Algebraic Topology Lecture Notes

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Abstract

We classify finitely generated abelian groups and, using simplicial complex, describe various groups that can be associated to manifolds, such as homotopy, homology and cohomology. We present some theorems that are useful for calculating these groups, like the van Kampen theorem, the Mayer-Vietoris sequence, the universal coefficient theorem, Poincaré duality and the Künneth formula. We mention circle bundles and characteristic classes.

Prepared for the Third Modave Summer School in Mathematical Physics
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1 Introduction

A manifold is a collection of subsets of the Euclidean space $\mathbb{R}^n$ glued together using transition functions. Sometimes there is additional structure, like a metric, vector of spinor fields or a symplectic or complex structure. Subsets of $\mathbb{R}^n$ come with coordinates, and this coordinate-based description is awkward to work with, for example, the same manifold can be presented using two entirely different coordinate descriptions. Also, while locally differential geometry lets you extract information about, for example, the curvature, global properties are encoded nonlocally in the transition functions.

Global information is important. For example, global data can tell you that it is impossible to define a spinor on a certain space, and so a given spacetime cannot be inhabited by fermions. In general there are both local and global obstructions, with the local obstructions determined by differential geometry and the global obstructions determined by nonlocal information from the transition functions. Also integers, like the rank of the gauge group of number of generations of matter than arise in a dimensional reduction are encoded entirely in global features of a manifold, and are independent of deformations of the coordinates or even the metric.

While the global information is difficult to extract from the transition functions, it is encoded in a series of groups, rings and modules which, once calculated, are often easily manipulated. The association of global information about manifolds to these algebraic structures is known as algebraic topology. There are similar algebraic structures that incorporate some local information, about for example the complex structures. Those structures are the subject of algebraic geometry, and will appear in many of the other series of lectures at this school, such as Chethan’s lectures on derived categories.

In these lectures we will introduce several of the most common groups studied in algebraic topology. We will begin with the fundamental group, which is a generally nonabelian group which classifies all of the loops in a manifold. Then we will introduce a series of groups called homology groups, which roughly classify the submanifolds. Finally we will introduce cohomology, which is a ring, and we will see that homology is a module of cohomology. If there is time we will describe certain manifolds which are called circle bundles, and we will see that they are classified by an element of cohomology known as the Chern class. We will introduce several computational tools,
such as exact sequences, the Mayer-Vietoris sequence, the van Kampen theorem, the universal coefficient theorem, the Künneth formula and Poincaré duality.

Many excellent textbooks on algebraic topology exist which cover much more ground and detail than you will find here. For more background on general topology, one can consult e.g. [1]. Most of what is covered in these notes (and much more) can be found in [1] - [3].

2 Group theory

Algebraic topology associates groups, and elements of groups, to manifolds. Some of these groups have additional structure, and are called rings or modules. A group is a set of elements together with a rule for how to multiply two elements $a$ and $b$ in the set to get a third element $c$

$$a \ast b = c. \quad (2.1)$$

There must be a special element, called the identity $e$, which when multiplied any given element $a$ gives back $a$

$$a \ast e = e \ast a = a. \quad (2.2)$$

In addition each element $a$ must have an inverse $b$ such that

$$a \ast b = b \ast a = e. \quad (2.3)$$

If the multiplication $\ast$ is commutative

$$a \ast b = b \ast a \quad (2.4)$$

for every pair of elements $a$ and $b$ then the group is said to be abelian, otherwise it is said to be nonabelian. In the abelian case $\ast$ is often called addition, by analogy with the fact that the addition of numbers is abelian

$$a + b = b + a. \quad (2.5)$$

An example of an abelian group is the group $\mathbb{Z}$ of integers. The group multiplication law is addition and the identity is zero. Ordinary multiplication cannot be used as the group multiplication law because only $\pm 1$ have integral inverses. Similarly the rational and real numbers form groups $\mathbb{Q}$ and $\mathbb{R}$. However the nonnegative numbers do not form a group, as they have no inverses. The positive numbers doubly fail
to form a group as they also lack an identity element. Addition modulo a natural number $N$ also forms an abelian group $\mathbb{Z}_N$. For example, the elements of $\mathbb{Z}_3$ are 0, 1 and 2 and the group multiplication is just

$$0 + 0 = 1 + 2 = 2 + 1 = 0, \quad 0 + 1 = 1 + 0 = 2 + 2 = 1, \quad 0 + 2 = 2 + 0 = 1 + 1 = 2.$$ (2.6)

The integral lattice $\mathbb{Z}^N$ in the Euclidean space $\mathbb{R}^N$ is also an abelian group, whose identity is the origin and whose group multiplication is translation, as is $\mathbb{Q}^N$ and even $\mathbb{R}^N$ itself, or any combination $\mathbb{Z}^J \oplus \mathbb{Q}^K \oplus \mathbb{R}^{N-J-K}$ in which different lattices are taken in different directions. More generally, if $G$ and $H$ are groups, then $G \oplus H$ is the group of pairs $(g, h)$ where $g \in G$ and $h \in H$ using the multiplication rule

$$(g_1, h_1) \ast (g_2, h_2) = (g_1 g_2, h_1 h_2).$$ (2.7)

An example of a nonabelian group is the dihedral group $D_3$ of symmetries of an equilateral triangle, which also appears as a coxeter group in Daniel and Nassiba’s lectures. This group consists of rotations by 120 degrees $r$, by 240 degrees $r^2$, reflections $s$ and compositions of rotations and reflections. Notice that three rotations by 120 degrees yields the identity

$$r^3 = e$$ (2.8)
as do two reflections

$$s^2 = e$$ (2.9)
and that rotations and reflections do not quite commute

$$rs = sr^2$$ (2.10)

which is why the group is nonabelian. Using these three rules we can construct the entire group multiplication table

From the table one can see that each element has a unique inverse.

We described $D_3$ as the group whose elements are compositions of powers of $r$ and $s$, and in general their inverses, constrained by 3 relations (2.8,2.9,2.10). Any group may be presented in this way, although not uniquely. The elements whose compositions form the group are called generators, and many of the groups, such as integral (co)homology that we will describe which are associated to compact spaces can always be defined using a finite set of generators. For example, the group $\mathbb{Z}$ of
Table 1: Group multiplication table for $D_3$

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</table>

Integers is generated by the element 1, or alternately it is generated by $-1$. It is also generated by any two relatively prime numbers, so the number of generators depends on the choice of generators. $\mathbb{Z}^N$ can be generated by only $N$ elements of the lattice, for example the $N$ elements whose coordinates are all 0 except for a single 1. On the other hand the groups $\mathbb{Q}$ and $\mathbb{R}$ require an infinite number of generators.

A group which can be generated by a finite number of elements is called a finitely generated group. For example, $D_3$, $\mathbb{Z}_N$ and $\mathbb{Z}^N$ are finitely generated, while $\mathbb{Q}$ and $\mathbb{R}$ are not finitely generated. Finitely generated abelian groups have been completely classified, they are all of the form

$$G = \mathbb{Z}^r \oplus_i \mathbb{Z}_{p_i^{n_i}}.$$  \hfill (2.11)

$r$ is a nonnegative integer called the rank of the group, and $\mathbb{Z}^r$ is called the free part, while the sum of the finite cyclic groups $\mathbb{Z}_{p_i^{n_i}}$ is called the torsion part.

You might wonder why the finite cyclic groups all have orders equal to powers of primes. The reason is that any finite cyclic group can be decomposed as the sum of groups of orders that are sums of powers of primes. For example,

$$\mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3.$$ \hfill (2.12)

where the elements of $\mathbb{Z}_6$ are written in terms of $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ elements as follows

$$0 = (0, 0), \quad 1 = (1, 2), \quad 2 = (0, 1), \quad 3 = (1, 0), \quad 4 = (0, 2), \quad 5 = (1, 1).$$ \hfill (2.13)

A ring is an abelian group where, in addition to addition, there is a multiplication rule such that any element multiplied by $e$ gives $e$. A ring is called abelian if the multiplication is abelian. The groups $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are all rings, where the multiplication rule is just ordinary multiplication.
3 The fundamental group

One of the main objectives of topology is to classify spaces up to continuous deformations, i.e. up to homeomorphism. Unfortunately, in general it is a quite difficult problem to show when two spaces are not homeomorphic. A classification which is usually easier to obtain is based on the rougher notion of homotopy. Given two spaces $X$ and $Y$, two continuous maps $f, g : X \to Y$ are called homotopic if there exists a continuous map $F : X \times I \to Y$, where $I = [0, 1]$, for which:

$$F(x, 0) = f(x), \quad F(x, 1) = g(x), \quad \forall x \in X. \tag{3.1}$$

The map $F$ is called a homotopy and continuously deforms the map $f$ into the map $g$. It’s easy to show that this defines an equivalence class of maps, the equivalence classes being called homotopy classes of maps $[f]$. We denote this by,

$$[f] = [g] \text{ iff } f \sim g. \tag{3.2}$$

Given $f : X \to Y$ and $g : Y \to X$, $f$ and $g$ define a homotopy equivalence if $g \circ f \sim \text{Id}_X$ and $f \circ g \sim \text{Id}_Y$. As the name suggests, this at its turn defines equivalence classes of spaces. Two spaces which are in the same equivalence class are said to be of the same homotopy type. Clearly, two homeomorphic spaces are always of the same homotopy type, but the reverse is generically not true. Although this thus results in a rougher classification, it still turns out to be quite nontrivial and very useful. The notion of homotopy classes of maps is also the starting point of the most easily defined algebraic structure one can associate with a topological space called the fundamental group, to which we now turn.

3.1 Definitions

We now specialize to the classification of maps from the closed interval $I$ to a space $X$. A path in $X$ is defined as a continuous map $f : I \to X$ with initial point $x_0 = f(0)$ and final point $x_1 = f(1)$. A path homotopy between two paths $f$ and $f'$ is a homotopy $F : I \times I \to X$ which keeps the initial and final points fixed. That is,

$$F(s, 0) = f(s), \quad F(s, 1) = f'(s) \quad \forall s \in I \tag{3.3}$$

$$F(0, t) = x_0, \quad F(1, t) = x_1 \quad \forall t \in I \tag{3.4}$$
where \( x_0 = f(0) = f'(0) \) and \( x_1 = f(1) = f'(1) \). The homotopy classes of paths can almost be given a group structure by first defining a product of paths. Given two paths \( f \) and \( g \) with \( f(1) = g(0) \), we define the product \( f \ast g \) by,

\[
(f \ast g)(s) = \begin{cases} 
  f(2s) & \text{for } s \in [0, \frac{1}{2}] \\
  g(2s - 1) & \text{for } s \in [\frac{1}{2}, 1]
\end{cases}
\]  

(3.5)

In other words, \( f \ast g \) is defined by first going through \( f \) and then through \( g \). This induces a product on homotopy classes of paths by

\[
[f] \ast [g] \equiv [f \ast g],
\]  

(3.6)

which is well defined because if \( F \) is a homotopy relating \( f \) to \( f' \) and \( G \) a homotopy relating \( g \) to \( g' \), then

\[
H(s,t) = \begin{cases} 
  F(2s,t) & \text{for } s \in [0, \frac{1}{2}] \\
  G(2s - 1, t) & \text{for } s \in [\frac{1}{2}, 1]
\end{cases}
\]

(3.7)

continuously relates \( f \ast g \) to \( f' \ast g' \). The problem is that this product of classes does not satisfy all relations needed for a group structure. Namely, if \( f(1) \neq g(0) \), the product \([f] \ast [g]\) is not even defined.  

This is easily solved by only looking at paths for which initial and final points coincide, \( f(0) = f(1) \equiv x_0 \), which will be called loops with base point \( x_0 \). Now for any two loops with the same base point, their product is defined. Furthermore, the space of homotopy classes of loops with base point \( x_0 \) together with the product defined above forms group called the fundamental group \( \pi_1(X, x_0) \). The identity element is the homotopy class of the constant path \( c_0 \), for which \( c_0(s) = x_0 \) for all \( s \). The elements in this class are called nullhomotopic. Given the class \([f]\) represented by some loop \( f \), its inverse element is the class of \( \bar{f} \), defined by \( \bar{f}(s) = f(1 - s) \). Summarizing, we have,

\[
[f] \ast [\bar{f}] = [\bar{f}] \ast [f] = [c_0].
\]  

(3.8)

Finally, the product is also associative

\[
([f] \ast [g]) \ast [h] = [f] \ast ([g] \ast [h]).
\]

(3.9)

Although it takes some time to prove these assertions rigorously (see e.g. [1] or [2]), they seem intuitively clear.

\[\text{An algebraic structure satisfying all group axioms but the one which asserts that for any } f \text{ and } g \text{ their product is defined, is called a groupoid.}\]
Whenever $X$ is path connected, given any other base point $x_1$, it’s easily shown that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$. This is because for any $x_1$ there always exists a path $\alpha$ from $x_0$ to $x_1$. This yields a well-defined map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$, given by
\[
\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].
\] (3.10)
This is clearly a loop at $x_1$. The map $\hat{\alpha}$ is easily shown to be a homomorphism. That this is also an isomorphism follows from the fact that the inverse loop $\bar{\alpha}$ induces an inverse map. This implies that we can simply speak of the fundamental group of a path connected space $X$ without referring to its base point. Some care is in order when changing base point however, because the isomorphism depends on the path $\alpha$.

