D-Branes, Sheaves and Categories

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Abstract

These notes are based on my lectures at the Third Modave Summer School in Mathematical Physics and are a pedestrian introduction to sheaf-theoretic models for D-branes. After motivating the necessity for an abstract description of D-branes and setting up some basic mathematical background, I review the topological B-model on Calabi-Yau manifolds as a toy model where such a description can be made fairly explicit. From there, the primary focus of these lectures is in tracing the sequence of physical arguments that leads one to a picture of B-model D-branes in terms of derived categories. Some of the necessary background in mathematics is introduced along the way.

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1 Why Sheaves and Categories?

Perturbative string theory is defined in terms of maps of the worldsheet to a target space, and the images of the boundaries of the worldsheet are supposed to live on D-branes. When the target is nearly flat spacetime, and the length-scales involved are large so that we may ignore the α' corrections, D-branes have a straightforward interpretation as the subspaces on which open strings can end. In such a situation, we can work with a supergravity description on the target space and put in D-branes by hand. But in general, if we believe that strings are more fundamental than spacetime, then we need a more general mathematical formalism where this geometric picture is an emergent feature of the theory at large radius rather than a fundamental notion. This is one reason to study D-branes in a more abstract setting. The language of sheaves and categories that seem to be relevant for this description, ultimately trace their origins back to mathematical structures that arise in the conformal field theory of the open string worldsheet.

Once we understand the mathematical structures that are necessary for the description of D-branes, the hope is that questions that are otherwise formidable or impossible might get stripped off of their irrelevant details, and computations might have a chance of succeeding. In fact, this philosophy works both ways across the physics-mathematics divide: the Homological Mirror Symmetry conjecture of Kontsevich takes on a much more concrete form when it is re-interpreted in terms of D-branes in topological string theory. As it turns out, one of the premises for the belief that a sheaf-theoretic model for D-branes is true outside of toy models where explicit checks are possible, is mirror symmetry.

Another reason to study the mathematics underlying branes is the hope that this understanding will somehow be useful in finding a full definition of string theory itself, starting from some basic principle. A historical precedent is provided by Riemannian geometry in the context of General Relativity. At present, we don't have an analogous geometrical(?) structure that encodes the flavor of string theory. Of course, Einstein came to General Relativity not through geometry, but through a far more intuitive and physical path based on the Principle of Equivalence. But since we have not yet managed to understand the physical foundations for string theory, it might be a reasonable exercise to absorb all the hints that we can, from the mathematics. And branes, being non-perturbative objects, are a natural place to start.

One possible way to study D-branes is to look at target spaces which have an exact conformal field theory description on the worldsheet. The general philosophy of this approach is to think of D-branes as boundary states (see [1] for an introduction to D-brane boundary states). The idea is that open string loops with D-brane boundary conditions can be thought of as tree-level closed string propagators between those boundary states. That is, figure 1 could be interpreted as either the left hand side or the right hand side of

$$\langle B|\text{Closed string propagator}|B\rangle = \text{Open string vacuum loop}, \tag{1.1}$$

and by calculating the r.h.s and knowing the closed string propagator, we can compute the D-brane boundary state $|B\rangle$. One problem with this approach is of course that situations where exact string backgrounds are known are only a handful. Another disadvantage is that the geometric nature of D-branes and their moduli spaces is obscured by the boundary state approach. The sheaf theoretic

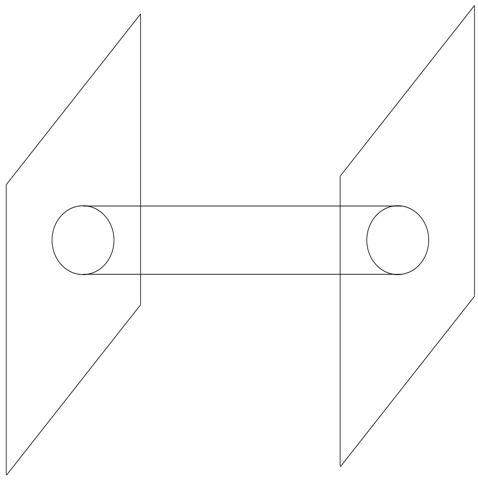

Figure 1: An open string loop stretched between two D-branes can be thought of as a closed string propagating between boundary states.

approach, which tries to get a handle on D-branes by looking at *generic* properties of CFTs as opposed to *specific* string backgrounds, provides us with a more abstract handle on the problem where the geometric/non-geometric origins are manifest throughout.

As an aside, we mention that the description of branes through sheaves is more powerful than their description in terms of K-theory, which, cavalierly speaking, is nothing more than a book-keeping device in terms of RR-charges and their conservation. The categorical picture contains information about the location of branes and other moduli, whereas K-theory is blind to such details. In other words, K-theory cannot (for example) distinguish between a system containing nothing from a system containing a brane and an anti-brane even when they are separated, but categories (in principle) can.

The task of constructing a fully consistent mathematical model for generic branes in the full string theory is a Herculean one: it lies right smack in the middle of all the usual things that make non-perturbative string theory (and field theory) hard. Instead, in these lectures we will be working with branes in topological string theory. So among other things, the worldsheet has $\mathcal{N} = (2,2)$ supersymmetry, the target space is Calabi-Yau, and the Hilbert space is finite dimensional. Even more drastically, we will also tune the string coupling g_s to zero. So we are living in the world of conformal field theory on a fixed Riemann surfface. These things together make the problem more tractable while still retaining many of the essentially stringy features like mirror symmetry.

As we will explore in detail in these lectures, it has turned out that in topological string theories, it is reasonable to think of D-branes in terms of categories, sheaves and related objects in algebraic geometry. Sheaves can work as models for D-branes, and certain groups connecting pairs of sheaves have a natural interpretation in terms of the open strings that connect D-branes. The claim is ultimately that B-type branes on a Calabi-Yau manifold X can be described by the "derived category of coherent sheaves" on X. The purpose of these lectures is to flesh out these words. For the most part, we will focus on B-model topological string theory, and explore the mathematics and physics of B-branes[1]. Along the way, I will develop the various cohomology theories and the necessary algebraic geometry that we will need.

A generic familiarity with the philosophy of worldsheet string theory and a nodding acquaintance with D-branes is enough as the necessary physics pre-requisite for these lectures. There are standard text-books for both [2]. As far as the mathematical background goes, a familiarity with the basic facts about complex manifolds and the language of fiber bundles will be useful. A practical introduction to these topics can be found in chapters 8 and 9 of Nakahara [4]. An especially concise introduction to line bundles which contains many of the relevant ideas is [5], and a more detailed review on fiber bundles in the context of gauge theories is [6].

The lectures collected here start out at a leisurely pace, but towards the end I was forced to hurry to fit the story within the allotted time. So some of the later discussions (in particular the last two subsections) are written in a whirlwind style, which will hopefully get corrected in a future arXiv version. Other reviews which discuss the topics covered in these lectures from a more advanced viewpoint are [13, 14, 15].

2 Algebra and Geometry

In this section we introduce some of the ingredients from algebra and topology/geometry that will be necessary for our discussions later on. The fact that geometry arises in string theory is not a surprise, but the reason why algebraic methods are powerful is less obvious. One way to understand this abundance of algebraic geometry is to notice that the spectrum of the theory (in particular, in topological sigma models that are the focus of these lectures) is determined by the cohomology of a BRST operator Q. We will see this in more detail in the next section.

The aim of the rest of this section is almost exclusively to set up the backdrop in mathematics, and can be read independently of the rest of these lectures. This inevitably means that the material presented here is going to be dry and without motivation at this point.

[1] But it should be mentioned that another "categorical" story involving the so-called Fukaya category can be pursued for the A-model branes.

2.1 Algebraic Preliminaries

Rings, Modules, Ideals

First we make some preliminary algebraic definitions which will be used repeatedly during the course of these lectures. Despite (or because of?) the fact that these definitions are very primitive, it is surprisingly hard (at least for the author of these notes) to remember which name corresponds to which algebraic object. So collecting them in one place seems like a useful thing to do.

In algebra, a ring is roughly a set in which addition and multiplication are defined with the usual property that multiplication distributes over addition. The precise definition is as follows, even though the intuition is all one really needs.

A ***ring*** is a set R with two binary operations $+ : R \times R \to R$ and $\cdot : R \times R \to R$ called addition and multiplication such that

- $(R, +)$ is an Abelian group. That is, $\forall a, b, c \in R$
    1. we have $a + b \in R$ (closure),
    2. $(a + b) + c = a + (b + c)$ (associativity of addition),
    3. $a + b = b + a$ (commutativity of addition),
    4. $\exists \ 0 \in R$ such that $0 + a = a + 0 = a$ (identity of addition),
    5. $\exists \ -a \in R$ such that $a + (-a) = (-a) + a = 0$ (additive inverse).
- $(R, \cdot)$ is a monoid. This means that $\forall a, b, c \in R$
    1. $a \cdot b \in R$ (closure),
    2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity of multiplication),
    3. $\exists \ 1 \in R$ such that $1 \cdot a = a \cdot 1 = a$ (multiplicative identity).
- Multiplication distributes over addition. That is,
    1. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,
    2. $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

The prototypes to keep in mind when thinking of rings are integers, integers with modulo n addition, and polynomials. One caveat to this intuition is that the multiplication of ring elements does not necessarily have to be commutative. If they are, we have a ***commutative ring***, an example for a non-commutative ring is the ring of matrices with the usual matrix multiplication and addition. Notice also that rings do not require the existence of a multiplicative inverse. If a multiplicative inverse exists for every element other than 0 (and $0 \neq 1$ in the ring) then we have a ***division ring***. A commutative division ring is also called a ***field***. The set of real or complex numbers is the quintessential prototype for a field[2].

[2]As a curiosity, lets note as an aside that an example for a division ring that is *not* commutative is provided by the set of quarternions H.

The ring of polynomials is a very useful notion in algebraic geometry, so we elaborate it here a little. The set of all polynomials (in say, x) with coefficients in a ring R, together with the ring addition and multiplication forms itself a ring (as can be easily checked using the usual rules of polynomial addition and multiplication). This ring of polynomials over R is denoted by $R[x]$. In the case $R = \mathbb{C}$, the polynomial ring is denoted $\mathbb{C}[x]$, and it is also an algebra. We can generalize this to more variables, $x, y, ..$ by first constructing the ring $R[x]$ and then defining $R[x, y] \equiv (R[x])[y]$. This is tantamount to identifying, for example, $x^3y^2 - yx^3 + 7xy^2 + 3xy - 6y + 4x^2 + 9 = (x^3 + 7x)y^2 + (-x^3 + 3x - 6)y + (4x^2 + 9)$. It turns out that $\mathbb{C}[x, y, ..]$ and spaces constructed from it are a recurring theme in algebraic geometry.

