} \times S^1_\tau \times X, \tag{5.43}$$

where the $\bowtie$ should not be interpreted as a simple product, but rather as a fibration. The eleventh M-theory direction which appears after doing our lift is the S^1_{M} circle. However, we could make a paradigm shift, by taking this eleven-dimensional geometry and interpreting the S^1_τ circle as the new M-theory circle, and S^1_M as the new Euclidean time circle. The fact that one can switch which one of the two circles one interprets as M-theory circle and which one as the Euclidean time circle allows for a beautiful observation: One interpretation leads to the GV picture of the TS, the other leads to the DT interpretation!

Interpreting the S^1_{M} as the M-Theory circle and shrinking the size to describe the system in type IIA ST, leads back to the DT interpretation. Here, S^1_τ serves as the Euclidean time circle. This is simply undoing the lift we just described. However, choosing the S^1_τ as the M-Theory circle and shrinking it, effectively leads to a IIA ST description on $\text{TN} \times X$, leading to the GV interpretation. For a justification of this statement, check [10]. In this case the S^1_{M} can be seen as the new Euclidean time.

This exchange of the 11th M-Theory dimension can also be entirely seen as a duality transformation within the framework of type IIA ST, namely by performing the sequence of a T, an S and a T duality. This operation is also known as the '$9 - 11$' flip in the literature, as it is associated to 'switching' the role of the 9th and the 11th dimension, interchanging the role of M-Theory circle and thermal time circle. Starting

from the D-brane, or 'DT' side, one can note the following transitions of interest. One starts off with type IIA ST on $\mathbb{R}^3 \times S^1 \times X$, performs a T duality, leading to type IIB ST on $\mathbb{R}^3 \times S^1 \times X$ at string coupling g_s. The following S-duality leads to type IIB ST at coupling $\frac{1}{g_S}$. The transversal T-duality along another circle leads to type IIA ST on TN $\times X$. Under this sequence of dualities, the D6-brane gets mapped to a (Euclidean) D5, to an NS5 and eventually to a Taub-NUT background.
So, what does the new TST free energy – or partition function – look like? The DT partition function, $\mathcal{Z}_{\mathrm{DT}} = e^{F_{\mathrm{DT}}}$ can be written as a generating function. It reads as follows:

$$\mathcal{Z}_{DT}(u,v) = \sum_{\beta,n} N_{\mathrm{DT}}(\beta,n) u^n v^\beta. \tag{5.44}$$

Making the identification of variables $u = e^{\pm i\lambda}$ and $v = e^{2\pi i t}$, this form has been conjectured to be related to the GV form (and to the GW form through the identification made earlier) via

$$\begin{aligned} \mathcal{Z}_{DT}^{\beta=0} &= (\mathcal{Z}_{GV}^{\beta=0})^2, \\ \mathcal{Z}_{DT}^{'}(u,v) &= \mathcal{Z}_{GV}^{'}(-u,v). \end{aligned} \tag{5.45}$$

The MacMahon form of the GV partition function thus is already very suggestive of an D-brane world-volume gauge theory interpretaion. By the suggested identification one can for instance read off the $\beta = 0$ part to be

$$\mathcal{Z}_{DT}^{\beta=0} = \sum_n N_{DT}(0,n) u^n = \prod_{k=1}^{\infty} (1 - (-e^{\pm i\lambda k}))^{-k\chi(X)}. \tag{5.46}$$

This should count D6-D0 states, and indeed it can be retrieved by counting D0 particles in a D6 background. Similarly, it can be shown how to reproduce the form of $\mathcal{Z}_{DT}^{'}$ by counting D6-D2-D0 BPS states, check [12] and [13].
We close by pointing out that the relation of the GW, the GV and the DT interpretations of topological string theory and their application to microstate counting of BPS black holes is an current topic of research. The reader is invited to pick up the thread of the relation of topological strings and black holes with the OSV conjecture, [14] (named after Ooguri, Strominger and Vafa). Finally, we provide a list of useful references to continue studying this topic.
First of all, there is a recent introduction to TST by Marcel Vonk, [15]. It is quite basic and explains a lot of the technical background. For more advanced reading material, there are two reviews by Marcos Mariño [16] and [17], one on the A-model, one on the B-model, both of them discussing a duality between closed topological strings and open topological strings and topological string field theory (Chern-Simons Theory and Matrix Theory). There is also a review containing a lot of information on physical applications by A. Neitzke and C. Vafa, [18]. Another set of lecture notes is provided by B. Pioline [19] that discusses OSV, and related topics. Last but not least, the big book (ca. 1000 pages!) on mirror symmetry called 'Mirror Symmetry' [2], which provides the reader with a nearly complete education on geometry, quantum field theory, supersymmetric QFT, string theory, also gives a detailled technical account of TST.

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