That the fundamental group provides us with topological information about $X$ is already clear from the elementary observation that if $X$ is simply connected – i.e. if every loop in $X$ is contractible (and if $X$ is path connected to begin with) – its fundamental group is the trivial one element group consisting only of the class of nullhomotopic loops. The full power of the fundamental group is however much more refined. To show this we introduce the important notion of an induced homomorphism. Suppose that $\varphi : X \to Y$ maps the base point $x_0$ to $y_0$. This is denoted as
\[
\varphi : (X, x_0) \to (Y, y_0).
\] (3.11)
This map induces a homomorphism
\[
\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0),
\] (3.12)
by the equation,
\[
\varphi_*([f]) = [\varphi \circ f],
\] (3.13)
where $f$ is a path in $X$. If $F$ is a homotopy between $f$ and $f'$, $\varphi \circ F$ is a homotopy between $\varphi \circ f$ and $\varphi \circ f'$, so that the induced homomorphism $\varphi_*$ is well-defined. That it is indeed a homomorphism follows from the fact that $(\varphi \circ f) * (\varphi \circ g) = \varphi \circ (f * g)$.

Two basic but important properties of the induced homomorphism are that, given $\varphi : X \to Y$ and $\psi : Y \to Z$,
\[
(\psi \circ \varphi)_* = \psi_* \circ \varphi_*,
\] (3.14)
and that when $\text{Id} : X \to X$ is the identity map, then $\text{Id}_*$ is the identity homomorphism. These properties turn the fundamental group into a covariant functor from the category of topological spaces and continuous maps to the category of groups and
homomorphisms. More importantly for our purposes, they immediately imply that when \( \varphi : (X, x_0) \to (Y, y_0) \) is a homeomorphism, \( \varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \) is an isomorphism. That is, the fundamental group is a topological invariant. It can in fact be shown that the fundamental group cannot distinguish between spaces of the same homotopy type [2].

### 3.2 Covering spaces

Although the fundamental group is perhaps the most straightforward construction of algebraic topology, calculating it for specific spaces is not that straightforward in general. One piece of machinery which can be quite useful in this regard uses the concept of a covering space.

Let \( p : \tilde{X} \to X \) be a continuous surjective map. If there exists an open cover \( \{U_\alpha\} \) of \( X \) such that for each \( \alpha \), \( p^{-1}(U_\alpha) \) is the disjoint union of open sets in \( \tilde{X} \), each of them mapped homeomorphically onto \( U_\alpha \) by \( p \), then \( p \) is called a covering map and \( \tilde{X} \) is called a covering space of \( X \). Clearly, covering spaces are not unique. A very trivial example is to take for \( \tilde{X} \) simply \( n \) disjoint copies of \( X \) and \( p \) such that its restriction to one component of \( \tilde{X} \) is the identity map. To exclude this rather trivial possibility, one usually restricts to path connected covering spaces.

A less trivial example is the map \( p : \mathbb{R} \to S^1 \) from the real line to the circle, given by

\[
p(x) = (\cos 2\pi x, \sin 2\pi x)
\]

It is easily checked that the inverse image of each open proper subset of \( S^1 \) is indeed a disjoint union of (a countably infinite set of) open intervals in \( \mathbb{R} \). Restricted to any of these open sets, \( p \) is a homeomorphism. Given two covering maps \( p : \tilde{X} \to X \) and \( p' : \tilde{X}' \to X' \), one can also show that

\[
p \times p' : \tilde{X} \times \tilde{X}' \to X \times X'
\]

is a covering map. For instance \( \mathbb{R}^n \) is a covering space of the \( n \)-torus \( T^n \). We will now explain how to use such covering maps to compute fundamental groups.

Given some other space \( Y \) and a map \( f : Y \to X \), the map \( \tilde{f} : Y \to \tilde{X} \) is called a lift of \( f \) if \( p \circ \tilde{f} = f \), i.e. if the diagram
commutes. The nice thing about covering spaces is that any path in \( X \) has a unique lift starting at a given point of \( \tilde{X} \) in the preimage of \( x_0 \). More precisely, given a path \( f : I \to X \) starting at \( x_0 \) and a point \( \tilde{x}_0 \) such that \( p(\tilde{x}_0) = x_0 \), one can show that there is a unique lift to a path \( \tilde{f} : I \to \tilde{X} \) beginning at \( \tilde{x}_0 \). Furthermore, if \( f \) and \( f' \) are path homotopic, their lifts \( \tilde{f} \) and \( \tilde{f}' \) are also path homotopic and will end at the same point, i.e. \( \tilde{f}(1) = \tilde{f}'(1) \). If \( f \) is a loop in \( X \) at \( x_0 \), it is clear that this common endpoint is an element of \( p^{-1}(x_0) \). This means that the set map

\[
\phi : \pi_1(X, x_0) \to p^{-1}(x_0),
\]

where \( \phi([f]) = \tilde{f}(1) \), is well-defined. Since \( \tilde{X} \) is path connected (by choice), this map is surjective. This is because given any \( \tilde{x}_1 \in p^{-1}(x_0) \), there exists a path \( \tilde{f} \) from \( \tilde{x}_0 \) to \( \tilde{x}_1 \). Then \( f = p \circ \tilde{f} \) is a loop at \( x_0 \) for which \( \phi([f]) = \tilde{x}_1 \) by definition. If furthermore \( \tilde{X} \) is simply connected, the map is bijective. This again easily follows from previous statements: given \( [f] \) and \( [g] \) such that \( \phi([f]) = \phi([g]) \), it follows that \( \tilde{f}(1) = \tilde{g}(1) \) (and of course they both start at \( \tilde{x}_0 \)). Since \( \tilde{X} \) is simply connected, there is a homotopy \( \tilde{F} \) between the two. Then \( F = p \circ \tilde{F} \) is a homotopy between \( f \) and \( g \), implying \( [f] = [g] \).

This allows us to easily compute the fundamental group for some simple examples. Take for instance the covering map (3.15) from the real line to \( S^1 \). Take \( \tilde{x}_0 = 0 \) and let \( x_0 = p(0) = (1,0) \). Then \( p^{-1}(x_0) \) is the set \( \mathbb{Z} \). Since \( \mathbb{R} \) is simply connected, this implies that the lifting correspondence,

\[
\phi : \pi_1(S^1) \to \mathbb{Z}
\]

is bijective. It’s not very difficult to show that this map is in fact a homomorphism. This then proves that,

\[
\pi_1(S^1) \cong \mathbb{Z}.
\]

Another example is

\[
p : SU(2) \to SO(3),
\]

\(^2\)The reason for this is that the path from \( \tilde{f}(1) \) to \( \tilde{f}'(1) \) is the unique lift of the constant path at \( f(1) \) which is necessarily constant.
where the inverse image of any point of \( SO(3) \) is a two element set. Since again \( SU(2) \) is simply connected, the same line of argument implies that,

\[
\pi_1(SO(3)) = \mathbb{Z}_2.
\] (3.21)

This is indeed also an isomorphism between groups. Viewing \( SO(3) \) as the three dimensional ball with opposite points on its bounding 2-sphere identified, the non-trivial element of \( \mathbb{Z}_2 \) is identified with a path starting at one point on the boundary and ending at the opposite point (which is identified under the covering map). In more rough terms, the previous conclusions can be stated as follows. Given a simply connected space \( \tilde{X} \) and a quotient space \( X = \tilde{X}/\Gamma \), where \( \Gamma \) is a discrete group which acts freely on \( \tilde{X} \), then \( \pi_1(X) = \Gamma \).

The covering map (as any continuous map) induces a homomorphism

\[
p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0).
\] (3.22)

Since any nullhomotopic loop in \( X \) lifts to a nullhomotopic loop in \( \tilde{X} \), this map is injective. This roughly means that the fundamental group of a covering space can never be bigger than the fundamental group of the original space. In fact, a more refined statement then the ones previously made, is that

\[
\pi_1(X, x_0)/H \cong p^{-1}(x_0),
\] (3.23)

where \( H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \). This means that \( \phi([f]) = \phi([g]) \) if and only if \( [f] \in H \ast [g] \). For \( [g] \) homotopic to the constant loop, this implies that \( \phi([f]) = \tilde{f}(1) = \tilde{x}_0 = \tilde{f}(0) \) iff \( [f] \in H \). So elements of \( H \) are exactly those loops in \( X \) which lift to loops in \( \tilde{X} \). The other (non-equivalent) loops of \( \pi_1(X, x_0) \) lift to paths ending at other points of \( \tilde{X} \), which indeed exactly make up \( p^{-1}(x_0) \).

These considerations lead to the intuitive picture that there exists a hierarchy of covering spaces. At each step in the hierarchy the fundamental group becomes smaller until a simply connected covering space is reached. This simply connected covering space is called the universal covering space of \( X \), since it is a covering space for all other covering spaces of \( X \). This intuition can be made precise. Fundamental groups can even be used to classify covering spaces. In this classification, two covering spaces \( p : \tilde{X} \to X \) and \( p' : \tilde{X}' \to X \) are considered equivalent if there exists a homeomorphism \( h : \tilde{X} \to \tilde{X}' \) such that \( p = p' \circ h \).
The above intuition is made precise by the following classification theorem for covering spaces: Given two covering spaces \( \tilde{X} \) and \( \tilde{X}' \) and \( p(\tilde{x}_0) = p'(\tilde{x}'_0) = x_0 \), the covering spaces are equivalent if and only if the subgroups

\[
H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)), \quad H' = p'_*(\pi_1(\tilde{X}', \tilde{x}'_0)),
\]

of \( \pi_1(X, x_0) \) are conjugate [2].

For example, since the fundamental group of the circle is abelian, its subgroups are conjugate iff they are equal. This implies that the subgroups of \( \mathbb{Z} \) classify all covering spaces of \( S^1 \) up to equivalence. These are all isomorphic to \( n\mathbb{Z} \) for some strictly positive \( n \). The case \( n = 1 \) corresponds to the real line as we discussed before. For the other cases, consider the map

\[
p : S^1 \to S^1 : z \mapsto z^n,
\]

(3.25)

where \( z \) is a unimodular complex number. This can be shown to be a covering map. In this case, \( p \) clearly sends a generator of \( \pi_1(S^1, \tilde{z}_0) \) to \( n \) times itself, so that this covering space corresponds to the subgroup \( n\mathbb{Z} \). This shows that all covering spaces of \( S^1 \) are equivalent to either \( \mathbb{R} \) or \( S^1 \) together with the map (3.25).

### 3.3 The van Kampen theorem

Up till now we have only seen examples where the fundamental group turns out to be abelian. This is however not the case in general. For example, consider the figure eight space \( \infty \), with the point common to both circles as base point. Let the loops around the left circle be generated by the loop \( a \) and the loops around the right circle by the loop \( b \). A general loop starting and ending at the base point takes the form of a sequence of powers of \( a \) and \( b \), e.g. \( a^2b^{-3}ab^4a^{-1} \) and \( abab \) are just two examples of representations of loops in the figure eight space. This is an example of a non-abelian free product of two copies of \( \mathbb{Z} \), which we will denote by \( \mathbb{Z} \ast \mathbb{Z} \).

In general, given a finite set of groups \( G_\alpha \), one forms the free product \( \ast_\alpha G_\alpha \) as follows. Any element of the free product is a word made out of the generators of each
Generators coming from groups with different $\alpha$ do not commute or interact, so one can only simplify the word by performing group operations on consecutive generators of the same group. A word which can not be simplified any further is called reduced. For example, for the figure eight space the word $aa^{-1}b$ is not reduced, its reduced form being $b$. The product is defined by juxtaposing the two words and if possible reducing it by ‘local’ group operations. For instance, the product of the two words displayed above is $a^2b^{-3}ab^4a^{-1}abab \rightarrow a^2b^{-3}ab^5ab$. Clearly, every element has an inverse and the product is associative. The identity element (which necessarily equals the identity element of every $G_\alpha$) is simply the ‘empty’ word.

A basic property of free products is that any collection of homomorphisms $\varphi_\alpha : G_\alpha \to H$ extends uniquely to a homomorphism $\varphi : *_\alpha G_\alpha \to H$. The value of $\varphi$ on a word $g_1 \ldots g_n$, where $g_i \in G_{\alpha_i}$ for some $\alpha_i$, is simply

$$\varphi(g_1 \ldots g_n) = \varphi_{\alpha_1}(g_1) \ldots \varphi_{\alpha_n}(g_n).$$

Since each $\varphi_\alpha$ is a homomorphism, the value of $\varphi$ on a word does not depend on whether it is reduced or not. That $\varphi$ is a homomorphism also follows immediately. This is called an extension property for free products of groups.