A ***module*** over a ring is intuitively like a vector space except that instead of living in a field, the scalars of a module live on a ring. More formally, a ***left R-module*** over the ring R consists of an Abelian group $(M, +)$ and an operation $R \times M \to M$ (scalar multiplication) such that $\forall r, s \in R$ and $\forall x, y \in M$,

- $r(x + y) = rx + ry$
- $(r + s)x = rx + sx$
- $(rs)x = r(sx)$
- $1x = x$

which are the usual rules for scalar multiplication familiar from vector spaces. Thus a module is a ring action on an Abelian group, and therefore generalizes representation theory which talks about group actions on vector spaces. A right S-module is defined analogously except the ring S acts on the left, i.e., $M \times S \to M$. If the ring R is commutative, then the left and right modules are identical (if we make the identification $rx \equiv xr$) and the module is simply referred to as an R-module. A ***bimodule*** (or more precisely an R-S-bimodule) is a left R-module and a right S-module such that $\forall r \in R, s \in S, m \in M$ we have the compatibility condition $(rm)s = r(ms)$. If $R = S$, then we call the bimodule an R-bimodule. In most our examples we will drop the qualifications and just use the term module. Other natural definitions like submodules, module-homomorphisms etc. can be extended exactly analogous to the corresponding definitions in the case of vector spaces. A ***free module*** is a module with a linearly independent basis (or $free$ basis as it is often referred to in mathematics). The precise definition is as follows. For an R-module M, the set $E = \{e_1, e_2, ...e_n\}$ is a free basis for M if:

1. E generates M, i.e., every element of M can be written in the form $\sum_i r_i e_i$ with $r_i \in R$.

2. E is a free set, i.e., if $r_1e_1 + r_2e_2 + ... + r_ne_n = 0_M$, then $r_1 = r_2 = ... = r_n = 0_R$ (where 0_M is the zero of the Abelian group M and 0_R, that of the ring R). If R-is well-behaved enough (which will be the case for the modules we will consider), then the cardinality of the basis is well-defined number called the rank of the module, and M is isomorphic to R^n.

Now we turn to the definition of an ideal [9]. Let $(R, +)$ be the additive group in the ring R. A subset I of R is called ***right ideal*** of R if

- $(I, +)$ is a subgroup of $(R, +)$

- $xr \in I, \forall x \in I$ and $\forall r \in R$.

An analogous definition can clearly be made for left ideals. An ideal that is both a left ideal and a right ideal is called a two-sided ideal (or sometimes simply as an ideal). Notice that if we take the Abelian group M in the construction of a (right) R-module to be R itself, then the (right-)ideal I is nothing but an R-submodule of the module R. (A submodule of M is a subgroup of the group M which stays in M itself after multiplication by elements of R - this is exactly like the definition of a vector subspace, the subspace is left unchanged under multiplication by the scalars.).

Now we present some examples for ideals, some of which will turn out to be useful later. The simplest ideal that can be constructed from any ring is as follows. If p is in R, then pR is a right ideal and Rp is a left ideal of R and they are called called the principal right and left ideals generated by p. A standard example for an ideal of this form (in the ring of integers $\mathbb{Z}$) is the set of all integers divisible a specific integer n. This ideal is often denoted by $n\mathbb{Z}$. In fact, it turns out that in the set of integers, every ideal can be generated by a single number and conversely, that the number is determined uniquely upto sign by the ideal. The former characteristic (namely that all ideals of the ring are principal ideals) makes the set of integers into what is called a *principal ideal domain.* Analogous to the case of integers, we can construct ideals in the ring of polynomials as polynomials which contain a given polynomial factor.

A notion from the theory of ideals which is useful in the context of algebraic geometry is that of an ***ideal generated by a set***. A formal definition can be given as follows: the left ideal of R generated by a subset X of R is the set of all elements of the form $r_1a_1 + ... + r_na_n$ with each $r_i \in R$ and each $a_i \in X$. It turns out that the intersection of all left ideals of R containing the set X is precisely the left ideal generated this way, so this is an alternative definition. Analogous definitions can be made for right and two-sided ideals. If an ideal I of R is such that there exists a finite subset X of R generating it, then the ideal I is said to be ***finitely generated***.

An important construction using ideals is that of the ***quotient ring*** (also called the factor ring). Roughly speaking, the quotient ring R/I is constructed by identifying two elements of a ring R when they differ (according to the ring addition) by an element of the ideal I. More precisely, Given a ring R and a two-sided ideal I in R, we first define a relation $\sim$ on R by the condition that $a \sim b$ iff $a - b \in I$. This is an equivalence relation, and the set of all such equivalence classes is denoted by R/I, and one can give it a ring structure by defining

- $(a + I) + (b + I) = (a + b) + I$,
- $(a + I)(b + I) = (ab) + I$.

Here the equivalence class of the element $a \in R$ is given by $[a] = a + I \equiv \{a + r : r \in I\}$. R/I, with this ring structure, defines the quotient ring. The zero-element of R/I is $(0 + I) = I$, and the multiplicative identity is $(1 + I)$. It is easy to show that the ring structure defined on R/I is well-defined by trying it out on distinct elements of the equivalence class and noticing that the resulting equivalence classes are identical.

A simple example for quotient rings is provided by $\mathbb{Z}/2\mathbb{Z}$. We see that all odd numbers $\in \mathbb{Z}$ differ from each other by an element of $2\mathbb{Z}$ (i.e., an even number) and so does all even numbers $\in \mathbb{Z}$.

Together the odd and even numbers exhaust the set of integers, and clearly they are in different equivalence classes modulo $2\mathbb{Z}$ because (odd $-$ even) $\notin 2\mathbb{Z}$. This means that the set {[even], [odd]} is the quotient ring $\mathbb{Z}/2\mathbb{Z}$ (the square brackets stand for equivalence classes), which has a natural isomorphism with the group $\mathbb{Z}_2$ with elements $\{0, 1\}$. An obvious generalization exists for $\mathbb{Z}/n\mathbb{Z}$, and this leads us to the usual modular arithmetic in elementary number theory.

Another instructive example for a quotient ring can be constructed from the ring $\mathbb{R}[x]$ of polynomials in the variable x with *real* coefficients, and the ideal $I = (x^2 + 1)$ consisting of all multiples of the polynomial x^2+1. The reader should convince him/her-self that the quotient ring $\mathbb{R}[x]/\{x^2+1\}$ is naturally isomorphic to $\mathbb{C}$. Here, the equivalence class $[x]$ is the analogue of $i = \sqrt{-1}$. This is because in the sense of equivalence classes, $x^2 + 1 \sim 0$ so "$x^2 = -1$", which is what defines i.

One way in which these seemingly trivial definitions become useful is when they are used to define interesting spaces in algebraic geometry. The basic premise of algebraic geometry is the idea that spaces can be defined using the rings of (algebraic) functions that are well-defined over them[3]. We now review some oft-cited examples [7] for such algebraic constructions of spaces as a warm-up for later. I start by giving an algebraic definition of the space $\mathbb{C}^2$. The set of algebraic functions over $\mathbb{C}^2$ is identical to the ring of polynomials $\mathbb{C}[x, y]$ (think of the x, y variables as coordinates). So $\mathbb{C}^2$ can be thought of as the minimal set of points over which the function ring $\mathbb{C}[x, y]$ is well-defined. To take the next simplest example, lets look at $(\mathbb{C}^*)^2$, where $\mathbb{C}^*$ is the set of all non-zero complex numbers. Since the program is to define a space through the space of functions that are well-defined on it, in the present case, we can allow functions which are *not* well-defined even when (at least) one of the coordinates is zero. This means that the space $(\mathbb{C}^*)^2$ that we are after can be thought of as the set of polynomials over x, x^{-1}, y, y^{-1}, because clearly, polynomials constructed from these variables can behave badly only when one of the coordinates goes to zero. Another way to write the same thing is to say that $(\mathbb{C}^*)^2$ is the minimal set of points over which the quotient ring $\mathbb{C}[x, y, z, w]/\{xz-1, yw-1\}$ is well-defined. (Any polynomial defines an ideal in the ring of polynomials just like any $n \in \mathbb{Z}$ defines an ideal in the ring of integers through $n\mathbb{Z}$, so modding out by a polynomial is well-defined, and the quotient ring is legitimate.). More general algebraic varieties[4] can also be constructed algebraically from quotient rings. For instance, take the variety $V = \{(x, y)|x^4 + y^4 = 1\}$ in $\mathbb{C}^2$. The ring of polynomials over V is the quotient ring $\mathbb{C}[x, y]/\{x^4 + y^4 - 1\}$, and this algebraic object therefore gives a handle on the variety V.

Complexes, Exact Sequences

Now we turn to some algebraic constructions which are very useful in algebraic topology and in the various cohomological ideas that we will soon encounter.

A ***complex*** is a collection of algebraic objects $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ along with maps $\phi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ such that $\phi_{i+1} \circ \phi_i = 0$. This last condition means that the image of any map in a complex is in the

[3]The ring of functions over a space is the starting point for many generalizations. One could consider various function rings and be lead to various geometries: smooth, analytic and algebraic functions lead respectively to differential, complex and algebraic geometry whereas if one modifies the multiplicative structure of the function ring by making it non-commutative, the result is non-commutative geometry.

[4]An algebraic variety is just the zero locus of a bunch of polynomials.

kernel of the next map. Here is how one represents a complex:

$$... \xrightarrow{\phi_{i-1}} \mathcal{A}_i \xrightarrow{\phi_i} \mathcal{A}_{i+1} \xrightarrow{\phi_{i+1}} ... \tag{2.1}$$

The cohomology of $\mathcal{A}$ is defined to be the direct sum $H(\mathcal{A}) = \oplus_{n\in\mathbb{Z}} H^n(\mathcal{A})$ with

$$H^n(\mathcal{A}) = (\ker\ \phi_{n+1})/(\mathrm{im}\ \phi_n). \tag{2.2}$$

A map $f : \mathcal{A} \to \mathcal{B}$ between two complexes is called a ***chain map*** if the maps in the respective complexes $\phi_{\mathcal{A}}, \phi_{\mathcal{B}}$ satisfy $f\ \phi_{\mathcal{A}} = \phi_{\mathcal{B}}\ f$. In pictures, and in terms of the ingredient maps (i.e, $f_i : \mathcal{A}_i \to \mathcal{B}_i$), this means that the diagram

$$\begin{array}{ccc} \mathcal{A}_i & \xrightarrow{\phi_{\mathcal{A}_i}} & \mathcal{A}_{i+1} \\ \downarrow f_i & & \downarrow f_{i+1} \\ \mathcal{B}_i & \xrightarrow{\phi_{\mathcal{B}_i}} & \mathcal{B}_{i+1} \end{array} \tag{2.3}$$

should commute for all values of i.

A complex is called an ***exact sequence*** iff $\ker\ \phi_{n+1} = \mathrm{im}\ \phi_n$ for all n, that is, if the cohomology is trivial. An important special case is the ***short exact sequence***:

$$0 \longrightarrow A \xrightarrow{\phi_1} B \xrightarrow{\phi_2} C \longrightarrow 0. \tag{2.4}$$

It is an easy exercise to check from the condition of exactness that for a short exact sequence, ϕ_1 is injective (i.e., its kernel is just the zero of A, or in other words ϕ_1 is one-to-one[5]), and ϕ_2 is surjective (i.e., its image is all of C, or in other words ϕ_2 is onto).

2.2 Geometric Preliminaries

We will see soon that in the topological B-model, D-branes are naturally described by bundles over holomorphic manifolds. So, an algebraic way of describing holomorphic vector bundles using locally free sheaves will be widely used in later sections, so here we make some relevant definitions. The reader is assumed to be familiar with the general notion of a bundle.

Holomorphic Vector Bundles

A ***holomorphic vector bundle*** is a complex vector bundle over a complex manifold X such that the total space E is complex manifold with the condition that the transition functions are holomorphic.

[5] The fact that the map is one-to-one follows because the kernel is just the zero of A. To see this, note that if the map were not one-to-one then we would have two elements a_1, a_2 with $a_1 \neq a_2$ in A which both mapped to the same element b in B. But then, the element $a_1 - a_2$ which is not zero, would have to be mapped to 0 in B. This would mean that the kernel contains non-zero elements in A, which is a contradiction. Notice that in coming to this conclusion we had to assume an additive structure in A as well as linearity of maps, which are all true for the cases of interest: namely when A, B, C are rings, modules, vector spaces, etc. and the maps are their corresponding homomorphisms.

If the fiber is one (complex-) dimensional, then we have a ***holomorphic line bundle***. It should be kept in mind that despite the name "line" bundle, the holomorphic line bundle is really a $\mathbb{C}$ bundle. The ***rank*** of a line bundle is the (complex-) dimensionality of its fiber.

The ***canonical line bundle*** is an example for a holomorphic vector bundle that can be defined on every complex manifold. On a chart spanned by the coordinates $\{z_1, ..., z_n\}$ (where n is the complex dimension of the manifold), the basis for the fiber of the canonical line bundle is the single element $dz_1 \wedge ... \wedge dz_n$. Since it depends only on the the z's and not on the $\bar{z}$'s, it is a holomorphic vector bundle. Because the dz_i's form a basis for the holomorphic cotangent bundle, the canonical line bundle can be thought of as its n-th exterior power. The transition function across charts is easily obtained from the definition of the fiber-basis: it is the Jacobian for the holomorphic change of coordinates between two charts. An analogous definition for anti-canonical line bundle can clearly be made in terms of the $\bar{z}_i$.

Line Bundles and Chern Classes

Topological classification of bundles is a non-trivial problem because bundles can be described in many ways (different trivializations for example), but still be the same topological space. Characteristic classes (which are elements of the cohomology groups of the base space) are topological invariants of bundles and are therefore useful for distinguishing them. Roughly speaking, they keep track of the "twisting" of a bundle. One such characteristic class for complex vector bundles is the Chern class. For complex line bundles, the Chern class (more precisely, the first Chern class which is an element of the second integral cohomology of the base space) provides a complete classification. This means that there is one-to-one correspondence between the elements of $H^2(X, \mathbb{Z})$ and the topologically distinct line bundles. For the case of line bundles over complex projective spaces (which will be interesting to us later on),

$$H^2(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}, \tag{2.5}$$

which means that the line bundle is specified by an integer. For this reason, line bundles over projective spaces are denoted typically as $\mathcal{O}(n)$, where n is the first Chern class. The first Chern class of a space (bundle) V is written as $c_1(V)$, so we write $c_1(\mathcal{O}(n)) = n$.

A precise definition of the Chern class[6] can be made in various ways, but the most natural definition that directly invokes the twisting of a bundle is too much of a sideroad for our purposes and we will refer the reader instead to p. 45-49 of [21]. In the following we will content ourselves with giving a definition the Chern class starting from a metric on the space. Of course, the choice of the metric is irrelevant since the quantity we are computing is topological, but this formulation is quite often a handy tool for calculations. So here is the metric-based definition of the first Chern class:

$$c_1 = i\frac{\Omega}{2\pi}, \tag{2.6}$$

where Ω is the curvature two-form calculated from the metric.

[6] By this we mean the precise choice of the (integral) homology class on the base for any given bundle over it.

Before we conclude this section, I give a simple proof of the hairy ball theorem for the projective sphere $\mathbb{CP}^1$ (which states that there are no nowhere vanishing vector fields on a sphere) as an elementary application of the Chern class to amuse the less cynical reader. The more business-like ones are invited to skip ahead.

The problem is essentially to show that the tangent bundle of $\mathbb{CP}^1$ is non-trivial, because if the bundle is trivial, we can always choose a nowhere vanishing section, which will be the required vector field. We will show the non-triviality of the bundle by showing that the first Chern class of $T\mathbb{CP}^1$ is non-trivial. We know that we can write down the following Fubini-Study metric on $\mathbb{CP}^1$:

$$g = \frac{dz d\bar{z}}{(1+|z|^2)}. \tag{2.7}$$

The curvature 2-form is computed to be

$$\Omega = \frac{2dz \wedge d\bar{z}}{(1+|z|^2)^2}. \tag{2.8}$$

To show that the cohomology class of $c_1 = \frac{i}{2\pi}\Omega$ is non-zero and to thereby prove the non-triviality of $T\mathbb{CP}^1$ all we need to do is integrate it over $\mathbb{CP}^1$ and find that the result is non-zero[7]. This latter is easily done by a change of variables to polar coordinates. Hairy ball theorem is proved.

3 Topological String Theory

Since D-branes in the full string theory are complicated beasts, throughout these lectures we will be working in the context of a simpler scenario, namely topological strings. The basic idea is to modify the worldsheet action for string theory so that the theory is simplified, but we will see that this is not such an over-simplification that we end up throwing the baby out with the bathwater. The reader who is unfamiliar with the details of string sigma models should still be able to understand the later sections if he-or-she is willing to believe the outline of the story I present here. Our true starting point is the action presented in (3.1).

3.1 The $\mathcal{N} = 2$ Non-Linear Sigma Model

When we define string theory in flat spacetime, the worldsheet action is taken as a free theory. But the consistency of the theory does not require that the worldsheet theory be free, it requires only that it be superconformal. In general, it is known that when the target space is a Kähler manifold, the theory has $\mathcal{N} = 2$ supersymmetry, and if in addition it is Calabi-Yau, then it is also superconformal[8]. So, as long as we are working with a target space that is Calabi-Yau, we can generalize the free worldsheet theory, and work with more general non-linear sigma models[9] on the worldsheet. Indeed,

[7] By Stokes' theorem an element of trivial homology would have integrated to zero.

[8] The Calabi-Yau condition comes from the vanishing of the Ricci tensor, which is necessary for the one-loop (in α') beta-function to be zero. There are no new conditions that arise at higher loops because of the high supersymmetry of the theory.

[9] The term non-linear sigma model is usually used for theories where the bosonic fields have an interpretation as the coordinates of a target manifold (with curvature).

type II string theory is formulated as an $\mathcal{N}=2$ superconformal field theory for the maps from the worldsheet to the target Calabi-Yau (times flat space), $\phi : \Sigma \to X$. We can construct generators of the superconformal algebra from both the left and right moving sectors (i.e., the holomorphic and anti-holomorphic sectors), and they satisfy the algebra separately, so sometimes we say that the worldsheet theory is an $\mathcal{N}=(2,2)$ superconformal field theory. A very readable overview of sigma models and the $\mathcal{N}=2$ superconformal algebra can be found in the first few chapters of [7].

The easiest way to write down the general form of the action for the non-linear sigma-model is to start with the $\mathcal{N}=2$ superspace formalism. Just as the $\mathcal{N}=1$ Kähler potential defines the kinetic terms for the component fields in four dimensions, we can use an $\mathcal{N}=2$, $D=2$ version of the Kähler potential to define the sigma-model action on the worldsheet (upto a minor complication involving the antisymmetric B-field which needs to be implemented on the worldsheet as a θ-term). A description of this superspace approach can be found in section 3 of Distler's Les Houches lecture notes [10]. Expanding the superspace action in component fields, one can write the final version of the non-linear sigma model action in the form

$$S = \frac{i}{4\pi\alpha'}\int_\Sigma d^2z\Big\{g_{i\bar{j}}\Big(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}\Big) + iB_{i\bar{j}}\Big(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}\Big)$$
$$+ig_{i\bar{j}}\psi_-^{\bar{j}}D\psi_-^i + ig_{i\bar{j}}\psi_+^{\bar{j}}\bar{D}\psi_+^i + R_{i\bar{i}j\bar{j}}\psi_+^i\psi_+^{\bar{i}}\psi_-^j\psi_-^{\bar{j}}\Big\}. \tag{3.1}$$

This action is our true starting point, and in the rest of this subsection we will explain its content. A standard reference on sigma models in the context of topological twisting (which we will get to in a minute) is [11]. In particular, our convention for the measure follows the convention of [11], so $d^2z = -idz \wedge d\bar{z}$.

Since the target space is a complex-manifold, we have split its coordinates into ϕ^i and its complex-conjugate, $\phi^{\bar{j}}$. We will use the same letter for the superfield corresponding to the map, and its lowest component. Since the target space is Kähler, there exists a function $\mathcal{K}(\phi, \bar{\phi})$ on X called the Kähler potential, and the sigma-model metric that we use is defined as,

$$g_{i\bar{j}} = \partial_i\partial_{\bar{j}}\mathcal{K}. \tag{3.2}$$

$\mathcal{K}$ is not uniquely defined across charts, but transforms as $\mathcal{K} \to \mathcal{K} + f(\phi) + \bar{f}(\bar{\phi})$ for some f, but it drops off from the metric, so the metric is globally well-defined on X.