To find the fundamental group of the figure eight space, we more or less guessed the result from our knowledge of the fundamental group of the parts which make it up, namely two circles. This intuition is made precise and is generalized considerably by the van Kampen theorem (sometimes also referred to as the Seifert-van Kampen theorem). Suppose that $X = \bigcup_\alpha A_\alpha$, where each $A_\alpha$ is open, path connected and contains the base point $x_0$. The inclusion $A_\alpha \hookrightarrow X$ induces the homomorphism $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$. From the extension property for free products it follows that there is a homomorphism

$$\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X).$$

The first assertion of the van Kampen theorem is that if each intersection $A_\alpha \cap A_\beta$ is path connected, this map is surjective. But even more can be said. Indeed, denoting the homomorphisms induced by $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ by $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$, it is clear that we should have $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$. From this it follows that the kernel of $\Phi$ at least contains elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$. The second part of the van Kampen theorem says that under mild conditions this in fact makes up the total kernel of $\Phi$. More precisely ([1] [2]), if in addition to the previous conditions each $A_\alpha \cap A_\beta \cap A_\gamma$ is also path connected, the kernel of $\Phi$ is the least normal
subgroup $N$ generated by elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$. In other words, $\Phi$ induces an isomorphism,

$$\pi_1(X) \cong \ast_\alpha \pi_1(A_\alpha)/N.$$  

(3.26)

As a first example, we compute the fundamental group of the wedge sum (or reduced join) of a finite number circles. In general, given two spaces $X$ and $Y$ and two points $x \in X$ and $y \in Y$, the wedge sum of $X$ and $Y$ (with respect to $x$ and $y$) is the quotient $X \vee Y$ of the disjoint union of $X$ and $Y$ obtained by identifying $x$ and $y$ to a single point. For example, the figure eight space from the beginning of this subsection is nothing but $S^1 \vee S^1$. In the same way, one defines the wedge sum of a collection of spaces $\{X_\alpha\}$ as the quotient of their disjoint union obtained by identifying the set $\{x_\alpha\}$, where $x_\alpha \in X_\alpha$, to a single point. Now let each $X_\alpha$ be a circle $S^1_\alpha$ and let $a_\alpha \subset S^1_\alpha$ be an open arc containing the common base point $x_0$ for each $\alpha$. A neighborhood of $S^1_\alpha$ is $A_\alpha = S^1_\alpha \cup_{\beta \neq \alpha} a_\beta$ and the intersection of two or more such neighborhoods is always simply $\cup_\alpha a_\alpha$, which is of the same homotopy type as a point and therefore simply connected. By the van Kampen theorem, it follows that we have the isomorphism,

$$\pi_1\left(\bigvee_\alpha S^1_\alpha\right) \cong \ast_\alpha \pi_1(S^1_\alpha) \cong \ast_\alpha \mathbb{Z}.$$  

(3.27)

This shows that the fundamental group of the figure eight space is indeed $\mathbb{Z} \ast \mathbb{Z}$, as we guessed at the beginning of this subsection. Under some mild conditions it is true in general that $\pi_1(\bigvee_\alpha X_\alpha) = \ast_\alpha \pi_1(X_\alpha)$, so that the fundamental group of a wedge sum of spaces is quite generally simply the free product of the fundamental groups of the individual spaces.

From the van Kampen theorem it is clear that in general the fundamental group is not simply a free product. The deviation from a free product is encoded in the normal subgroup $N$ we mentioned before. The quotient by this subgroup can always be represented by imposing a set of relations between generators of the ‘subgroups’ of the free product. Unfortunately, in the non-abelian case there is no general classification theorem like the one for finitely generated abelian we discussed in the first section, even if the group is finitely generated. The best one can do is describe a finitely generated non-abelian group as a free product of groups together with an equivalence defined by a set of relations among the generators. It also turns out to be very difficult to establish when two different free products with a different set of relations
are actually representations of the same group. This lack of an effective procedure to determine whether or not two different representations correspond to isomorphic groups is usually called the “unsolvability for the isomorphism problem” for groups.

As an example of this more general structure of the fundamental group, let us compute it for an orientable compact genus $g$ surface $M_g$. Just as the torus is topologically constructed by identifying opposite edges of a square, a genus $g$ surface is constructed by pairwise identification of edges of a polygon with $4g$ sides (see figure 1). Under this identification, all $4g$ vertices of the polygon get identified, so that the identification maps the boundary of the polygon to a wedge of $2g$ circles. The surface $M_g$ is then obtained by attaching a 2-dimensional space homeomorphic to an open 2-disc to this wedge of circles. Such a two dimensional surface is called a 2-cell (in the language of cell- or CW complexes [2]). Each circle is the result of attaching a 1-cell (homeomorphic to an open interval) to the common base point, which in this case is called a 0-cell. $M_g$ thus contains a 2-cell, $2g$ 1-cells and one 0-cell. Since we already computed the fundamental group of a wedge of circles, the van Kampen theorem now allows us to compute the fundamental group of the space obtained by attaching a 2-cell to a wedge of $2g$ circles, namely $M_g$.

Let $X$ denote the wedge of $2g$ circles. An open cover for $M_g$ is $\{A, B\}$ where $A = M_g - m$, where $m$ is a point in the interior of the 2-cell (which is thus not mapped to $X$ by the map which attaches the 2-cell to $X$), and $B = M_g - X$. Note that $A$ is of the same homotopy type as $X$ and $B$ is contractible, since it is homeomorphic to an open 2-disc. Since $A \cap B = M_g - m - X$ it is path connected, but not simply connected. Since $\pi_1(B)$ is trivial, the factor $i_{BA}(\omega)^{-1}$ in the van Kampen theorem is not there and $N$ is simply generated by elements in the image of the map $\pi_1(A \cap B) \to \pi_1(A)$.
The van Kampen theorem then states that

\[ \pi_1(M_g) = \pi_1(X)/N = \pi_1 \left( \bigvee_{2g} S^1 \right)/N = \ast_{2g} \mathbb{Z}/N, \]  

(3.28)

where \( N \) is generated by elements of \( \pi_1(A \cap B) \) (at least their images under \( i_{AB} \)). Since \( A \cap B \) is homeomorphic to a disc with one point removed, we have that \( \pi_1(A \cap B) = \mathbb{Z} \), generated by the image of the loop that goes once around the polygon. This means that \( N \) is infinite cyclic generated by \( [a_1, b_1], [a_2, b_2], \ldots [a_g, b_g] \), where the commutator is defined by \( [a, b] = aba^{-1}b^{-1} \). Summarizing, the fundamental group of \( M_g \) is the free group generated by \( 2g \) elements \( a_1, b_1, \ldots a_g, b_g \) subordinate to the single relation \( [a_1, b_1] \ldots [a_g, b_g] = 0 \). This is denoted by

\[ \pi_1(M_g) = \langle a_1, b_1, \ldots a_g, b_g | [a_1, b_1] \ldots [a_g, b_g] \rangle. \]  

(3.29)

For example \( \pi_1(T^2) = \langle a, b | aba^{-1}b^{-1} \rangle \), which clearly implies that \( ab = ba \). In other words \( \pi_1(T^2) \) is the abelianization \( \mathbb{Z} \ast \mathbb{Z} \), which is simply \( \mathbb{Z} \oplus \mathbb{Z} \), providing us with yet another derivation of the familiar result. In this case, \( N \) is called the commutator subgroup. More generally, the commutator subgroup \( [G, G] \) of a group \( G \) is generated by all possible commutators of generators and the abelian group \( G/[G, G] \) is called the abelianization of \( G \). For higher genus \( \pi_1(M_g) \) is clearly non-abelian and not free.

4 Simplicial complexes and homology

As the previous section showed, computing the fundamental group of relatively simple spaces can already become quite involved. In addition, the lack of a general classification of finitely generated non-abelian groups and the non-abelian nature of \( \pi_1 \) restricts the practical use of the fundamental group in many cases. Besides these restrictions, the fundamental group is insensitive to higher dimensional cells (\( n \)-cells for \( n > 2 \)). This is solved by considering higher homotopy groups. Namely, instead of looking at loops, one looks at images of higher dimensional \( n \)-spheres in the space under consideration. Despite of being abelian, the resulting higher homotopy groups \( \pi_n(X) \) are much more difficult to compute than \( \pi_1(X) \) in general. For instance, even for spheres it turns out that \( \pi_i(S^n) \) for \( i > n \) is generically nonzero and extremely hard to compute. This is the subject of higher homotopy theory [2].

Here, we will follow a different route and define another notion called homology. Although it takes a bit more time to define homology groups, they will turn out to all
be abelian and we will discover some very useful ways to compute them. The general idea of homology is to find an algebraic way to classify a space by the structure of the ‘holes’ inside the space. This can be done by first studying a very rigid construction called a simplicial complex, which allows for a very concrete definition of what is meant by a hole in a space. In fact, we will be working with a slightly less restrictive construction called a \( \Delta \)-complex \[2\].

### 4.1 Simplicial complexes

The idea is to build up a space out of smaller building blocks called simplices. To see how this works, consider the case of a torus in figure 2. In this figure, the torus is built up out of one vertex \( v \), three lines \( a, b \) and \( c \), and two triangles \( A \) and \( B \). These are all examples of simplices. An \( n \)-simplex is the smallest convex subset of a Euclidean space containing \( n + 1 \) points \( v_0, \ldots, v_n \) which do not all lie in a hyperplane of dimensionality less than \( n \). It is denoted by \([v_0, \ldots, v_n]\), where the order of the vertices is important because it defines an orientation of the simplex. An example is the so-called standard \( n \)-simplex

\[
\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \middle| \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.
\]

(4.1)

So a (standard) 0-simplex is a point, a 1-simplex a line segment, a 2-simplex a triangle, a 3-simplex a tetrahedron and so on. The torus in figure 2 is thus constructed out of one 0-simplex, three 1-simplices and two 2-simplices. A face of an \( n \)-simplex \([v_0, \ldots, v_n]\) is an \((n - 1)\)-simplex obtained by deleting one of the points \( v_i \), which is denoted by \([v_0, \ldots, \hat{v}_i, \ldots, v_n]\). The boundary of an \( n \)-simplex \( \Delta^n \) is the union of all its faces and is denoted by \( \partial \Delta^n \). The interior of \( \Delta^n \) is \( \Delta^n = \Delta^n - \partial \Delta^n \).
A ∆-complex structure on a space $X$ is a collection of maps $\sigma_\alpha : \Delta^n \to X$, where $n$ depends on $\alpha$, such that

1. The restriction $\sigma_\alpha|\hat{\Delta}^n$ is injective and each point of $X$ is in exactly one such restriction.

2. Each restriction of $\sigma_\alpha$ to a face of $\Delta^n$ is a map $\sigma_\beta : \Delta^{n-1} \to X$ which also belongs to the collection.

3. A set $A \subset X$ is open iff $\sigma^{-1}_\alpha(A)$ is open in $\Delta^n$ for each $\sigma_\alpha$.

The first two conditions roughly mean that the whole space is covered by images of simplices and no such images overlap. The third condition is of a more technical nature and among other things rules out trivialities like regarding all the points of $X$ as individual vertices. To form a simplicial complex one also requires that each simplex be uniquely specified by its vertices. This condition is omitted in the definition of a ∆-complex. For instance, the ∆-complex structure given to the torus in figure 2 clearly does not correspond to a simplicial complex. Note that an orientation (i.e. an ordering of vertices) of a simplex implies an orientation of its faces by keeping track of the ordering. This fact will be important when we define simplicial homology.

### 4.2 Simplicial homology

Denote the images $\sigma_\alpha(\hat{\Delta}^n)$ in a ∆-complex structure on $X$ by $e^n_\alpha$. Let $\Delta_n(X)$ be the free abelian group generated by the set $\{e^n_\alpha\}$. An element of $\Delta_n(X)$ is called an $n$-chain and can thus be written as a finite formal sum $\sum_n n_\alpha e^n_\alpha$, with coefficients $n_\alpha \in \mathbb{Z}$. The basic ingredient for defining homology is the boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$, defined by its action on basis elements,

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha[[v_0, \ldots, \hat{v}_i, \ldots, v_n]]. \quad (4.2)$$

To see how this boundary operator enables us to find holes is the complex, consider the example of figure 2. Consider the upper triangle $A$. The chain $a+b-c$ corresponds to a loop around the triangle. According to the definition, $\partial_1(a + b - c) = 0$, so that the boundary of this chain is zero. However, it does not encircle a hole. This is reflected by the fact that it is the boundary of a 2-simplex which is also part of the complex, namely $\partial_2 A = a + b - c$. On the other hand, the chains $a$ and $b$ still have no
boundary, while there is no 2-simplex $C$ for which $\partial_2 C = a$ or $\partial_2 C = b$. The existence of chains with these properties is exactly what is expected for a space with holes.