The fermions are worldsheet fermions, but the index i (respectively, $\bar{j}$) indicates that they are valued in the holomorphic (anti-holomorphic) tangent bundle of X. To be precise, we can write

$$\psi_+^i \in \Gamma(K^{1/2} \otimes \phi^*TX), \qquad \psi_-^i \in \Gamma(\bar{K}^{1/2} \otimes \phi^*TX), \tag{3.3}$$
$$\psi_+^{\bar{i}} \in \Gamma(K^{1/2} \otimes \phi^*\overline{TX}), \qquad \psi_-^{\bar{i}} \in \Gamma(\bar{K}^{1/2} \otimes \phi^*\overline{TX}). \tag{3.4}$$

Here Γ means that the field is a section of the corresponding bundle - in quantum field theory, we work with the path integral where we integrate over all possible field histories, and field histories are naturally thought of as sections of the appropriate bundle over the spacetime manifold. In the present case, the idea is that the left-movers with the plus subscript live in the worldsheet chiral spinor-bundle[10] which is denoted by $K^{1/2}$ (and the right-movers live in the anti-chiral spinor

[10] I will explain what spinor-bundles are momentarily.

bundle $\bar{K}^{1/2}$), but then we need to tensor them with the holomorphic (i) and anti-holomorphic ($\bar{j}$) coordinates of the target space in order to define the spinor component-field of the superfields ϕ^i and $\phi^{\bar{j}}$. Since the ψ-s are worldsheet fields, the i and $\bar{j}$ indices need to be pulled back to the worldsheet from X before doing the tensoring. And for doing the pullback, we use the only available map, namely (the bosonic part of) ϕ.

Now, lets turn to $K^{1/2}$ and $\bar{K}^{1/2}$. These objects are the world-sheet spinor bundles, and their standard construction is as follows. We start with K, the canonical line bundle over Σ. The worldsheet Riemann surface is 1-complex dimensional, so its canonical line bundle is the same thing as the holomorphic cotangent bundle, i.e., $K = T^*\Sigma$. The "square root" of the bundle means that we look at the same base space (Σ) and the fiber for the bundle, but with the transition functions between patches now taken to be the square roots of the transition functions for the same patches on K. There are some ambiguities associated with globally consistent choices of signs for these square roots, but they will not concern us here[11] - we will assume that such a consistent choice has been made. The fibers of this new bundle $K^{1/2}$ are again complex lines[12], but this time, the sections will be interpreted not as left-moving covectors, but as left-moving worldsheet fermions. An exactly analogous construction starting with the anti-canonical line bundle $\bar{K}$ gives rise to right-moving (anti-holomorphic) fermions. Incidentally, this "square-rooting" is a standard procedure when defining spinors on a manifold: transformation matrices in the spin 1/2 representation are the square roots of those for spin 1.

The covariant derivative in (3.1) cannot mix spinors that live in different bundles, so it takes the form,

$$D\psi^i_- = \partial\psi^i_- + \partial\phi^j \Gamma^i_{jk}\psi^k_- \quad (3.5)$$

$$\bar{D}\psi^i_+ = \bar{\partial}\psi^i_+ + \bar{\partial}\phi^j \Gamma^i_{jk}\psi^k_+ \quad (3.6)$$

Notice that the affine connections need to be pulled back to the worldsheet. Exactly the same formulas hold for any field with one upper target-space index.

The action (3.1) has a local $N = 2$ worldsheet supersymmetry on both the left and right moving sectors. Which means there are four independent fermionic parameters in the SUSY-variations. We will exhibit the supersymmetry transformations for the various component fields here for future reference, but to explicitly check that these are invariances of the action, it is far easier to use

[11] For a discussion of these signs ("spin structures"), see sections 12.1, 12.2 of Green, Schwarz and Witten volume 2, or sections 5.2, 5.3 of [12].

[12] All complex lines are isomorphic. We can think of gamma matrices as defining an isomorphism between the cotangent fiber and the spinor fiber.

superfields.

$$\delta\phi^i = i\alpha_-\psi_+^i + i\alpha_+\psi_-^i \tag{3.7}$$

$$\delta\phi^{\bar{\imath}} = i\tilde{\alpha}_-\psi_+^{\bar{\imath}} + i\tilde{\alpha}_+\psi_-^{\bar{\imath}} \tag{3.8}$$

$$\delta\psi_+^i = -\tilde{\alpha}_-\partial\phi^i - i\alpha_+\psi_-^j\Gamma^i_{jk}\psi_+^k \tag{3.9}$$

$$\delta\psi_+^{\bar{\imath}} = -\alpha_-\partial\phi^{\bar{\imath}} - i\tilde{\alpha}_+\psi_-^{\bar{\jmath}}\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\psi_+^{\bar{k}} \tag{3.10}$$

$$\delta\psi_-^i = -\tilde{\alpha}_+\bar{\partial}\phi^i - i\alpha_-\psi_+^j\Gamma^i_{jk}\psi_-^k \tag{3.11}$$

$$\delta\psi_-^{\bar{\imath}} = -\alpha_+\bar{\partial}\phi^{\bar{\imath}} - i\tilde{\alpha}_-\psi_+^{\bar{\jmath}}\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\psi_-^{\bar{k}} \tag{3.12}$$

To make sure that both the left-hand sides and the right hand sides live on the same bundles, we need to have $\alpha_-, \tilde{\alpha}_-$ living in $K^{-1/2}$ and $\alpha_+, \tilde{\alpha}_+$ in $\bar{K}^{-1/2}$.

3.2 Twisting and the Topological B-model

Topological string theory is constructed by changing the field content of the sigma model of the last section, while retaining the same form for the action. There are two standard choices for the new field contents that are known to give topological theories, and they are called the A-model and the B-model. We will be concerned exclusively with the B-model twist[13] in these lectures, so we will describe that here. I will describe the A-model in the final section.

The B-model twist is defined by taking the new fields to live in new bundles:

$$\psi_+^i \in \Gamma(K\otimes\phi^*TX), \qquad \psi_-^i \in \Gamma(\bar{K}\otimes\phi^*TX), \tag{3.13}$$

$$\psi_+^{\bar{\imath}},\ \psi_-^{\bar{\imath}} \quad\in\quad \Gamma(\phi^*\overline{TX}). \tag{3.14}$$

It is customary to introduce the new variables,

$$\eta^{\bar{\imath}} \quad=\quad \psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}}, \tag{3.15}$$

$$\theta_i \quad=\quad g_{i\bar{\imath}}(\psi_+^{\bar{\imath}} - \psi_-^{\bar{\imath}}), \tag{3.16}$$

$$\rho^i \quad \text{with} \quad \rho_z^i = \psi_+^i \text{ and } \rho_{\bar{z}}^i = \psi_-^i. \tag{3.17}$$

The first two are worldsheet scalars, and $\rho^i \equiv \rho_z^i dz + \rho_{\bar{z}}^i d\bar{z}$ is a one form. The action in these variables becomes

$$S_B = \frac{i}{8\pi\alpha'}\int_\Sigma d^2z\Big\{g_{i\bar{\jmath}}\Big(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{\jmath}} + \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{\jmath}}\Big) + iB_{i\bar{\jmath}}\Big(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{\jmath}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{\jmath}}\Big)$$
$$+ig_{i\bar{\jmath}}\eta^{\bar{\jmath}}(D\rho_{\bar{z}}^i + \bar{D}\rho_z^i) + i\theta_i(\bar{D}\rho_z^i - D\rho_{\bar{z}}^i) + R_{i\bar{\imath}j\bar{\jmath}}\rho_z^i\rho_{\bar{z}}^j\eta^{\bar{\imath}}\theta^{\bar{\jmath}}\Big\}. \tag{3.18}$$

Since the fields on which they act have different spins after the twist, the corresponding "supersymmetry" charges from the last subsection, instead of being spinors, are now scalars and vectors (two

[13]The word twist is used for the change in the field content because we replace the spinors in the sigma model with scalars and vectors on the worldsheet.

of each). A specific linear combination of the scalar variations will be relevant for our purposes, so we explictly write it down:

$$\delta_Q \phi^i = 0 \tag{3.19}$$

$$\delta_Q \phi^{\bar{i}} = i\alpha\eta^{\bar{i}} \tag{3.20}$$

$$\delta_Q \eta^{\bar{i}} = 0, \qquad \delta_Q \theta_i = 0, \tag{3.21}$$

$$\delta_Q \rho^i_z = -\alpha\partial_z\phi, \qquad \delta_Q \rho^i_{\bar{z}} = -\alpha\partial_{\bar{z}}\phi. \tag{3.22}$$

These transformations arise straightforwardly from the SUSY variations presented at the end of the last section, written in terms of the new fields, with the caveat that the variation parameters are now to be interpreted globally as sections of appropriate new bundles. For instance α_- and α_+ are now sections of K^{-1} and $\bar{K}^{-1}$ respectively, whereas $\tilde{\alpha}_-$ and $\tilde{\alpha}_+$ are simply functions. The α above corresponds to the specific choice with $\tilde{\alpha}_- = \tilde{\alpha}_+ \equiv \alpha$ while at the same time setting $\alpha_- = 0 = \alpha_+$. The reason for this choice is that this specific symmetry and its generator will be of great use to us momentarily.

We are allowed to set functions to non-zero constants canonically, but not so with sections because in the case of sections, there is no way of canonically identifying distinct fibers: the fibers are vector spaces and any isomorphism between them is as good as any other[14]. In any event, the upshot is that since α is a function, we can consider the specific case where it is constant, as a global symmetry of the system and talk about the Noether charge that generates it. It is easily checked from the variations written down above that the Noether charge for this transformation, Q, defined by $\delta_Q(...) = -i\alpha\{Q, ...\}$ is nilpotent (i.e., $Q^2 = 0$), so it will be called the BRST charge. The charge Q can be explictly constructed in terms of the component fields using the Noether procedure, but we will not bother to do so because we are only after the general story to set the backdrop for the the rest of the lectures.

It turns out that the B-model action can be rewritten in a very useful form using this BRST charge as follows:

$$S_B = -\frac{1}{8\pi\alpha'}\int \{Q, V\} + \frac{i}{8\pi\alpha'}W, \tag{3.23}$$

with

$$V = g_{i\bar{j}}(\rho^i_z \partial_{\bar{z}}\phi^{\bar{j}} + \rho^i_{\bar{z}}\partial_z\phi^{\bar{j}}), \tag{3.24}$$

and

$$W = \int_\Sigma (-\theta_i \mathcal{D}\rho^i - \frac{i}{2}R_{i\bar{i}j\bar{j}}\rho^i \wedge \rho^j \eta^{\bar{i}}\theta^{\bar{j}}). \tag{3.25}$$

Here $\mathcal{D}$ is the covariantized exterior derivative (i.e., the covariant version of $d = \partial + \bar{\partial}$), and so $\mathcal{D}\rho^i = (D\rho^i_{\bar{z}} - \bar{D}\rho^i_z)dz \wedge d\bar{z}$ with D and $\bar{D}$ as defined earlier.

The advantage of this split is that now we can easily see that the theory is topological, i.e., the worldsheet metric has no say in the partition function of the theory. To see this, first note that the

[14]The zero-element on the other hand is preserved under *any* isomorphism, so setting a section to zero is acceptable.

partition function Z is given by

$$Z = \int \mathcal{D}\Phi \exp(-S_B), \tag{3.26}$$

where the functional integral over all the fields, schematically denoted here by Φ, is computed in a specific background worldsheet metric in a given complex-structure class[15]. Our task is to show that Z is independent of the choice of this complex structure. Now W, being written purely in terms of differential forms (which include *both* holomorphic and antiholomorphic pieces symmetrically) is manifestly independent of the choice of complex structure, because the complex structure is nothing but a choice of split between holomorphic and anti-holomorphic coordinates on the Riemann surface. On the other hand, the fact that the first piece is Q-exact means that when we change the metric, it can only change by Q-exact pieces, which do not contribute to correlation functions. Putting these together, the total theory is therefore independent of the worldsheet metric.

It is also possible to show that the whole action is independent of the Kähler metric of the target space X (for the first piece this is evident due to the Q-exactness, but for W, it is not immediately obvious). One way to show this is to use an off-shell formulation of the B-model action as constructed in [16, 17] through the introduction of auxiliary fields. In this formulation, the entire action can be expressed easily as the integral of just a Q-exact piece and therefore the Kähler independence is obvious. The fact that the B-model depends only on the complex structure of the target space X implies that we can define X as a complex algebraic variety (i.e., as the intersection of homogeneous polynomials in a complex projective space), and that the B-model is completely defined by the defining polynomials of the variety. This is another reason (see also the beginning of section 2) why algebraic methods play such a crucial role in these lectures.

Another important conclusion that can be drawn from the form of the action above is that the theory is independent of the coupling constant ($\sim 1/\alpha'$). The reason is that the changes in $1/\alpha'$ make only irrelevant contributions to the first piece because of its Q-exactness as before, whereas the coupling dependence of the second piece can be absorbed in a redefinition of θ_i and so it is irrelevant. This in particular implies that weak coupling computations done in the B-model are exact! In fact, the weak coupling limit can also be thought of as the semi-classical limit: the action gets divided by $\hbar$ in the path integral, so only the combination $1/(\alpha'\hbar)$ is relevant, and therefore the smallness of α' is the same thing as the smallness of $\hbar$ as far as the theory is considered. This means that we can look at the classical solutions and they contain the whole story. Now, the classical solutions of (3.18) can be checked to be of the form ϕ =constant, i.e., they map Σ to a single point on X (the fermionic fields are zero classically). The moduli space of solutions is therefore the space of all points on X, which is X itself. There is a more elegant way of looking at this using the idea of "localization" of the path integral, we refer the reader to section 5 of [11].

3.3 Spectrum and Ghost Numbers

Now we turn to a description of the spectrum of observables of the B-model. The first thing to note is that the observables of the theory are the elements of Q-cohomology: they need to be closed under

[15] In string theory, the integration over worldsheet metrics reduces to integration over complex structures after making allowance for the Diff×Weyl redundancy.

Q for them to be topological, but they cannot be exact because otherwise the correlation functions would vanish. From the δ_Q variations listed earlier in this subsection, we see that objects of the form

$$W_V = V^{j_1 \dots j_m}_{\bar{i}_1 \dots \bar{i}_n}(\phi)\eta^{\bar{i}_1}\dots\eta^{\bar{i}_n}\theta_{j_1}\dots\theta_{j_m} \tag{3.27}$$

are the Q-closed operators. It turns out that to make sure that these elements are in the cohomology of Q, we need to restrict the coefficients such that

$$V \equiv V^{j_1 \dots j_m}_{\bar{i}_1 \dots \bar{i}_n}(d\bar{z}^{\bar{i}_1} \wedge \dots \wedge d\bar{z}^{\bar{i}_n}) \otimes \Big(\frac{\partial}{\partial z^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial z^{j_m}}\Big), \tag{3.28}$$

which is a $(0,n)$-form valued in the m-th exterior power of the holomorphic tangent bundle (i.e., $V \in \wedge^n \overline{T^*X} \otimes \wedge^m TX$), is a cohomology element for the operator $\bar{\partial}$. That is, W_V is in the Q-cohomology iff V is an element of $H^n_{\bar{\partial}}(X, \wedge^m TX)$. The simple reason for this is that (as easily checked from the Q-variations written down earlier),

$$\{Q, W_V\} = -W_{\bar{\partial}V}, \tag{3.29}$$

and therefore, W_V is BRST closed if $\bar{\partial}V = 0$ and exact if $V = \bar{\partial}\chi$ for some χ. The cohomology groups $H^n_{\bar{\partial}}(X, \wedge^m TX)$ go by various names: Hochschild cohomology, twisted Dolbeault cohomology (see Nakahara p.416), or the sheaf cohomology with coefficients in exterior powers of the holomorphic tangent bundle of X.

The general story that we want to take away from the above description of observables is that the spectrum of bulk states (i.e., closed string vertex operators) is described by cohomology with coefficients in some bundle. Another example for the same general structure is provided by heterotic compactifications [18], where one finds that some of the massless states are accounted for by $H^n(X, \wedge^m \mathcal{E})$, where $\mathcal{E}$ is the holomorphic gauge bundle for the compactification. One of the claims of the derived categories program is that this cohomology picture gets modified for open (topological) strings and that the spectrum of strings stretched between branes is counted not by cohomology, but by the so-called Ext groups.

Before we leave the closed string B-model, we would like to add another important structure to it, namely the notion of ghost number. This involves assigning a ghost charge to each of the fields: zero for ϕ, -1 for ρ and +1 for η and θ, with the understanding that ghost numbers are additive when we construct operators out of the elementary fields. At this point, these charge assignments seem completely ad=hoc (except for the one fact the action is indeed a ghost number scalar). The thing that makes this contrivance useful is that it serves as a book-keeping device. From the explicit Noether construction of the BRST charge (which we have not attempted here), it is possible to show that Q has ghost number 1, and we will need this fact later.

4 A Mathematical Interlude

This section is dedicated to the construction of some of the machinery that we will need in describing D-branes using algebraic geometry and homological algebra.

4.1 Sheaves and Categories

Sheaves are a generalization of the notion of a bundle, where we work with the set of sections over the base space (base+sections), instead of working with the total space (base+fibers). D-branes have world-volume gauge fields living on them, so in the large radius limit, they are modelled by vector bundles over submanifolds. So to generalize D-branes, it is natural to generalize bundles.

The definition of a sheaf might seem a bit baroque at first because of its length. But the ideas involved are much simpler than those involved in say the definition of a manifold or a fiber bundle, partly because a sheaf is defined purely algebraically. For $\mathcal{F}$ on X to be a ***sheaf*** we first assign to each open set $U \subset X$ an Abelian group $\mathcal{F}(U)$. The elements of these groups are called *sections* over U, and generalize the notion of the set of sections over a bundle trivialization. For this assignment to qualify as a sheaf it has to satisfy the following set of conditions.

- For every $V \subset U$ we assign a *restriction map* which is a group homomorphism, $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, such that

  (a) for triple inclusions $W \subset V \subset U$, the restrictions follow the composition rule[16] $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$,

  (b) $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map, and

  (c) $\mathcal{F}(\emptyset) = 0$ (The section over the null set is the zero of the Abelian group).

- The gluing conditions:

  (i) If $\sigma \in \mathcal{F}(U), \tau \in \mathcal{F}(V)$ and $\sigma|_{U\cap V} = \tau|_{U\cap V}$, then there exists a section $\eta \in \mathcal{F}(U \cup V)$ such that $\eta|_U = \sigma$ and $\eta|_V = \tau$. The section η can be thought of as the the glued together version of σ and τ across patches. Here we use the notation $|_V$ to denote the restriction to V, since by (a) above, restrictions are consistent under inclusions and so we do not need to specify the open set *from* which we are restricting.

  (ii) If $\sigma \in \mathcal{F}(U \cup V)$ and $\sigma|_U = \sigma|_V = 0$, then $\sigma = 0$.

The first set of conditions define a ***presheaf***. So a sheaf is a presheaf that satisfies the gluing conditions, which essentially implies that a sheaf can be built starting from local pieces.

A very useful example for a sheaf is the sheaf of holomorphic functions, also called the ***structure sheaf***. It is defined by taking $\mathcal{F}(U)$ to be the additive group of holomorphic functions over U. The restriction maps to a patch U are given by the usual restriction of functions to an open set. It is trivial to check that with these definitions, the structure sheaf is indeed a sheaf, and we will denote it by $\mathcal{O}_X$. We can also construct $\mathcal{O}^*$, the sheaf oh nowhere-zero holomorphic functions with point-wise multiplication as the group addition. Another simple example is provided by the ***skyscraper sheaf*** over a point p on X. It is defined simply by stipulating that $\mathcal{F}(U) = \mathbb{C}$ if $p \in U$ and $\{0\}$ otherwise. Here $\mathbb{C}$ is sometime referred to as the *stalk* of the skyscraper. Notice that this sheaf (denoted $\mathcal{S}_X$) is a trivial example for a sheaf that generalizes the notion of a bundle over a submanifold.

[16] This is why they are called restrictions in the first place.

We can define more general sheaves that arise from bundles over submanifolds. Given a sheaf $\mathcal{E}$ associated with a vector bundle defined only on a submanifold S of X, we can construct a sheaf over X as follows. Consider the inclusion map $i : S \to X$. The sheaf defined by the sections $i_*\mathcal{E}(U) \equiv \mathcal{E}(i^{-1}(U))$ is defined on all of X with its support restricted to S. When S is the point p we get back the skyscraper sheaf.