These ideas are easily generalized once one notes the elementary relation

$$\partial_n \partial_{n+1} = 0,$$

(4.3)
a fact which is usually referred to by the phrase “the boundary of a boundary is always zero”. This implies that $\text{Im} \, \partial_{n+1} \subset \text{Ker} \, \partial_n$. Chains which are in the image of $\partial_{n+1}$ are called $n$-boundaries and those who are in the kernel of $\partial_n$ are called $n$-cycles. Since the cycles which are also boundaries are topologically trivial, one is interested in the quotient group,

$$H_n(X) = \frac{\text{Ker} \, \partial_n}{\text{Im} \, \partial_{n+1}},$$

(4.4)
called the $n$-th simplicial homology group of $X$. Algebraically, a sequence of homomorphisms of abelian groups,

$$\cdots \overset{d_{n+2}}{\longrightarrow} C_{n+1} \overset{d_{n+1}}{\longrightarrow} C_n \overset{d_n}{\longrightarrow} C_{n-1} \overset{d_{n-1}}{\longrightarrow} \cdots \overset{d_1}{\longrightarrow} C_0 \overset{d_0}{\longrightarrow} 0$$

with $d_n d_{n+1} = 0$, is called a chain complex. Here we added the map $d_0 \equiv 0$. Since $\text{Im} \, d_{n+1} \subset \text{Ker} \, d_n$, one can always define the $n$-th homology group $H_n = \text{Ker} \, d_n / \text{Im} \, d_{n+1}$ of the chain complex. Elements of $H_n$ are called homology classes. In the present case, $C_n = \Delta_n(X)$ and $d_n = \partial_n$, but clearly the algebraic construction is much more general. For a finite $\Delta$-complex, the homology groups clearly have the structure of a finitely generated abelian group, which we described in the first section.

As a first example, consider a $\Delta$-complex structure for the circle, namely one 0-simplex $v$ and one 1-simplex $e$, with $\partial_1 e = v - v = 0$. Both $\Delta_1(S^1)$ and $\Delta_0(S^1)$ are equal to $\mathbb{Z}$ and the other chain groups are identically zero. Since $\partial_1 = 0$, we have $H_n(S^1) = \Delta_n(S^1)$ for all $n$. So our first result is,

$$H_n(S^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(4.5)
Using the $\Delta$-complex structure for the torus in figure 2, we can also easily calculate the groups $H_n(T^2)$. We have $\Delta_2(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, $\Delta_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\Delta_0(T^2) = \mathbb{Z}$. Since $a$, $b$ and $c$ are all cycles, we have again that $\partial_1 = 0$ and $H_0(T^2) = \mathbb{Z}$. Since $\partial_2(A) = \partial_2(B) = a + b - c$ and $\{a, b, a + b - c\}$ can be taken as a basis for $\Delta_1(T^2)$, we find that $H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. Since there are no 3-simplices $H_2(T^2) = \text{Ker} \, \partial_2 = \mathbb{Z}$.
Summarizing,

$$H_n(T^2) = \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\
\mathbb{Z} & \text{for } n = 0, 2 \\
0 & \text{for } n \geq 3 
\end{cases} \quad (4.6)$$

A $\Delta$-complex structure for the $n$-sphere $S^n$ is obtained by taking two $n$-simplices $A$ and $B$ and identifying their boundary. Because there are no $(n+1)$-simplices $H_n(S^n) = \text{Ker } \partial_n$. The latter is infinite cyclic generated by $A-B$, so that $H_n(S^n) = \mathbb{Z}$. The other homology groups will be obtained below. Since $S^n$ (just as all previous examples) is path connected any two points are always homologous. Indeed, given two points (two vertices of the $\Delta$-complex, if you will) there always exists a path (a 1-chain) such that the difference between the two points is the boundary of this path. This implies that for any path connected space $X$, $H_0(X) = \mathbb{Z}$. For a more general space $Y$, each path component will give rise to a cyclic generator of infinite order. So that $H_0(Y) = \mathbb{Z}^n$, where $n$ is the number of path components of $Y$.

Homology groups need not be free, but can contain a torsion part. The most elementary example of this is the first homology group of the projective plane $\mathbb{R}P^2$. A possible $\Delta$-complex structure on $\mathbb{R}P^2$ is depicted in figure 3. Because $\partial_2(A) = a-b+c$ and $\partial_2(B) = -a+b+c$, we find that $\partial_2$ is injective, so that $H^2(\mathbb{R}P^2) = 0$. On the other hand, $\text{Ker } \partial_1 = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the cycles $a-b$ and $c$. A more convenient basis for $\text{Ker } \partial_1$ is $a-b+c$ and $c$ and for $\text{Im } \partial_2$ we can take $a-b+c$ and $2c = (-a+b+c) + (a-b+c)$. This shows that $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$, generated by $c$, with the relation $2c = 0$.

Like in homotopy theory, we can again define the notion of an induced map. Given a map $f : X \to Y$, this induces a map between the chain groups $f_* : \Delta_n(X) \to \Delta_n(Y)$ by composing $f$ with the maps which define the $\Delta$-complex

$$f_*(\sigma) = f \circ \sigma : \Delta^n \to Y, \quad \text{for } \sigma : \Delta^n \to X. \quad (4.7)$$
From the definition of the boundary homomorphism (4.2) it immediately follows that
\[ f_\sharp \partial = \partial f_\sharp. \]
A map between chain complexes which commutes with the boundary homomorphism is called a chain map. Clearly, a chain map takes cycles to cycles, since \( \partial a = 0 \) implies \( \partial(f_\sharp a) = f_\sharp(\partial a) = 0 \). It also maps boundaries to boundaries, since \( f_\sharp(\partial b) = \partial(f_\sharp b) \). This implies that \( f \) induces a well defined map \( f_* : H_n(X) \to H_n(Y) \). Like for the induced map in homotopy theory, this induced map has the basic properties that \( (fg)_* = f_*g_* \) and \( \text{Id}_* = \text{Id} \). These properties again imply that we are dealing with a covariant functor, this time from the category of topological spaces and continuous maps to the category of homology groups and homomorphisms. Like for the fundamental group, one can prove that two homotopic maps give rise to the same induced map [2]. In the same way as before, it follows that two spaces of the same homotopy type have the same homology groups.

Sometimes it is useful to define homology groups over other abelian groups than \( \mathbb{Z} \). Simply let the coefficients in front of the simplices in the definition of the chains be elements of an abelian group \( G \). The boundary operator and homology groups are defined in exactly the same way as before. The difference with homology over \( \mathbb{Z} \) only becomes apparent when one starts doing calculations. The zeroth homology group of a path connected space is now simply \( G \). For some constructions it is required that the coefficient group has the richer structure of a ring \( R \). In that case, one calls \( R \) the coefficient ring.

Clearly, for some simple spaces and once an adequate \( \Delta \)-structure is found, it is not very difficult to calculate the simplicial homology. But for increasingly complicated spaces, a corresponding \( \Delta \)-complex will also become increasingly complicated and the computation of homology groups increasingly cumbersome. The power of homology lies however in its rich algebraic structure, which allows for many calculations (and proofs) without having to resort to (for instance) a \( \Delta \)-complex structure. One of the most important algebraic tools is the existence of certain exact sequences of homology groups, to which we will now turn.

### 4.3 Exact sequences

A sequence of homomorphisms \( \alpha_n \) between abelian groups \( A_n \)
\[
\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots
\]
is called exact if $\ker \alpha_n = \text{im} \alpha_{n+1}$ for each $n$. The inclusions $\text{im} \alpha_{n+1} \subset \ker \alpha_n$ are equivalent to $\alpha_n \alpha_{n+1} = 0$, so that the sequence is a chain complex. The opposite inclusion implies that all homology groups of this chain complex are trivial. The following statements are almost immediate,

1. $0 \rightarrow A \xrightarrow{\alpha} B$ is exact iff $\alpha$ is injective.

2. $A \xrightarrow{\alpha} B \rightarrow 0$ is exact iff $\alpha$ is surjective.

3. $0 \rightarrow A \xrightarrow{\alpha} B \rightarrow 0$ is exact iff $\alpha$ is bijective.

4. $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is exact iff $\alpha$ is injective, $\beta$ is surjective and $\ker \beta = \text{im} \alpha$. This implies that $\beta$ induces an isomorphism $C \cong B/A$. An exact sequence of this kind is called a short exact sequence.

Let us now assume that we have three chain complexes of abelian groups $A_n, B_n$ and $C_n$ (each with a boundary homomorphism $\partial$) and that for each $n$ the sequence

$$0 \rightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \rightarrow 0$$

is exact. We also assume that $i$ and $j$ are chain maps, i.e. $i \partial = \partial i$ and $j \partial = \partial j$. Summarizing, we have the large commutative diagram called a short exact sequence of chain complexes:

This is commutative because $i$ and $j$ are chain maps. The claim is now that to this short exact sequence one can associate a long exact sequence of homology groups [2]
\[ \ldots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \rightarrow \ldots \]

Here \( i_* \) and \( j_* \) are induced by the chain maps \( i \) and \( j \). The homomorphism \( \partial_* : H_n(C) \rightarrow H_n(A) \) is defined as follows. Let \( c \in C_n \) be a cycle. Since \( j \) is onto, \( c = j(b) \) for some \( b \in B_n \). The boundary \( \partial b \in B_{n-1} \) is in the kernel of \( j \), since \( j(\partial b) = \partial j(b) = \partial c = 0 \). Since \( \text{Ker} j = \text{Im} i \), this implies \( \partial b = i(a) \) for some \( a \in A_{n-1} \).

The element \( a \) is also a cycle since \( i(\partial a) = \partial i(a) = \partial^2 b = 0 \) and \( i \) is injective. Intuitively, because \( c \) is a cycle, its boundary should be zero modulo some element \( a \) (remember that \( C = B/A \)). The map \( \partial_* \) is precisely defined to be \( \partial_*[c] = [a] \). It is not very hard to show that this map is well-defined, i.e. independent of the choice of representative of \( [c] \) and of intermediate element \( b \) as long as \( j(b) = c \).

The theorem that to every short exact sequence of chain complexes one can associate a long exact sequence of homology groups represents the beginnings of the subject called homological algebra, its main method of proof sometimes being referred to as “diagram chasing”. Starting from this algebraic structure, one can now define different exact sequences of homology groups depending on which short exact sequence of chain complexes one starts from. In the next subsection we will discuss one of the more important ones, namely the Mayer-Vietoris sequence.

### 4.4 The Mayer-Vietoris sequence

Let \( X = A \cup B \) be the union of two open sets \( A \) and \( B \). Let \( \Delta_n(A+B) \) be the subgroup of \( \Delta_n(X) \) consisting of chains that are sums of chains of \( A \) and chains of \( B \). The boundary map \( \partial \) takes elements of \( \Delta_n(A+B) \) to elements of \( \Delta_{n+1}(A+B) \) so that this defines a chain complex. To obtain the Mayer-Vietoris sequence, we then start from the following short exact sequence of chain complexes:

\[ 0 \rightarrow \Delta_n(A \cap B) \xrightarrow{\varphi} \Delta_n(A) \oplus \Delta_n(B) \xrightarrow{\psi} \Delta_n(A+B) \rightarrow 0 \]

The homomorphisms involved are \( \varphi(x) = (x, -x) \) for \( x \in A \cap B \), and \( \psi(x, y) = x + y \) for \( x \in A \) and \( y \in B \). Clearly, \( \text{Ker} \varphi \) is trivial and \( \text{Im} \psi = \Delta_n(A+B) \) by definition. To establish exactness of the sequence, the only thing left to show is \( \text{Ker} \psi = \text{Im} \varphi \). \( \text{Im} \varphi \subset \text{Ker} \psi \) follows from \( \psi \varphi = 0 \). On the other hand, \( \text{Ker} \psi \subset \text{Im} \varphi \), because \( \psi(x, y) = 0 \) implies \( x = -y \), which means that \( x \) is a chain on both \( A \) and \( B \), such that \( (x, y) = (x, -x) \in \text{Im} \varphi \).
As discussed in the previous subsection, this induces the following long exact sequence, called the Mayer-Vietoris sequence:

\[ \cdots \to H_n(A \cap B) \xrightarrow{\varphi_*} H_n(A) \oplus H_n(B) \xrightarrow{\psi_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \to \cdots \]

Here we used the fact that the chain map \( \Delta_n(A + B) \to \Delta_n(X) \) induces an isomorphism between the homology groups \( H_n(A + B) \) and \( H_n(X) \). One way to prove this is by using something called barycentric subdivision to show that a cycle in \( H_n(X) \) can always be written as a sum of a chain on \( A \) and a chain on \( B \). Let \( z = x + y \) be such a representation of a cycle \( z \) in terms of \( x \in \Delta_n(A) \) and \( y \in \Delta_n(B) \). The latter two need not be cycles, but they should satisfy the relation \( \partial x = -\partial y \), so that \( \partial x \) is a representative of \( \partial_\ast[z] \in H_{n-1}(A \cap B) \), as follows from the definition of \( \partial_\ast \).

In practice, it turns out to be very useful to work with reduced homology groups \( \tilde{H}_n(X) \). To obtain these, one defines the augmented chain complex,

\[ \cdots \to \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \]

where \( \varepsilon(\sum_\alpha n_\alpha \sigma_\alpha) = \sum_\alpha n_\alpha \). Because \( \varepsilon \partial_1(\sigma) = \varepsilon(\sigma[[v_1] - \sigma[[v_0]]) = 1 - 1 = 0 \), for any 1-simplex \( \sigma \), we have that \( \text{Im} \partial_1 \subset \text{Ker} \varepsilon \), so that this augmented sequence is still a chain complex. The zeroth reduced homology group is then \( \tilde{H}_0(X) = \text{Ker} \varepsilon / \text{Im} \partial_1 \). Since \( \varepsilon \partial_1 = 0 \), the map \( \varepsilon : \Delta_0(X) \to \mathbb{Z} \) induces a map \( H_0(X) \to \mathbb{Z} \) with kernel \( \tilde{H}_0(X) \). This means that

\[ H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z} \]

(note that \( \varepsilon \) is surjective for nonempty \( X \)). Of course, for \( n > 0 \) there is no change \( \tilde{H}_n(X) = H_n(X) \). The main reason why this is so useful, is because this implies that for a path-connected space, \( \tilde{H}_0(X) \) is trivial. In particular, for a contractible space all reduced homology groups are trivial.

For reduced homology the Mayer-Vietoris sequence is formally identical to the one for ordinary homology. From this fact, it follows immediately that if \( A \cap B \) is path connected, we have the isomorphism

\[ H_1(X) \cong (H_1(A) \oplus H_1(B)) / \text{Im} \varphi_* . \quad (4.8) \]

This shows very clearly that the Mayer-Vietoris plays a similar role for homology as the van Kampen theorem does for homotopy. In fact, (4.8) is just an abelianization of the van Kampen theorem for \( X = A \cup B \), which makes perfect sense, since the first
homology group is nothing but an abelianization of the fundamental group (see e.g. [2] for more details).

As a concrete example of an application of the Mayer-Vietoris sequence, consider $S^n = A \cup B$, where $A$ and $B$ are hemispheres homeomorphic to open balls, such that $A \cap B = S^{n-1}$. Since $\tilde{H}_n(A)$ and $\tilde{H}_n(B)$ are trivial for all $n$, we find the exact sequence,

$$0 \longrightarrow \tilde{H}_i(S^n) \longrightarrow \tilde{H}_{i-1}(S^{n-1}) \longrightarrow 0$$

In other words,

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}), \quad (4.9)$$

from which all homology groups of the spheres can be found by induction. Even for $n = i = 1$, we find $H_1(S^1) = H_0(S^0)$, where $H_0(S^0) = \tilde{H}_0(S^0) \oplus \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$ since $S^0$ is just the disjoint union of two points, so that we find the familiar result $H_1(S^1) = \mathbb{Z}$.

By induction this leads to,

$$H_i(S^n) = \begin{cases} \mathbb{Z} & \text{for } i = 0, n \\ 0 & \text{otherwise} \end{cases} \quad (4.10)$$

Another application is the computation of the first homology group of a compact orientable genus $g$ surface $M_g$. As for the fundamental group, this can be done by first computing it for a wedge of circles. Write the figure eight space $\infty$ as the union of two circles with one point in common. All relative homology groups of a point are trivial, so that the Mayer-Vietoris sequence gives

$$0 \longrightarrow H_1(S^1) \oplus H_1(S^1) \longrightarrow H_1(S^1 \vee S^1) \longrightarrow 0$$

By induction; we find for a wedge of $2g$ circles that $H_1(\bigvee_{2g} S^1) = \bigoplus_{2g} \mathbb{Z}$. Now, as was discussed in the previous chapter, the surface $M_g$ can be obtained by attaching a 2-cell to a wedge of $2g$ circles. Using the same open cover as before, so that $H_1(A) = H_1(\bigvee_{2g} S^1) = \bigoplus_{2g} \mathbb{Z}$ and $H_1(B) = 0$, equation (4.8) yields,

$$H_1(M_g) = \bigoplus_{2g} \mathbb{Z} / \text{Ker } \psi_* \quad (4.11)$$

Where $\psi_* : H_1(A) \to H_1(M_g)$ (because $H_1(B) = 0$) which can now simply be seen as the inclusion of a cycle on $A$ into $M_g$. None of these cycles on $A$ become trivial when included in $M_g$, so that $\text{Ker } \psi_* = 0$. This shows that

$$H_1(M_g) = \bigoplus_{2g} \mathbb{Z}. \quad (4.12)$$
This is indeed the abelianization of the fundamental group of $M_g$ computed in the previous section.

The Klein Bottle $K$ can be seen as the union of two Möbius strips $A$ and $B$ by gluing their boundary circles together. $A$, $B$ and $A \cap B$ are all of the same homotopy type as a circle. This implies the Mayer-Vietoris sequence

$$0 \rightarrow H_2(K) \rightarrow H_1(A \cap B) \xrightarrow{\varphi_*} H_1(A) \oplus H_1(B) \rightarrow H_1(K) \rightarrow 0$$

The map $\varphi_*$ has the form $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} : 1 \mapsto (2, -2)$, since the boundary circle winds twice around the base circle of the Möbius strip. This map is injective, implying that $H_2(K) = 0$. This reduces the above sequence to a short exact sequence, so that $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/\text{Im} \varphi_*$. Taking $(1, 0)$ and $(1, -1)$ as a basis for $\mathbb{Z} \oplus \mathbb{Z}$, we find $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$.

### 4.5 The Künneth formula

Given the homology groups $H_k(M)$ and $H_k(N)$ of the manifolds $M$ and $N$ over a coefficient ring $R$, the Künneth formula allows one to calculate the homology of the Cartesian product manifold $M \times N$, which is defined to be the manifold of pairs of points $(m, n)$ in $M$ and $N$ respectively. The $k$th homology group of the product is

$$H_k(M \times N) = \bigoplus_j H_j(M) \otimes_R H_{k-j}(N) \oplus \bigoplus_j \text{Tor}(H_j(M), H_{k-j-1}(N)).$$ (4.13)

We have introduced several new pieces of notation. The tensor product $G \otimes_R H$ of two groups $G$ and $H$ on which $R$ acts is defined to be the group generated by elements $gh$, where $g$ and $h$ are in $G$ and $H$ respectively. Unlike the direct sum $G \oplus H$, whose elements $(g, h)$ are all distinct, elements of the tensor product are identified to enforce that it is bilinear. More precisely, if $r$ is an element of the ring $R$ then one identifies the elements

$$r(gh) = (rg)h = g(rh)$$ (4.14)

of the tensor product. In contrast, the direct sum is merely linear

$$r(g, h) = (rg, rh).$$ (4.15)

For example

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_N = \mathbb{Z}_N.$$ (4.16)
To see this, note first by bilinearity (4.14) that the tensor product group is generated by the single element $gh$ where $g = 1$ and $h = 1$, as $G$ and $H$ are both generated by the element $1$. Next, we observe that

$$Ngh = g(Nh) = 1(N) = 1(e) = e$$

and so the generator of the tensor product group is of order $N$, really this shows that order divides $N$, but it is precisely $N$. So the tensor product group is generated by a single element of order $N$, identifying at as $\mathbb{Z}_N$ the finite cyclic group of order $N$.

The geometrical interpretation of the tensor product terms is quite straightforward. Consider a $j$-cycle $Z_j$ in $M$ and a $(k - j)$-cycle $Z_{k-j}$ is $N$. The Cartesian product $Z_j \times Z_{k-j}$ is a $k$-cycle in $M \times N$. The first term in the Künneth formula counts such cycles.

However not all cycles in $M \times N$ are just products of cycles in $M$ with cycles in $N$. Consider a $(j + 1)$-chain $C_{j+1}$ in $M$ and a $(k - j)$-chain $C_{k-j}$ in $N$ with boundaries

$$B_j = \partial C_{j+1}, \quad B_{k-j-1} = \partial C_{k-j}.$$  (4.18)

Then there is a new cycle

$$Z_k = C_{j+1} \times B_{k-j-1} - (-1)^j B_j \times C_{k-j}.$$  (4.19)

This is a cycle because

$$\partial(Z_k) = B_j \times B_{k-j-1} - B_j \times B_{k-j-1} = 0.$$  (4.20)

These cycles correspond to $j$-cycles in $M$ and $(k - j - 1)$-cycles in $N$ and so they are related to $\text{H}_j(M)$ and $\text{H}_{k-j-1}(N)$.

Unfortunately, our cycles $Z_k$ are just the boundaries of $C_{j+1} \times C_{k-j}$, and so do not actually contribute to the homology. However, it may be that for some integer $m$, there is a cycle $Y_k$ such that $Z_k = mY_k$ and $Y_k$ is not a boundary. In this case, $Y_k$ will generate a $\mathbb{Z}_m$ torsion subgroup of $\text{H}_k(M)$. These new subgroups are calculated by the Tor term in the Künneth formula (4.13). The fact that there are no more terms implies that all cycles in $M \times N$ are of one of these two types. If $Y_k$ exists, it means intuitively that there is a multiplication by $m$ in the boundary map from $C$ to $Z$. More concretely, it means that there is a $\mathbb{Z}_m$ torsion subgroup in both $\text{H}_j(M)$ and $\text{H}_{k-j-1}(N)$. This happens if the greatest common divisor of the degrees of their torsion parts is a multiple of $m$. Therefore

$$\text{Tor}(\mathbb{Z}_p, \mathbb{Z}_q) = \mathbb{Z}_{\text{gcd}(p,q)}, \quad \text{Tor}(\mathbb{Z}, G) = 0$$  (4.21)
where \( G \) is an arbitrary group. Tor is symmetric, as \( M \times N \) has the same homology as \( N \times M \). In addition it is additive under direct sums, as all of the operations here are linear.

As an example, we will calculate the homology of the Cartesian sum of lens spaces \( L_{4,1} \times L_{6,1} \). The homology of the lens space \( L_{p,1} \) can be easily calculated using the Gysin sequence, which we will discuss later, and the fact that it is a circle bundle contributes to the homology of the product at degree 1 + 1 + 1 = 1.

The terms without Tor are just tensor products of the homology of the factors and so Tor is only nonvanishing on \( H_1 \) of each factor, which contributes to the homology of the product at degree 1 + 1 + 1 = 1.

The terms without Tor are just tensor products of the homology of the factors

\[
\begin{align*}
H_0(L_{4,1} \times L_{6,1}) &= H_0(L_{4,1}) \otimes H_0(L_{6,1}) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z} \\
H_1(L_{4,1} \times L_{6,1}) &= H_0(L_{4,1}) \otimes H_1(L_{6,1}) \oplus H_1(L_{4,1}) \otimes H_0(L_{6,1}) = \mathbb{Z} \otimes \mathbb{Z}_6 \oplus \mathbb{Z}_4 \otimes \mathbb{Z} \\
&= \mathbb{Z}_6 \oplus \mathbb{Z}_4 = \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\
H_2(L_{4,1} \times L_{6,1}) &= H_0(L_{4,1}) \otimes H_2(L_{6,1}) \oplus H_1(L_{4,1}) \otimes H_1(L_{6,1}) \oplus H_2(L_{4,1}) \otimes H_0(L_{6,1}) \\
&= \mathbb{Z} \otimes 0 \oplus \mathbb{Z}_4 \otimes \mathbb{Z}_6 \oplus 0 \times \mathbb{Z} = 0 \oplus \mathbb{Z}_{gcd(4,6)} \oplus 0 = \mathbb{Z}_2 \\
H_3(L_{4,1} \times L_{6,1}) &= H_0(L_{4,1}) \otimes H_3(L_{6,1}) \oplus H_1(L_{4,1}) \otimes H_2(L_{6,1}) \oplus H_2(L_{4,1}) \otimes H_1(L_{6,1}) \\
&= \mathbb{Z}_6 \oplus \mathbb{Z}_4 \otimes \mathbb{Z}_6 \oplus 0 \times \mathbb{Z} = \mathbb{Z}_6 \oplus \mathbb{Z}_4 \otimes \mathbb{Z}_6 \oplus \mathbb{Z} \otimes \mathbb{Z} \\
&= \mathbb{Z} \oplus 0 \oplus 0 \oplus \mathbb{Z} = \mathbb{Z}^2 \\
H_4(L_{4,1} \times L_{6,1}) &= H_1(L_{4,1}) \otimes H_3(L_{6,1}) \oplus H_2(L_{4,1}) \otimes H_2(L_{6,1}) \oplus H_3(L_{4,1}) \otimes H_1(L_{6,1}) \\
&= \mathbb{Z}_4 \otimes \mathbb{Z} \oplus 0 \times \mathbb{Z} \oplus \mathbb{Z}_6 = \mathbb{Z}_4 \oplus \mathbb{Z}_6 = \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \\
H_5(L_{4,1} \times L_{6,1}) &= H_2(L_{4,1}) \otimes H_3(L_{6,1}) \oplus H_3(L_{4,1}) \otimes H_2(L_{6,1}) = 0 \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes 0 \\
&= 0 \oplus 0 = 0 \\
H_6(L_{4,1} \times L_{6,1}) &= H_3(L_{4,1}) \otimes H_3(L_{6,1}) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}.
\end{align*}
\]
These groups are easily interpreted. $H_0 = \mathbb{Z}$ because the product is path-connected. The generators of $H_1$ of the product space are just those of the original lens spaces, so it can be represented entirely in terms of sums of loops that exist in the two lens spaces considered separately.