It is clear from the definition of a sheaf that a bundle together with its sections, forms a sheaf. In fact, it can be shown that sheaves that are "locally free" have a one-to-one correspondence with holomorphic vector bundles. Here we will make the definition of a locally free sheaf, but refer the reader to p.33-34 of [15] for a simple demonstration of the above correspondence. To make the definition, we first note that the additive sheaf of holomorphic functions defined above (i.e., the structure sheaf $\mathcal{O}_X$) also has a ring structure because of pointwise multiplication of functions. So we can use the ring $\mathcal{O}_X(U)$ to define $\mathcal{O}_X(U)$-modules. These rings over the open sets $U \subset X$ make X into what is called a ***locally ringed space***. Now any sheaf $\mathcal{F}$ will be called a ***sheaf of*** $\mathcal{O}_X$***-modules*** (or simply an $\mathcal{O}_X$***-module*** if we trade succinctness for technical correctness) if the Abelian group $\mathcal{F}(U)$ is also an $\mathcal{O}_X(U)$-module for every open set U in X, and the restriction maps are compatible with the module structure. The simplest example of an $\mathcal{O}_X$-module is the sheaf $\mathcal{O}_X$ itself[17]. With these definitions at hand, we can construct new sheaves from old by taking more than one copy of the Abelian group over each open set. Thus we define,

$$\mathcal{O}_X^{\oplus n} \equiv \underbrace{\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X}_{n} , \tag{4.1}$$

which is called the ***free*** $\mathcal{O}_X$-module of rank n. Now we have all the ingredients necessary to understand what a locally free sheaf is. A sheaf $\mathcal{E}$ is ***locally free*** of rank n if there exists a covering $\{U_\alpha\}$ of X such that $\forall \alpha,\ \mathcal{E}(U_\alpha) \cong \mathcal{O}_X(U_\alpha)^{\oplus n}$. Because of the one-to-one correspondence mentioned above, we will use the terms "locally free sheaf" and "holomorphic vector bundle" interchangeably in these lectures.

It is possible to make a category out of sheaves, and since the word "category" occurs in the title of these lectures, it is only fair that we give a formal definition here. The general idea of a category is by now familiar to most physicists: they deal with abstract objects that have some kind of structure and operations on them that preserve that structure. An example is provided by the category of vector spaces and their homomorphisms. We make this notion precise as follows. A ***category*** consists of

- a class[18] of objects.
- a class of ***morphisms*** between objects. Morphisms between objects A an B form the set $Hom(A, B)$.
- if f is a morphism from A to B and g is a morphism from B to C, then the composite morphism $g \circ f$ from A to C is well-defined.

[17] Notice that here the Abelian group M and the ring R which makes M into an R-module are both the same *set*, namely the set of holomorphic functions over U, which is the underlying set of the Abelian group $\mathcal{O}_X(U)$.

[18] We can think of a class as a glorified set. The notion is necessary because categories usually talk about collections of sets, and we would like to avoid running into things like the Russell's paradox: does the set of all sets contain itself? But this is a technicality that we will not care enough to dwell on.

- Composition of morphisms is required to be associative.
- There is an identity morphism for every object, that is, there exists $1_A \in Hom(A, A)$ for every A.

It is easy to see that sets, groups, smooth manifolds, etc. with maps, homomorphisms, diffeomorphisms, etc. form well-defined categories. Sheaves also form a category if we define the morphisms $\phi : \mathcal{F} \to \mathcal{G}$ by associating group homomorphisms $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for open sets U. The restrictions of sections under inclusions have to map to restrictions of sections under inclusions, so the following diagram must commute:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho'_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V), \end{array} \tag{4.2}$$

where ρ' is the restriction map in $\mathcal{G}$.

When we model D-branes, we will find that they are modelled by what are called ***coherent sheaves***. To define what they are, we first familiarize ourselves with the notion of exact sequences between sheaves. The idea is to look at exact sequences defined between the groups over each (sufficiently "small") open set in X. We clarify this with an example:

$$0 \longrightarrow \mathcal{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}^* \longrightarrow 0. \tag{4.3}$$

Here $\mathcal{Z}_X$ is the sheaf of integer valued functions over X. Over U, $\mathcal{Z}_X(U)$ are locally constant, which means that $\mathcal{Z}_X(X)$ is a vector space with dimension equal to the number of disconnected pieces comprising X. The first (one-one) map in the short exact sequence is just inclusion of the integer function as a holomorphic function, whereas the second (onto) map takes a holomorphic function f to $\exp(2\pi i f)$. Note that the maps in this complex may *not* be exact if the open sets are not sufficiently small, so we will always assume the latter to be the case.

Now, a coherent sheaf has the property that it has a ***locally free resolution***. What this means is that if the sheaf is coherent, there exists an exact sequence of the form

$$0 \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{G}_{n-1} \longrightarrow ... \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \tag{4.4}$$

for some n, such that all the $\mathcal{G}$'s are locally free sheaves (i.e., holomorphic bundles). There are technologies for constructing such resolutions, like the Koszul construction, but we will not get into them here.

4.2 Cohomology

As seen before, the closed string spectrum is typically given by some cohomology group (essentially because the spectrum is given by some BRST cohomology) valued in some bundle. In the case of open strings, things are more complicated because of the presence of the boundary. Since we are modeling branes by sheaves which generalize the notion of a bundle, it is natural to expect that we will need some kind of sheaf cohomology to calculate the spectrum. But that cannot be the whole story, because strings can stretch between two distinct branes, and sheaf cohomology groups involve only one sheaf/brane. In fact, one of the central claims of the categorical view of branes is that the open string spectrum between branes is calculated using Ext groups between shaves. One piece of technology that will be useful in calculating Ext groups (indeed, cohomology groups in general [8])is that of a spectral sequence, where one uses successive approximations to get to the final answer. In this section we will explore sheaf-inspired cohomology theories that set the groundwork for understanding Ext groups, as well as explain how to do computations using spectral sequences.

Čech Cohomology

Čech cohomology is a practical way of defining cohomology of a space X with coefficients valued in sheaves. There is a more categorical and more abstract definition of sheaf cohomology, which is equivalent Čech, but it is not immediately useful for computations.

Cohomology is always defined as "cochains" that are closed modulo exact, where closed and exact are defined with respect to a "coboundary operator". We now specify these two terms for Čech cohomology. The space X under consideration has an open cover $\mathcal{U} = \{U_\alpha\}$ and it also has a sheaf $\mathcal{F}$ defined over it. We define a ***cochain*** of degree n as

$$C^n(\mathcal{F}) \quad = \quad \coprod_{\alpha_0 \neq \cdots \neq \alpha_n} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}). \tag{4.5}$$

What does this mean? The idea is to consider every possible intersection of the sets in the open cover, $n+1$ at a time[19]. Then, an n-cochain is defined by specifying a section in each of these intersections. Here $\coprod$ stands for a disjoint union: the collection of one section each from all the possible $(n+1)$-intersections is what defines the n-cochain.

Now we turn to the ***coboundary*** operator $\delta : C^n(\mathcal{F}) \to C^{n+1}(\mathcal{F})$.

$$(\delta\sigma)_{\alpha_0 \ldots \alpha_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \sigma_{\alpha_0 \ldots \hat{\alpha}_k \ldots \alpha_{n+1}}|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_{n+1}}}. \tag{4.6}$$

The term $\hat{\alpha}_k$ in the sum is supposed to imply that we omit α_k, and terms like $\sigma_{\alpha_0 \ldots \alpha_n}$ stand for the chosen section from the intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$. To get a better understanding of the coboundary operator, it is useful to explicitly write it down for low degrees. Degree zero cochains are given by a choice of a section over each open set in $\mathcal{U}$. Consider two of the open sets in $\mathcal{U}$, lets call them U_{α_0} and U_{α_1}. The choices of the sections over them will be σ_{α_0} and σ_{α_1}. From these we can calculate the degree one cochain that arises as a result of the action of the coboundary operator:

$$(\delta\sigma)_{\alpha_0\alpha_1} = \sigma_{\alpha_1}|_{U_{\alpha_0} \cap U_{\alpha_1}} - \sigma_{\alpha_0}|_{U_{\alpha_0} \cap U_{\alpha_1}}. \tag{4.7}$$

[19] Just to emphasize something that might already be obvious to the reader, we hasten to add that the notation $\alpha_0 \neq \cdots \neq \alpha_n$ means that *none* of the α's are equal to each other. Perhaps a better (notationally unambiguous) way to do this is to pick the indexing set to be ordered and choose $\alpha_0 < \cdots < \alpha_n$.

This gives the representative for $\delta\sigma$ over the 2-fold intersection $U_{\alpha_0} \cap U_{\alpha_1}$. Therefore, by knowing σ, by which we mean knowing the σ_α over all the U_α, we can compute $\delta\sigma_{\alpha\beta}$ in all double intersections of the form $U_\alpha \cap U_\beta$, thereby giving the full definition of $\delta\sigma$. For higher degrees this basic idea extends trivially.

With these definitions, one can check that $\delta^2 \equiv 0$, so we construct Čech cohomology as δ-closed cochains mod δ-exact cochains, and the resulting cohomology groups are denoted by $H^n(X, \mathcal{F})$. Notice that the coefficients according to our definitions live naturally on the sheaf $\mathcal{F}$. There is a subtlety here associated to the fact that the definitions we have made here depend upon the trivialization $\mathcal{U}$. But it is possible to show that as long as all finite intersections between the U_α are diffeomorphic to $\mathbb{R}^m$ (m =dimension of X), which happens for instance when the open sets are all "small and round"[20], Čech cohomology is independent of the choice of the trivialization. A cover of this form is called a ***good cover***, but we will not make it any more precise than this because its existence is more important to us than its construction.

Lets try to understand Čech cohomology groups by looking more closely at some examples. How, for instance, should we understand $H^0(X, \mathcal{F})$? This can be answered by looking at (4.7). The cohomology is defined by degree zero cochains (none of which are exact because there is no degree -1 cochain), which are closed under δ. From (4.7) it is clear that these are given by sections over open sets such that

$$\sigma_{\alpha_1}|_{U_{\alpha_0} \cap U_{\alpha_1}} = \sigma_{\alpha_0}|_{U_{\alpha_0} \cap U_{\alpha_1}}, \tag{4.8}$$

i.e., sections over open sets which agree on overlaps. Since this has to hold for *all* open sets and their intersections, such a section is nothing but a global section of the sheaf $\mathcal{F}$. That is,

$$H^0(X, \mathcal{F}) = \mathcal{F}(X). \tag{4.9}$$

Spectral Sequence of a Double Complex

Spectral sequences are a powerful way of computing cohomology groups and as we will see later on, Ext groups. So here we give an elementary user's manual to spectral sequences. The theory is given in [8].