The second homology group is more interesting. $H_2 = \mathbb{Z}_2$ has a single nontrivial element, which is the torus $a_1 \times b_1$ generated by two circles $a_1$ and $b_1$ which generate the fundamental groups of the two lens spaces. As these cycles are $\mathbb{Z}_4$-torsion and $\mathbb{Z}_6$-torsion respectively, there exist 2-chains $a_2$ and $b_2$ such that

$$\partial a_2 = 4a_1, \quad \partial b_2 = 6b_1.$$  \hfill (4.24)

We can construct chains in the product space by taking the Cartesian products of chains in the lens spaces, for example we can construct the 3-chains $a_1 \times b_2$ and $a_2 \times b_1$ which have boundaries

$$\partial(a_1 \times b_2) = (\partial a_1) \times b_2 - a_1 \times (\partial b_2) = 0 - 6a_1 \times b_1 = -6a_1 \times b_1, \quad \partial(a_2 \times b_1) = 4a_1 \times b_1.$$  \hfill (4.25)

In particular, twice our torus $a_1 \times b_1$ is a boundary

$$\partial(-a_2 \times b_1 - a_1 \times b_2) = 2a_1 \times b_1$$  \hfill (4.26)

and so the torus generates a $\mathbb{Z}_2$ in $H_2$ of the product space.

Another combination of these three-chains is a cycle

$$\partial(3a_2 \times b_1 + 2a_1 \times b_2) = (12 - 12)a_1 \times b_1 = 0.$$  \hfill (4.27)

Before concluding that this represents a nontrivial element of the third homology group of the product, we must check to see if it is a boundary. For example, we can check

$$\partial(a_2 \times b_2) = 6a_2 \times b_1 + 4a_1 \times b_2$$  \hfill (4.28)

which is twice our cycle. Thus if our three-chain is nontrivial in $H_3$, it will be a $\mathbb{Z}_2$ torsion element. This is just the $\mathbb{Z}_2$ in the Tor term calculated using Eq. (4.21)

$$H_3(L_{4,1} \times L_{6,1}) = \mathbb{Z}^2 \oplus \text{Tor}(H_1(L_{4,1}), H_1(L_{6,1})) = \mathbb{Z}^2 \oplus \text{Tor}(\mathbb{Z}_4, \mathbb{Z}_6) = \mathbb{Z}^2 \oplus \mathbb{Z}_2. \hfill (4.29)$$

The two generators of $H_3$ are just the two original lens spaces.

The last three homology groups are easier to interpret. $H_4$ is represented by the individual lens spaces each crossed with a 1-cycle in the other. There are no nontrivial
5-cycles. Finally $H_6 = \mathbb{Z}$ is represented by the entire space. The fact that it is $\mathbb{Z}$ reflects the fact that the product space is orientable, with an orientation given by the product of the orientations of the individual lens spaces.

5 Cohomology

Homology provides a series of groups that partially characterize a manifold. Roughly it classifies manifolds by classifying the various submanifolds, which represent the cycles, although technically some cycles may only be representable by a singular subset. Cohomology is a related classification scheme, but which classifies fluxes or field configurations instead of submanifolds. Thus homology is useful in physics for classifying extended objects, while cohomology is useful for example for classifying fields. Cohomology however turns out to possess some extra structure, it is a ring, instead of just an abelian group, which means that elements can be added and also multiplied. If we allow real weights for our chains then we can think of cohomology classes as being represented by differential forms, in which case addition is the usual addition while multiplication is exterior multiplication. This choice of representatives of real cohomology is known as de Rham cohomology. It contains less information than integral cohomology, in which chains are superpositions of simplices with integral weights.

5.1 What is cohomology

Cohomology is extraordinarily similar to homology. We have defined homology using chains, which are weighted sums of simplices with weights in some ring $R$, like the integers $\mathbb{Z}$ or the real numbers $\mathbb{R}$, which is called the coefficient ring. For a given simplex $\sigma_i$ we may define a special chain $c_i$ which consists of just $\sigma_i$ with weight 1. All chains are linear combinations of such elementary chains. A cochain is a map from the space of chains to the coefficient ring. All cochains are linear combinations of the cochains $c^i$ which just takes the coefficient of $\sigma_i$ from a chain, so that

$$c^i(c_j) = \delta^i_j. \quad (5.1)$$

An $n$-cochain acting on a chain counts a linear combination of the coefficients of the $n$-simplices in the chain. We will denote the group of $n$-cochains $C^n$. The action of cochains on chains is called the homology-cohomology pairing and in many contexts
it generalizes integration with the chain playing the part of the submanifold and the cochain the part of the integrand.

However there is one crucial difference between chains and cochains. As we have seen, chains are covariant objects and so they pushforward naturally. Intuitively, a chain $c_i$ is a subset of our manifold $M$ and so if one has a map $f : M \to N$ then the pushforward $f_*(c_i)$ of a chain is just the image of the subset under $f$. Cochains, on the otherhand, are contravariant, which means they pullback. The pullback of a cochain is defined using the pushforward of the chains. More precisely, if $c^i$ is an arbitrary cochain in $N$ and $c_j$ is an arbitrary chain in $M$ then the cochain $f^*(c^i)$ in $M$ is defined by

$$ (f^*(c^i))(c_j) = c^i(f_*(c_j)). \quad (5.2) $$

To define the simplicial cohomology of $M$, we will need to define a coboundary operator $\delta$ on the cochains. If we consider chains to be vectors whose entries correspond to simplices then the the boundary operator $\partial$ on the chains is a matrix. The coboundary operator is just the transpose of this matrix. So for example if $\partial(c_i) = 3c_j$ then $\delta(c^j) = 3c^i$. An $n$-cocycle is an $n$-cochain in the kernel of the coboundary operator $\delta$, while an $n$-coboundary is an $n$-cochain in its image. The coboundary operator squares to zero and so coboundaries are automatically cocycles. The $n$th cohomology group of $M$, denoted $H^n(M)$, can now be defined similarly to the definition of the $n$th homology group

$$ H^n(M) = \frac{\delta : C^n \rightarrow C^{n+1}}{\delta : C^{n-1} \rightarrow C^n} \quad (5.3) $$

as the quotient of the group of cocycles by the group of coboundaries. There is, as in the homology case, an implicit dependence on the coefficient ring $R$.

Calculating cohomology is just as easy or hard as calculating homology, one just needs to take the transpose of the boundary map. For example, recall that we described a circle using a 1-simplex $e$ and a zero simplex $v$ related by the boundary map $\partial(e) = v - v = 0$, while $\partial(v) = 0$ because the boundary would be $(-1)$-dimensional. Abusing notation by using the same symbols for the cochains, now $\delta(e) = 0$ because the coboundary would be a 2-cochain, but there are no 2-cochains in a 1-dimensional circle. Taking the transpose of the boundary operator, $\delta(v) = e - e = 0$ and so both $v$ and $e$ are cycles. Therefore

$$ H^0(S^1) = H^1(S^1) = \mathbb{Z} \quad (5.4) $$
where we have chosen the coefficient ring $R = \mathbb{Z}$, although more generally one may just replace this $\mathbb{Z}$ by $R$.

Similarly, we may calculate the cohomology of the 2-torus $T^2$ with integral coefficients. Recall that the boundaries of the 2-chains $A$ and $B$ are linear combinations of the 1-cycles $a$, $b$ and $c$ given by $\partial(A) = \partial(B) = a + b - c$. Taking the transpose of this boundary map, we find the coboundary

$$
\delta(a) = \delta(b) = A + B, \quad \delta(c) = -A - B, \quad \delta(v) = 0 \quad (5.5)
$$

where $v$ is a 0-cochain, which is a cocycle as we see by taking the transpose of the boundary map. The fact that the boundaries of the one-cycles in (5.5) are plus or minus $A + B$ means that, while the space of 2-cocycles is $\mathbb{Z}^2$ and is generated by $A$ and $B$, the space of coboundaries is $\mathbb{Z}$ and corresponds to the subgroup generated by $A + B$. The second cohomology group is just the quotient group $\mathbb{Z} = \mathbb{Z}^2/\mathbb{Z}$. Again, with coefficient ring $R$ we would have found $R$. There are no 1-boundaries, and so the first homology group is just the group of 1-cocycles, which is the $\mathbb{Z}$ generated by $a - b$ and $a + c$. The zeroth cohomology group is again generated by the 0-cycle $v$. Thus, as in the case of circle, the cohomology of $T^2$ is isomorphic to its homology.

Homology and cohomology groups are different when there is torsion around. Consider for example the real projective 2-sphere $\mathbb{RP}^2$, which is the 2-sphere $S^2$ with antipodal points identified. We can describe this space using a 2-chain $A$, a 1-chain $a$ and a 0-chain $v$. The 2-chain $A$ corresponds to the northern hemisphere, which is everywhere but the equator since points in the southern hemisphere are identified with their antipodal points which are in the northern hemisphere. The 1-chain $a$ is half of the equator, the other half is antipodal to it and so they are identified. However the boundary of the northern hemisphere $A$ is the entire equator $2a$. Again the 1-chain is a cycle, as its two boundary points are antipodal and so are identified, but they have opposite orientation. Summarizing, $\partial(A) = 2a$. Therefore there are no 2-cycles, and the group of 1-cycles is $\mathbb{Z}$ which is generated by $a$ but all even multiples are boundaries. This yields the homology

$$
H_2(\mathbb{RP}^2) = 0, \quad H_1(\mathbb{RP}^2) = \frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z}_2, \quad H_0 = \frac{\mathbb{Z}}{0} = \mathbb{Z} \quad (5.6)
$$

where 0 is the trivial group which consists of just the identity element.

To calculate the cohomology we need the coboundary map

$$
\delta(a) = 2A \quad (5.7)
$$
which implies that there are no 1-cocycles and all 2-cochains are 2-cocycles but the even ones are 2-coboundaries. Thus the first and second cohomology groups are isomorphic to the second and first homology groups

\[ H^2(\mathbb{RP}^2) = \frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z}_2, \quad H^1(\mathbb{RP}^2) = 0 = 0, \quad H^0 = \frac{\mathbb{Z}}{0} = \mathbb{Z}. \]  

(5.8)

If the top homology group, in this case the second homology group, of a connected manifold is \( \mathbb{Z} \) then the manifold is said to be orientable, and a choice of which homology element is \(+1\) and which is \(-1\) is called a choice of orientation. The top cohomology group of an orientable connected manifold will always be \( \mathbb{Z} \), in terms of differential forms it is generated by the volume form. A nonorientable connected manifold has top homology group equal to 0 and the top cohomology group is \( \mathbb{Z}_2 \). Thus torii are orientable but \( \mathbb{RP}^2 \) is not.

### 5.2 Cohomology is a ring and homology is a module

Consider a \( p \)-simplex \( \sigma_p \) with vertices \((0...p)\) and a \( q \) simplex \( \sigma_q \) with vertices \((p...p+q)\). Then we may define a \((p+q)\)-simplex \( \sigma_{p+q} \) with vertices \((0...p+q)\) such that the original two simplices are faces of the new simplex with an embedding map known as the canonical embedding. We can define a bilinear multiplication on chains that takes each pair of a \( p \)-simplex and a \( q \)-simplex to this \((p+q)\)-simplex if they have precisely one vertex in common, which is called \( p \) above, and otherwise gives zero for the product of the pair. Thus the product of a \( p \)-chain by a \( q \)-chain gives a \((p+q)\)-chain.

This product does not obey any nice relation with respect to the boundary map \( \partial \), but the corresponding operation on cochains, called the cup product \( \cup \) is better behaved. The coboundary operator is a superderivation with respect to the cup product, because it obeys the Leibnitz rule

\[ \delta(c^p \cup c^q) = (\delta c^p) \cup c^q + (-1)^p c^p \cup \delta c^q. \]  

(5.9)

This means in particular that if the cochains \( c^p \) and \( c^q \) are cocycles then so is the \((p+q)\)-cochain \( c^p \cup c^q \). Thus the cup product is a multiplication on the cohomology. It is well-defined, in other words it is independent of the choice of cocycle representative, because the cup product of a cocycle with a coboundary is again a coboundary and so changing the choice of representative of a factor \( c^p \) by a coboundary only changes
the product cocyle by a total coboundary. Armed with this multiplicative product, cohomology is a ring.