A basic object in our theory of spectral sequences is the ***double complex***. We denote the

[20] And not narrow and strangely shaped with multiple disconnected intersections, for example.

elements of a double complex by $E_0^{p,q}$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \iddots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
E_0^{0,2} & \xrightarrow{\delta} & E_0^{1,2} & \xrightarrow{\delta} & E_0^{2,2} & \xrightarrow{\delta} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
E_0^{0,1} & \xrightarrow{\delta} & E_0^{1,1} & \xrightarrow{\delta} & E_0^{2,1} & \xrightarrow{\delta} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
E_0^{0,0} & \xrightarrow{\delta} & E_0^{1,0} & \xrightarrow{\delta} & E_0^{2,0} & \xrightarrow{\delta} & \cdots
\end{array} \tag{4.10}$$

The differentials d and δ have to satisfy $d\delta + \delta d = 0$, so that $(d+\delta)^2 \equiv 0$. Spectral sequences are a recipe for evaluating the cohomology groups H_D^n associated to the operator $D = d + \delta$ using iterative approximations. Notice first that if we define

$$E^n = \bigoplus_{p+q=n} E_0^{p,q}, \tag{4.11}$$

then H_D^n are the cohomology groups associated with the exact sequence

$$0 \longrightarrow E^0 \xrightarrow{D} E^1 \xrightarrow{D} E^2 \xrightarrow{D} ... \tag{4.12}$$

The spectral sequence technology calculates H_D^n by iteratively constructing $E_r^{p,q}$ starting from $E_0^{p,q}$. The inductive definition of $E_{r+1}^{p,q}$ is as a cohomology group constructed at the r-th stage:

$$E_{r+1}^{p,q} = \frac{(\text{ker } d_r \ : E_r^{p,q} \to E_r^{p+r,q-r+1})}{(\text{im } d_r \ : E_r^{p-r,q+r-1} \to E_r^{p,q})}. \tag{4.13}$$

Clearly, d_0 is the same as d (we assume that p's increase along the horizontal axis in the figure above and q's along the vertical), and we can systematically construct the higher d_r's by starting with the original double complex.

The good news is that usually, the higher d_r's are identically zero after a few iterations, and the $E_r^{p,q}$'s become independent of r and we denote them by $E_\infty^{p,q}$. The main result of the spectral sequence method for us is then that the cohomology group H_D^n is given by,

$$H_D^n = \bigoplus_{p+q=n} E_\infty^{p,q}. \tag{4.14}$$

Notice that d and δ appearred symmetrically in the definition of the double complex (interchange of rows and columns), so our computation of H_D can be done starting with either. This gives us two ways of computing H_D and the result, depending on the choice of d and δ, can be different kinds of cohomology theories. Since there is only one H_D, this can be used as a way of showing the equivalence of various cohomology theories. See for example [15] for a demonstration of the equivalence between Dolbeault cohomology (the cohomology of the operator $\bar{\partial}$) and Čech cohomology using this idea.

4.3 Ext Groups

As promised, now we turn to Ext groups. They are defined between modules and we can use that definition to define Ext's between sheaves. The definition for modules can be made as follows. Let A, B be modules over a ring R, i.e., $A, B \in \text{Mod}_R$, the category of modules and their homomorphisms (module homomorphisms are defined analogous to vector space homomorphisms, with the field of scalars replaced by R). One fact we will need in defining Ext's for modules is that for any module, there always exists a ***free resolution***. A free resolution of a module M is defined as a possibly infinite exact sequence of the form

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M \longrightarrow 0, \tag{4.15}$$

where all the F_i are free modules. By slight abuse of a standard notation, we will sometimes denote the free resolution as

$$F(M) \longrightarrow M \longrightarrow 0. \tag{4.16}$$

Now the Ext groups between A and B, denoted $\mathbf{Ext}^{\mathbf{n}}_{\mathbf{R}}(A, B)$ is defined as the cohomology groups associated with the complex

$$Hom(F(A), B) \longrightarrow Hom(A, B) \longrightarrow 0, \tag{4.17}$$

where Hom denotes the morphisms in Mod_R. The maps in the complex between the various Hom's is determined from the maps in the resolutions in the standard way: if we start with a map $f : U \to V$, then the dual map $f^* : Hom(V, W) \to Hom(U, W)$ which takes $h \in Hom(V, W)$ to $f^* \circ h \in Hom(U, W)$ is defined by the condition that $(f^* \circ h)(u) = h(f(u))$ for each $u \in U$. It turns out that the Ext groups for modules computed according to our definitions so far is independent of the choice of the resolution.

It should be mentioned that a more standard definition of the Ext group for modules involves a *projective* (or *injective*, if one uses the dual definition) resolution and not a free resolution, but since we will be concerned with coherent sheaves which always have (locally) free resolutions, this will not be important to us.

We want to model D-branes, and D-branes are modeled by sheaves since they are a natural generalization of bundles. The spectrum of open string states (or analogously vertex operators) between D-branes then should be modelled by some natural vector spaces that can be constructed from sheaves. Ext groups provide such a natural definition of vector spaces that can be constructed from pairs of sheaves, and this is the basis for the claim that Ext groups capture open string excitations.

To go from the definition of Exts between modules to those between sheaves using first principles, we need what is called the local-to-global spectral sequence. A good reference for Ext groups constructions in the context of coherent sheaves is Griffiths and Harris [19].

Before explaining Ext groups for sheaves, we notice that using two coherent sheaves $\mathcal{E}$ and $\mathcal{F}$, we can construct a third sheaf, called a ***local Ext sheaf***, $\text{Ext}^n_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$, by starting with a locally-free resolution of $\mathcal{E}$:

$$0 \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{G}_{n-1} \longrightarrow ... \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{E} \longrightarrow 0. \tag{4.18}$$

Since a locally free resolution involves locally free sheaves (which are nothing but sheaves of $\mathcal{O}_X$-modules), we can work with the module definition of Ext groups and work patch by patch in X. Therefore it makes sense to define local Ext sheaves as the cohomology of the complex

$$0 \longrightarrow \underline{Hom}(\mathcal{G}_0, \mathcal{F}) \longrightarrow \cdots \longrightarrow \underline{Hom}(\mathcal{G}_n, \mathcal{F}) \longrightarrow 0. \tag{4.19}$$

The patch by patch definition of local Ext sheaves works because the $\underline{Hom}$'s denote morphisms in the category of sheaves which in turn are defined through homomorphisms between the Abelian groups (which are also $\mathcal{O}_X$-modules for locally free sheaves) defined over each open set of the sheaf (see section 4.1).

To get from the local Ext sheaf above to the ***global Ext group*** (or just Ext group for short) between the two sheaves, we use spectral sequences. The global Ext is obtained from the spectral sequence defined by

$$E_2^{p,q} = H^p(X, \mathrm{Ext}^q_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})), \tag{4.20}$$

as the sequence settles down to its convergent double complex. So,

$$E_\infty^{p,q} = \mathrm{Ext}_X^{p+q}(\mathcal{E}, \mathcal{F}). \tag{4.21}$$

The first principles computation of the global Ext groups as presented above is too complicated for most problems, so we will not describe the details on how to construct the differentials d_r in the spectral sequence computation. In the next section, we will introduce alternative methods which are useful for computing Ext groups and also demonstrate more generally that Ext groups do indeed reproduce open string spectra. In the rest of this section, as a demonstration of the first principles approach explained here, I will present a simple specific example where the computation of Ext groups can be done despite the technical details we have glossed over[21]. We will also find that the result agrees with the open string spectrum already known from the literature. This example is taken from [13].

The example we consider is the case where both $\mathcal{E}$ and $\mathcal{F}$ are bundles on X, which corresponds to the large radius limit where the usual description of branes is valid. By our earlier discussion, the locally free resolution of a bundle is trivial::

$$0 \longrightarrow \mathcal{E} \stackrel{=}{\longrightarrow} \mathcal{E} \longrightarrow 0$$

The cohomology of the exact sequence, $0 \longrightarrow \underline{Hom}\,(\mathcal{E}, \mathcal{F}) \longrightarrow 0$, gives us the local Ext sheaves:

$$\mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = \begin{cases} \underline{Hom}(\mathcal{E}, \mathcal{F}) = \mathcal{E}^* \otimes \mathcal{F} & n = 0 \\ 0 & n > 0 \end{cases}$$

[21] This happens because the spectral sequence happens to be trivial after the first iteration and therefore the higher differentials become irrelevant.

Global Ext's are then computed from the spectral sequence

$$E_2^{p,q} \;=\; H^p\left(X, \mathrm{Ext}^q\left(\mathcal{E},\mathcal{F}\right)\right) \implies \mathrm{Ext}_X^{p+q}\left(\mathcal{E},\mathcal{F}\right).$$

Since local Ext vanishes for $q > 0$, $E_2^{p,q} = 0$ for $q > 0$. All the differentials d_r map between different q for $r \geq 2$ and we end up with a trivial spectral sequence. Therefore,

$$\mathrm{Ext}_X^n\left(\mathcal{E},\mathcal{F}\right) \;=\; H^n\left(X, \mathcal{E}^* \otimes \mathcal{F}\right).$$

On the other hand, as we keep reiterating, Ext groups capture the open string states between two D-branes. In the present case these branes wrap all of X, with gauge bundles $\mathcal{E}$ and $\mathcal{F}$. So we expect that in the sense that the space of open string states should be the vector space $\bigoplus_n \mathrm{Ext}_X^n\left(\mathcal{E},\mathcal{F}\right)$. In fact, the open string spectrum in this case was computed directly by Witten [20], and it was found that the open string spectrum is given by $H^n(X, \mathcal{E}^* \otimes \mathcal{F})$. So we find that our result matches perfectly with the Ext group computation.

5 Complexes of Coherent Sheaves

The fact that solutions of the B-model are holomorphic submanifolds (or in a more fancy language, that the B-model localizes on holomorphic maps) suggests the following. As already mentioned, we expect an algebraic characterization to work for B-model objects. Since this is a statement from conformal field theory, it should hold even away from the large radius limit. This suggests that we should look for locally free sheaves as the more general description of "branes wrapping holomorphic submanifolds", because bundles are not naturally algebraic objects.

Another piece of information that we can use to sharpen our picture of D-branes is that we expect them to be describable locally by a set of equations. This translates to the statement that mathematically we expect our branes to be coherent sheaves.

In fact it turns out that the more natural mathematical object for describing branes in CFTs is complexes of coherent sheaves rather than sheaves themselves. The idea roughly is that the complex is to be thought of as a sequence of branes and anti-branes with the complex maps being modeled by tachyons stretched between them. We expand on this idea in this section: first we describe the prescription for fixing the grading of branes in the complex, and then we explain how the above picture in terms of tachyons and complexes of sheaves emerges. In the next section we will see that even these complexes are not quite the full story and we have to mod out by certain equivalences that exist between different complexes. This modded out complex is what is called the derived category.

5.1 Spectral Flow and Grading

The $N = 2$ superconformal algebra has a free paremeter (lets call it η, $0 < \eta < 1/2$) that characterizes the boundary conditions for the fermions. We can use this parameter as an indexing set for the algebra, and it turns out that the $N = 2$ algebras for different values of this parameter are isomorphic, and that the generators of the algebra for generic η can be written as linear combinations of the

$\eta = 0$ generators. The map that takes the states and the operators at $\eta = 0$ to those at generic η is called the ***spectral flow*** operator. We refer the reader to section 2 of [7] for a quick and clear discussion of spectral flow.