A simple example is the cohomology ring of the 2-torus. The product of any class with the generator 1 of the zeroth cohomology is just the original class. The product of the two generating 1-classes is the 2-class

$$ (a - b)(a + c) = (a - b)(b + c) = A = -B $$

and all other products of generators vanish. For example, the square of any 1-class is the trivial cohomology class.

The cap product induces a map on (co)homology because it satisfies a kind of Liebnitz rule

$$ \partial(a \cap b) = (-1)^p((\delta a) \cap b - a \cap (\partial b)) $$

where $p$ is the degree of the cocycle $a$. For example, the cap product of a first cohomology generator of $T^2$ with the top homology class gives the other first homology class.
5.3 Poincaré duality

If a connected $n$-manifold is orientable then the $n$th homology group is $\mathbb{Z}$. Let $A_n$ represent the generator 1 of $H_n$. Then the cap product of any $p$-cocycle $B^p$ with $A_n$ is an $(n-p)$-cycle $B^p \cap A_n$, called the Poincaré dual of $B^p$. As the cap product takes cycles and cocycles to cycles, and the cap product of a boundary or coboundary with anything is a boundary, the Poincaré duality map (the cap product with the top class $A_n$) is a map from the $p$th cohomology to the $(n-p)$th homology. This map turns out to be an isomorphism, and so Poincaré duality provides an isomorphism

$$H^p(M) = H_{n-p}(M)$$  \hspace{1cm} (5.13)

over any coefficient ring.

In the case of $\mathbb{R}P^2$, whose homology and cohomology we calculated above, (5.13) is not satisfied. This comes as no surprise, as $M$ is not orientable and so $A_n$ does not exist. Thus the orientability assumption is really necessary. On the other hand the homology and cohomology of the circle do satisfy this relation, as all of the relevant groups are $\mathbb{Z}$. Likewise $T^2$ provides a working example, as the 0th and 2nd groups are all $\mathbb{Z}$ while the first groups are exchanged.

Torii provide trivial examples of Poincaré duality, because there is no torsion in the (co)homology so the homology and cohomology are isomorphic. A less trivial example is $\mathbb{R}P^3$, which is the 3-sphere $S^3$ quotiented by the antipodal $\mathbb{Z}_2$ action. There is a simplicial decomposition similar to that of $\mathbb{R}P^2$, it can be constructed using a single simplex of each dimension $e_0$, $e_1$, $e_2$ and $e_3$ with the boundary map

$$\partial e_3 = \partial e_1 = \partial e_0 = 0, \quad \partial e_2 = 2e_1.$$  \hspace{1cm} (5.14)

Therefore all 0-chains, 1-chains and 3-chains are cycles but the even 1-chains are also boundaries, yielding the homology groups

$$H_0(\mathbb{R}P^3) = H_3(\mathbb{R}P^3) = \mathbb{Z}, \quad H_1(\mathbb{R}P^3) = \mathbb{Z}_2, \quad H_2(\mathbb{R}P^3) = 0.$$  \hspace{1cm} (5.15)

In particular $\mathbb{R}P^3$ is orientable and so Poincaré duality should apply. Name the fundamental cochains $e^i$, then the coboundary map is just the transpose of (5.14)

$$\delta e^0 = \delta e^2 = \delta e^3 = 0, \quad \delta e^1 = 2e^2.$$  \hspace{1cm} (5.16)

Thus all 0-cochains, 2-cochains and 3-cochains are also cocycles but even 2-cochains are also coboundaries, and so the cohomology is

$$H^0(\mathbb{R}P^3) = H^3(\mathbb{R}P^3) = \mathbb{Z}, \quad H^2(\mathbb{R}P^3) = \mathbb{Z}_2, \quad H^1(\mathbb{R}P^3) = 0.$$  \hspace{1cm} (5.17)
and Poincaré duality (5.13) is satisfied.

5.4 Universal coefficient theorem

We have repeatedly claimed that homology and cohomology contain the same information, and the calculations of the two are very similar. Thus, one might suspect that there is a simple algorithm for calculating one given the other. In the orientable case Poincaré duality does this, but there is a very different theorem which always relates them even in the unoriented case, known as the universal coefficient theorem.

To see how this works, consider an arbitrary generator of the $k$th integral homology, represented by a cycle $a_k$. There are two possibilities, either some multiple of $a_k$ is a boundary $\partial b_k = ma_k$ or it is not. In the first case $a_k$ generates a $\mathbb{Z}_m$ subgroup of $H_k$, in the second it generates a $\mathbb{Z}$ subgroup. Taking the transpose of the boundary map we find the coboundary map. In the first case $\delta a_k = mb_k + 1$. It turns out that $b_k + 1$ can always be taken to be a cocycle, and so it generates a $\mathbb{Z}_m$ subgroup of $H^{k+1}$. If this were not true, we would find only a proper subgroup of $\mathbb{Z}_m$, meaning that the torsion part of cohomology would be smaller than that of homology. But then running this argument in the other direction we would find that the torsion part of homology is smaller than itself and so create a paradox. Now let us consider the second case, in which no multiple of $a_k$ is a boundary. Now $\delta a_k = 0$ and so $a^k$ generates a $\mathbb{Z}$ subgroup of $H^k$.

Summarizing, the torsion part of $H_k$ also appears in $H^{k+1}$ whereas the free part appears in $H^k$. This means that $H^k$ is equal to the direct sum of the free part of $H_k$ plus the torsion part of $H_{k-1}$. Conversely, $H_k$ is the direct sum of the free part of $H^k$ plus the torsion part of $H^{k+1}$. This is the content of the universal coefficient theorem when applied to integral (co)homology. Notice that the universal coefficient theorem is a relation between different homology and cohomology groups than those related by Poincaré duality. Combining them one finds that the free parts of $H_k$, $H^k$, $H_{n-k}$ and $H^{n-k}$ of an oriented $n$-manifold are all the same, as are the torsion parts of $H_k$, $H^{k+1}$, $H_{n-k-1}$ and $H^{n-k}$.

It is easy to check that this works for $\mathbb{R}P^2$ and $\mathbb{R}P^3$, the free parts of the homology and cohomology is the same, as are the torsion parts of $H_1$ and $H^2$, so it is one degree higher for cohomology as it should be. In the case of $L_{4,1} \times L_{6,1}$ we may calculate the cohomology using Poincaré duality and then check that it obeys the universal
coefficient theorem. By Poincaré dualizing (4.23) with the $\mathbb{Z}_2$ Tor term (4.29) added to $H_3$ one finds the cohomology groups

\[
\begin{align*}
H^6(L_{4,1} \times L_{6,1}) &= \mathbb{Z} \\
H^5(L_{4,1} \times L_{6,1}) &= \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\
H^4(L_{4,1} \times L_{6,1}) &= \mathbb{Z}_2 \\
H^3(L_{4,1} \times L_{6,1}) &= \mathbb{Z}^2 \oplus \mathbb{Z}_2 \\
H^2(L_{4,1} \times L_{6,1}) &= \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \\
H^1(L_{4,1} \times L_{6,1}) &= 0 \\
H^0(L_{4,1} \times L_{6,1}) &= \mathbb{Z}
\end{align*}
\]

which agrees with the universal coefficient theorem.

6 Circle bundles

6.1 What is a circle bundle

A circle bundle is a manifold $E$, called the total space of the bundle, equipped with a free circle action, which means that the manifold contains a series of loops, called orbits, none of which degenerate, such that each point lies on precisely one loop. The collection of these loops is another manifold, $M$, called the base space. There is a projection map, $\pi : E \to M$ which takes each point in the total space $E$ to the loop on which it lies, which is a point in $M$, and the preimage of each point $m$ in $M$ is a circle called the fiber over $m$. All of this data is summarized in the picture

\[
\begin{array}{c}
S^1 \to E \\
\downarrow \pi \\
M
\end{array}
\]  

which is a short exact sequence.

A choice of a point in each orbit is called a section of $E$, and when there exists a continuous section using all of the orbits then the bundle (6.1) is said to be trivial. In this case the section is the manifold $M$, which is therefore a submanifold of $E$. In
fact, topologically a trivial bundle is just \( E = M \times S^1 \), in other words, there exists a continuous map between \( E \) and \( M \times S^1 \) and this map has a continuous inverse, if the bundle is trivial. All bundles are trivial when fibered over a topologically trivial space, which is a space like a ball or disk that can be continuously deformed to a point. Any manifold can be covered by topologically trivial neighborhoods, and so any given bundle can be trivialized over each neighborhood. The fibers in the different neighborhoods are related by transition functions on the overlaps.

The simplest example of a nontrivial circle bundle is called the Hopf fibration, in which \( E = S^3 \) and \( M = S^2 \)

\[
S^1 \longrightarrow S^3 \\
\downarrow \pi \\
S^2
\]

As one realization of this bundle, consider the complex vector space \( \mathbb{C}^2 \) with coordinates \( a + ib \) and \( c + id \). The three-sphere \( E = S^3 \) is the subset such that

\[
a^2 + b^2 + c^2 + d^2 = 1.
\]

The free circle action consists of a simultaneous phase rotation, by an equal phase, on the complex numbers \( a + ib \) and \( c + id \). The space of orbits, \( M = S^2 = \mathbb{C}\mathbb{P}^1 \), is then the (Bloch) sphere. To see that the space of orbits is a sphere, notice that the ratio \( (a + ib)/(c + id) \) is invariant under the circle action, and that every orbit is entirely determined by this ratio. These ratios can assume any value in the complex plane plus infinity, and so by the stereographic projection they describe an \( S^2 \). The overall phase gives the circle fiber. The bundle is not trivial because \( S^3 \neq S^2 \times S^1 \), as one finds by calculating their homologies using the Künneth formula.

\[
\begin{align*}
H_0(S^3) &= H_0(S^2 \times S^1) = H_3(S^3) = H_3(S^2 \times S^1) = H_1(S^2 \times S^1) = H_2(S^2 \times S^1) = \mathbb{Z} \\
H_1(S^3) &= H_2(S^3) = 0.
\end{align*}
\]

Although the Hopf fibration is nontrivial, it can be described in terms of local trivializations and transition functions. To do this, we first cover \( S^2 \) with two discs, \( S^2_S \) and \( S^2_N \) which are all of \( S^2 \) minus the north pole and minus the south pole respectively. The bundle can be trivialized over these disks, giving \( S^2_S \times S^1 \) and \( S^2_N \times S^1 \). Now these need to be pasted together. Let \( \theta_S \) and \( \theta_N \) by the coordinates of the circles in the two trivializations and let \( \phi \) be the latitude of the \( S^2 \) and therefore of \( S^2_S \) and \( S^2_N \). Then we glue the two patches together via the identification

\[
(\phi, \theta_S) \sim (\phi, \theta_N + \phi).
\]
As a check, notice that before the identification the fundamental group of each trivialized patch is $\mathbb{Z}$, because a loop may wrap the $S^1$ any number of times. However after the identification this circle is identified with the $\phi$ circle, which is contractible, in particular it can be shrunk to zero by pulling it over one of the poles. Therefore the fundamental group of the total space of the bundle is trivial, as it should be for $S^3$, as $\text{H}_1(S^3) = 0$ and so its abelianization (the fundamental group) is the trivial group $0$. We will see in subsection 6.3 how to calculate the cohomology of the total space, which together with the Poincaré conjecture (Perelman theorem?) proves that $E = S^3$.

As a slightly more complicated example, we could consider the transition function

$$ (\phi, \theta_S) \sim (\phi, \theta_N + k\phi) \quad (6.6) $$

for some integer $k$. Now the $\theta$ circle is no longer identified with the trivial $\phi$ circle. However $k$ multiples of the $\theta$ circle is now trivial, and so the first homology group is

$$ \text{H}_1(E) = \mathbb{Z}_k. \quad (6.7) $$

This supports the claim that the total space is $E = L_{k,1} = S^3/Z_k$, which it is. In the case $k = 2$ we find $E = L_{2,1} = \mathbb{R}P^3$ and so the first homology group is the $\mathbb{Z}_2$ that we calculated for $\mathbb{R}P^3$ in (5.15).

### 6.2 The Chern class

The above characterizations of circle bundles are a bit *ad hoc*, and as a result it is difficult to see that the different descriptions of lens spaces as circle bundles are equivalent. Furthermore, it is hard to see that they are independent of the choice of covering in terms of topologically trivial spaces. Just as algebraic topology allows us to partially characterize manifolds in terms of purely algebraic data like (co)homology groups, the theory of characteristic classes allows us to characterize bundles in terms of cohomology classes. In the case of circle bundles, which is the case we are discussing in which the fiber is a circle, there is a complete classification. Circle bundles are entirely determined by a single invariant, an element of the second cohomology group with integer coefficients called the Chern class. Conversely, every element of the second integral cohomology of $M$ is the Chern class of some circle bundle $E$ over the base space $M$. 

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The Chern class of an arbitrary circle bundle can be constructed as follows. Replace each circle fiber by the cone over the circle. In other words, replace it with the complex plane such that it is a circle and constant radius. Let the transition functions act on all of the circles at different radii just as it acted on the original circle, which is now at radius one. The resulting noncompact space is called the associated complex line bundle, because the fiber over every point in $M$ is now the complex line $\mathbb{C}$. It will appear again in Chethan’s lectures. Complex line bundles, via this construction, are in one-to-one correspondence with the circle bundles that you get by restricting each fiber $\mathbb{C}$ to the circle at radius one. Therefore complex line bundles will also be classified entirely by their Chern classes.

A section of a complex line bundle is a continuous choice of a point in the fiber $\mathbb{C}$ over each point $m \in M$. While circle bundles only have global sections when the bundle is trivial, complex line bundles always have global sections. For example, the origin $0 \in \mathbb{C}$ is invariant under the transition functions, and so there is always a section, called the zero section, in which one chooses the point $0 \in \mathbb{C}$ in each fiber. In fact any section of a nontrivial complex line bundle will necessarily have some zeroes.

To construct the Chern class, we will need to consider a generic section, by which we will mean a section whose zeroes are all transversal. Transversal roughly means that they will be codimension two in $M$, which is almost always the case as 0 is codimension two in $\mathbb{C}$. Thus if $M$ is an $n$-manifold, then the 0-set of a section, that is the subset of $M$ for which the section is equal to zero, is an $(n-2)$-cycle. This cycle is called the divisor associated with the line bundle. A divisor contains more information than just its homology class, it also comes with an embedding, and so divisors contain some information which is not just topological, and so often appear in geometry as in Chethan’s talk. However we will be less ambitious, and consider only the topology of a bundle, which will be characterized entirely by the homology class of the divisor. The Poincaré dual of this class in $H_{n-2}(M)$ is a cohomology class in $H^2(M)$, the Chern class. Although we have not proved it, this class is independent of our choice of section, it depends only on the topology of the bundle.

As simple examples, consider the lens spaces $E = L_{k,1}$, which we have seen have base $M = S^2$. The associated complex line bundle is called $O(k)$. If we use complex coordinates $z$ for $S^2_3$ and $w$ for $S^2_N$ such that $z = 1/w$ then the transition functions identifies

$$f(z) \sim z^{-k} f(z) = u^k f(1/w).$$  \hspace{1cm} (6.8)
For example, consider the trivial bundle $k = 0$, where $E = S^2 \times S^1$. In this case, for example, the constant functions are global sections. In fact, they are the only holomorphic global sections, as any polynomial would have some poles, but in this talk we will not restrict our attention to holomorphic sections.

More generally, if $k$ is negative there are no holomorphic sections, however topologically positive and negative $k$ are the same, so we may restrict our attention to positive $k$. For positive $k$ any degree $k$ polynomial in $z$ gives a global section since the $w^k$ on the right hand side of (6.8) kills any poles at $w = 0$, which is the only point where the $z$ coordinates are ill-defined. The divisors are now the roots of $f$ and the genericity condition implies that all $k$ roots are distinct, and so the divisor consists of $k$ points. While we do not need the section to be holomorphic, we have claimed that the homology class of the divisor is independent of the section, and so it suffices to use holomorphic sections in this case. Although of course had the base not been complex then no holomorphic section would have been available. The fact that the divisor consists of $k$ points, all of which turn out to have the same orientations, implies that the divisor represents the class

$$[D] = k \in \mathbb{Z} = H_0(S^2).$$

(6.9)

By Poincaré duality, the Chern class is

$$c(E) = k \in \mathbb{Z} = H^2(S^2).$$

(6.10)

### 6.3 The Gysin sequence

Given the Chern class of a circle bundle and the cohomology of the base $M$ one can determine the cohomology of the total space $E$, or given partial information about all three one can often determine the rest. This is not surprising, as the cohomology of $E$ is completely characterized by that of $M$. The relation between the cohomology groups however is quite simple, they are related by a long exact sequence known as the Gysin sequence. In fact, the Gysin sequence can be generalized to sphere bundles, in which the fiber is an $n$-sphere $S^n$. These bundles are partially characterized by a degree $(n + 1)$ cohomology class called the Euler class $e$, which is the Chern class in the case $n = 1$. Although the Euler class only partially characterizes the topology of $E$ when $n > 1$, via the Gysin sequence it completely characterizes the cohomology.
We will describe the Gysin sequence for circle bundles, but the Gysin sequence for general sphere bundles may be obtained by simply replacing \( c \) with \( e \) and adjusting the degrees of the classes respectively. The Gysin sequence is

\[
\begin{align*}
\ldots \rightarrow \pi^* H^n(E) \overset{\pi_*}{\rightarrow} H^{n-1}(M) \overset{\cup c}{\rightarrow} H^{n+1}(M) \overset{\pi_*}{\rightarrow} H^{n+1}(E) \overset{\pi_*}{\rightarrow} \ldots
\end{align*}
\]

where \( \pi^* \) and \( \pi_* \) are the pullback and pushforward of the projection map \( \pi : E \rightarrow M \) and \( \cup c \) is the cup product with the Chern class.

Now we can apply the Gysin sequence to derive the homology of the lens space \( L_{p,1} \), stated in Eq. (4.22). The base space is \( M = S^2 \), the total space is \( E = L_{p,1} \) and the Chern class is

\[ c = p \in \mathbb{Z} = H^2(S^2). \]  

Putting this all together we get the Gysin sequence for the cohomology of a lens space. It starts with

\[
0 \overset{p \cup}{\rightarrow} H^0(S^2) = \mathbb{Z} \overset{\pi^*}{\rightarrow} H^0(L_{p,1}) \overset{\pi_*}{\rightarrow} H^{-1}(S^2) = 0
\]

and so the pullback \( \pi^* \) is an isomorphism, yielding

\[ H^0(L_{p,1}) = H^0(S^2) = \mathbb{Z} \]

which means that the lens space is connected.

The next nontrivial block of the Gysin sequence is

\[
0 \overset{p \cup}{\rightarrow} H^1(S^2) = 0 \overset{\pi^*}{\rightarrow} H^1(L_{p,1}) \overset{\pi_*}{\rightarrow} H^0(S^2) = \mathbb{Z}
\]

\[
\overset{p \cup}{\rightarrow} H^2(S^2) = \mathbb{Z} \overset{\pi^*}{\rightarrow} H^2(L_{p,1}) \overset{\pi_*}{\rightarrow} H^1(S^2) = 0.
\]

The kernel of the cup product of \( H^0(S^2) \) with \( p \cup \) is trivial, as the cup product with \( p \) is nonvanishing on each element of \( H^0(S^2) = \mathbb{Z} \), it just gives the corresponding element of \( H^2(S^2) \) multiplied by \( p \neq 0 \). Therefore

\[ \pi_*(H^1(L_{p,1})) = 0. \]

Since the image of \( \pi^* \) on \( H^1(S^2) = 0 \) is zero, as its domain is zero, \( H^1(L_{p,1}) \) gets contributions neither from the left nor from the right, and so

\[ H^1(L_{p,1}) = 0. \]
The image \( p \cup H^0(S^2) \) is not all of \( H^2(S^2) = \mathbb{Z} \), instead it consists of only the elements which are divisible by \( p \). Therefore these elements are the kernel of \( \pi^* : H^2(S^2) = \mathbb{Z} \longrightarrow H^2(L_{p,1}) \). Thus the image of this map is \( \mathbb{Z}_p \). As \( H^2(L_{p,1}) \) gets no contribution from the right

\[
H^2(L_{p,1}) = \mathbb{Z}_p.
\] (6.18)

The last nontrivial piece of the Gysin sequence is

\[
0 = H^3(S^2) \stackrel{\pi^*}{\longrightarrow} H^3(L_{p,1}) \stackrel{\pi^*}{\longrightarrow} H^2(S^2) = \mathbb{Z} \longrightarrow H^4(S^2) = 0
\] (6.19)
and so

\[
H^3(L_{p,1}) = H^2(S^2) = \mathbb{Z}
\] (6.20)

establishing that the lens spaces are orientable.

A more complicated example is the set of 5-dimensional Sazaki-Einstein spaces \( Y^{pq} \), which are used in the context of AdS/CFT, where it provides gravity duals to cascading gauge theories. These spaces are circle bundles over \( M = S^2 \times S^2 \). The Chern class is an element of \( H^2(S^2 \times S^2) = \mathbb{Z}^2 \) and so is a pair of integers, \( p \) and \( q \). Actually this description of \( Y^{pq} \) only applies when \( p \) and \( q \) are relatively prime, otherwise \( Y^{pq} \) is singular. More generally this circle bundle describes Romans’ spaces \( T^{pq} \), but we will ignore this distinction and refer to all of these spaces as \( Y^{pq} \). We can find the cohomology groups of a general \( Y^{pq} \) using the Gysin sequence, even when \( p \) and \( q \) are not relatively prime.

The first nontrivial part of the Gysin sequence is

\[
0 \stackrel{(p,q)}{\longrightarrow} H^0(S^2 \times S^2) = \mathbb{Z} \quad \stackrel{\pi^*}{\longrightarrow} \quad H^0(Y^{pq}) \stackrel{\pi^*}{\longrightarrow} H^{-1}(S^2 \times S^2) = 0
\] (6.22)
and so the pullback \( \pi^* \) is an isomorphism, yielding

\[
H^0(Y^{pq}) = H^0(S^2 \times S^2) = \mathbb{Z}
\] (6.23)

which means that \( Y^{pq} \) is connected.

The next piece is again similar to the lens space case

\[
0 \stackrel{(p,q)}{\longrightarrow} H^1(S^2 \times S^2) = 0 \quad \stackrel{\pi^*}{\longrightarrow} \quad H^1(Y^{pq}) \stackrel{\pi^*}{\longrightarrow} H^0(S^2 \times S^2) = \mathbb{Z}
\]
\[
\stackrel{(p,q)}{\longrightarrow} \quad H^2(S^2 \times S^2) = \mathbb{Z}^2 \quad \stackrel{\pi^*}{\longrightarrow} \quad H^2(Y^{pq}) \stackrel{\pi^*}{\longrightarrow} H^1(S^2 \times S^2) = 0.
\] (6.24)
Again, assuming that $p$ and $q$ are not both equal to zero, we find that $(p,q)\cup$ has no kernel and so
\[ H^1(Y^{pq}) = 0. \] (6.25)

However the image of $(p,q)\cup$ in $H^2(S^2 \times S^2) = \mathbb{Z}^2$ is more complicated. Again it is only a proper sublattice of $\mathbb{Z}^2$, but this time it misses an entire free group $\mathbb{Z}$ plus anything which when multiplied by a constant gives the element $(p,q)$. Such elements form a finite cyclic subgroup whose order is $\gcd(p,q)$, the greatest common divisor of $p$ and $q$. Therefore
\[ H^2(Y^{pq}) = \mathbb{Z} \oplus \mathbb{Z}_{\gcd(p,q)}. \] (6.26)

The next useful piece is slightly longer than last time
\[ 0 = H^3(S^2 \times S^2) \xrightarrow{\pi^*} H^3(Y^{pq}) \xrightarrow{\pi^*} H^2(S^2 \times S^2) = \mathbb{Z}^2 \xrightarrow{\pi^*} H^4(S^2 \times S^2) = \mathbb{Z} \]
\[ \xrightarrow{\pi^*} H^4(Y^{pq}) \xrightarrow{\pi^*} H^3(S^2 \times S^2) = 0. \] (6.27)

The kernel of $(p,q)\cup : H^2(S^2 \times S^2) = \mathbb{Z}^2 \longrightarrow H^4(S^2 \times S^2) = \mathbb{Z}$
is $\mathbb{Z}$, which is generated by $(q,-p)/\gcd(q,p)$ and so
\[ H^3(Y^{pq}) = \mathbb{Z}. \] (6.29)

The image of (6.28) on the other hand is not all of $\mathbb{Z}$, but just the subset consisting of numbers with are sums of multiples of $p$ by multiples of $q$, which is the same as the subset of multiples of $\gcd(p,q)$. This subset is the kernel of the next map, the pullback to $H^4(Y^{pq})$, and so the image of that map is
\[ H^4(Y^{pq}) = \mathbb{Z}_{\gcd(p,q)}. \] (6.30)

The last useful part of the Gysin sequence is
\[ 0 = H^5(S^2 \times S^2) \xrightarrow{\pi^*} H^5(Y^{pq}) \xrightarrow{\pi^*} H^4(S^2 \times S^2) = \mathbb{Z} \xrightarrow{(p,q)\cup} H^6(S^2 \times S^2) = 0 \] (6.31)
and so
\[ H^5(Y^{pq}) = H^4(S^2 \times S^2) = \mathbb{Z} \] (6.32)
establishing that the $Y^{pq}$ spaces are orientable.
References