Since the NS and R sector of the string theory are nothing but different choices of the fermion boundary conditions, the existence of the spectral flow operator suggests that once we know one of these Hilbert spaces, the other can be obtained through spectral flow by 1/2. This means that worldsheet spectral flow can be interpreted as the spacetime supersymmetry operator.

In the case of closed strings there are worldsheet spectral flow operators that act both on the right and the left, and they are decoupled. But when one talks about open strings, then these two cease to be independent because there is mixing at the worldsheet boundary. If the D-brane at the boundary preserves half of the supersymmetry, then the linear combination of the spectral flow operators that vanishes at the boundary gives rise to the spacetime supercharge that is unbroken.

This suggests a definition for the ***grade*** of a D-brane. We can define the grade of the D-brane to be the phase difference between the left-moving and right-moving spectral flow operator on the open string worldsheet, at the boundary ending on the brane. The $N = 2$ algebra has a $U(1)$-current (the "R-current") as one of its generators, and it can be written as

$$J(z) = i\sqrt{\frac{c}{3}}\partial_z\phi \tag{5.1}$$

in terms of a free scalar[22]. Here c is the central charge of the algebra. It turns out that the spectral flow (by η) operator can explicitly be written down in terms of this ϕ as

$$U_\eta = \exp\Big(- i\sqrt{\frac{c}{3}}\eta\phi\Big) \tag{5.2}$$

From this definition, it is clear that the grade is $U(1)$-valued.

Usually, what one does is to define the phase, divided by π, to be the grade so that there is a shift by 2 gauge symmetry. Also, notice that an overall shift in grade of all the D-branes simultaneously is physically irrelevant, because only the phase difference is significant. Another important point is that if one shifts the grade by 1 (or an odd number), it corresponds to a shift in the phase by π, and the brane changes into an anti-brane because the opposite linear combination of supercharges is the one preserved at the boundary. In other words, shift by one in grade, together with an interchange of branes with antibranes is another way to implement the gauge symmetry mentioned above. If we have a set of branes that preserve spacetime supersymmetry, then from the above discussion it is clear that we can make the grades of all the branes even, and those of the anti-branes odd, by means of an physically irrelevant overall shift.

So far everything we have talked about is for physical string theory. But inspired by the above-mentioned notion of grading, we can define a notion of grading for (supersymmetric) D-branes in the topological B-model. We will do this by decomposing our collection of branes as:

$$E = \bigoplus_{n=-\infty}^{\infty} E^n \ . \tag{5.3}$$

[22] This is sometimes referred to as bonozization.

Since E is finite dimensional, only a finite number of the bundles E^n are nonzero. The index $n \in \mathbb{Z}$ defines the grading associated to the bundle E^n. The end of a given open string must be associated to a definite grading and hence a single summand E^n.

5.2 Deformations, Tachyons and the Graded Complex

One situation where we can see the emergence of the derived category is in the context of deformations of Witten's B-model. This scenario is sufficiently rich, yet sufficiently simple that we can explicitly see the emergence of the derived category as arising from the modified cohomological structure of the deformed theory.

The deformations that we will consider correspond to the addition of tachyon vertex operators to the theory. We give a vev to the tachyon between top-dimensional (which is six in the compactified situation we are considering) branes and antibranes in the open string B model. In this section we follow closely the work of [22].

We deform the action by a BRST closed boundary piece:

$$\delta_\varphi S = \int_{\partial\Sigma} \{G, \varphi\} \tag{5.4}$$

G here is the "other" supersymmetry charge, the one that we suppressed when we wrote down the BRST variations of the B-model earlier. It is the generator of the fermionic $U(1)$ current. The $\mathcal{N} = 2$ algebra dictates that $\{G, Q\} \sim \partial_\tau$ (the worldsheet Hamiltonian). We will denote the BRST charges of the theory, before and after the deformation, by Q_0 and Q. The new BRST operator Q can be determined by letting the variation parameter α defined in section 3.2 to have a boundary coordinate (τ) dependence and doing the Noether procedure. The result is (there is an integration by parts which results in contributions only from the boundary):

$$Q = Q_0 + \varphi|_{\sigma=0} - \varphi|_{\sigma=\pi}. \tag{5.5}$$

This can also be seen more directly by demanding that BRST exact correlation functions are zero: $\langle\{Q, V\}\rangle = 0$ results in the same expression above upon moving the BRST operator through the deformation piece in the action.

Since we want the new BRST charge to have ghost number 1, we need to impose that φ has ghost number 1. It turns out that all the relevant deformations are captured by operators $\varphi \in \mathrm{Ext}^0(\mathcal{E}^n, \mathcal{E}^n)$. This follows from the assignment of grading and ghost number for the various operators stretched between the graded branes of the last subsection. It turns out that all the other operators of ghost number 1 add nothing really new to our final result. Since $\mathrm{Ext}^0(\mathcal{E}^n, \mathcal{E}^n) = \mathrm{Hom}(\mathcal{E}^n, \mathcal{E}^{n+1})$ this means that our deformations are described by

$$\varphi_n : \mathcal{E}^n \to \mathcal{E}^{n+1} \tag{5.6}$$

The nilpotency of the modified BRST operator gives us one very important condition: namely that

$$\varphi_{n+1}\varphi_n = 0, \quad \forall n. \tag{5.7}$$

This also guarantees that the theory is topological. So now we see that our deformed theory is described by a *complex* of locally free sheaves,

$$... \xrightarrow{\varphi_{-2}} \mathcal{E}^{-1} \xrightarrow{\varphi_{-1}} \mathcal{E}^{0} \xrightarrow{\varphi_{0}} ... \tag{5.8}$$

This complex, denoted $\mathcal{E}^*$, is *bounded* because the original vector bundle was finite-dimensional.

Since the deformation is not neutral under the $U(1)_R$ charge, this naturally ties in with the fact that the open strings (i.e., the fields φ) stretch between branes with relative charge 1. This is as expected from the discussion of the previous section, and the physical intuition that the adjacent elements are branes and antibranes, and that the complex is graded.

From all these conditions imposed by worldsheet $\mathcal{N} = 2$ boundary supersymmetry, we see how the tachyon vevs are related to graded complexes of coherent sheaves: the composition of neighboring tachyons vanishes, (i.e., the composition of adjacent maps in the complex must vanish), and each tachyon results in a shift of the $U(1)_R$ charge by one unit (grading of the complex). Therefore we come to the conclusion that D-brane boundary states are nothing but graded complexes of coherent sheaves - the one extra subtlety is that we need to mod out by a large number of equivalences, and then we will have the derived category.

In the context of complexes of sheaves, the modified BRST operator results in a new cohomology. We start with D-branes that wrap the entire Calabi-Yau since they are nothing but locally free sheaves. To obtain D-branes on submanifolds, we can consider their resolutions. Indeed, it turns out that the cohomology of the modified BRST operator, acting on differential forms on X valued in the bundles appearing in the locally-free resolutions, gives us precisely the Ext groups. So Ext groups are precisely the natural objects arising from the cohomology of graded complexes. The basic statement is that for a complex $\mathcal{E}^*$ generated as the locally-free resolution of a sheaf $\mathcal{E}$,

$$\mathrm{Ext}^n_X\left(\mathcal{E}^*, \mathcal{F}\right) = \mathrm{Ext}^n_X\left(\mathcal{E}, \mathcal{F}\right). \tag{5.9}$$

If the complexes are resolutions of sheaves, then our BRST cohomology calculation computes the Ext groups between the complexes (which is the same as the Ext groups for the sheaves).

We are almost at the stage of derived categories: we describe the equivalences that we need to mod out the complex by, in the next section.

6 The Derived Category

When the complexes of the previous section are modded out by (1) chain homotopies and (2) quasi-isomorphisms, what we end up with is called the derived category. The objects in this category are therefore equivalence classes of complexes.

We described chain maps in section 2.1[23], as maps between complexes such that they commute in an obvious way with the maps (differentials) in the complex. It is easy to check that for a map f between complexes, imposing that it is consistent with BRST closure implies two conditions (this is analogous to the conditions arising on the modified BRST operator Q in the last section). One is

[23]We follow the same notation here.

that f is holomorphic, and the other is that it is a chain map. The latter condition can be written as,

$$0 = \varphi_{\mathcal{B}} \ o \ f_i - f_{i+1} \ o \ \varphi_{\mathcal{A}}, \quad \forall i. \tag{6.1}$$

This is the condition for closure, and now we want to mod out our graded complex by BRST exact maps. In the language of chain maps, an exact map f will be such that

$$f = \varphi_{\mathcal{B}} \ o \ g - g \ o \ \varphi_{\mathcal{A}}, \tag{6.2}$$

where g is a map from the complex $\mathcal{A}$ to complex $\mathcal{B}$. A map of this form is called a chain homotopy, and this is the rationale for modding out by chain homotopies on our way to the derived category.

The idea of the quasi-isomorphism is more tricky. To explain it fully, we really need to get into the bowels of the derived categories programme, which we will not do. Instead we settle here for a qualitative explanation. Quasi-isomorphisms are (roughly) chain maps that induce isomorphisms at the level of sheaf cohomology, and we need to identify complexes that have isomorphic cohomology sheaves. This statement is not quite correct, but to make them more precise directly is subtle. One problem is that we don't fully understand the physical universality classes of D-branes. Aspinwall and Lawrence define the "physical equivalence" of complexes, by introducing a test complex. That is, two complexes are called physically equivalent if they have the same spectrum with any arbitrary (but fixed) test complex. The equivalence classes of complexes under these test branes, gives us a handle on the derived category.

One of the details we glossed over was that when we spoke about maps in the complex, was that we spoke only about maps in the neighboring elements in the complex. This ties in with the fact that when we wanted ghost number 1 operators for φ, we took them to be elements of $\mathrm{Ext}^0(\mathcal{E}^n, \mathcal{E}^{n+1})$. But we could have equally well taken elements of $\mathrm{Ext}^1(\mathcal{E}^n, \mathcal{E}^n), \mathrm{Ext}^2(\mathcal{E}^n, \mathcal{E}^{n-1}), ...$ and they would have had unit ghost number too, according to our grading of branes. The reason why we stuck to the simpler case is because we wanted to give a physical motivation for the origin of ordinary complexes. In fact, it turns out that considering more general Ext groups elements leads to the idea of a 'generalized complex' (with more arrows between elements), but this also ultimately gives us derived categories.

So finally after a long, circuitous and *still* somewhat handwaving route, we have reached the promised interpretation that D-branes correspond to objects in the derived category. This is of course just the beginning. We will not undertake the multitude of questions that can and need to be addressed in this framework. Our focus has only been to give a flavor of this subject without getting into too much detail. For those the reader will find the reviews cited in the introduction a useful navigational tool.

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