Lie and Kac-Moody algebras

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Abstract

In these lectures, we present in a selfcontaining way an introduction to the theory of Lie and Kac-Moody algebras. Throughout the text, we mainly focus on the concepts that are important in applications to mathematical physics, especially in the context of hidden symmetries.

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1 Introduction

This introduction to Lie and Kac-Moody algebras is meant to contain tools that are necessary when using this theory in mathematical physics, and in particular in the context of hidden symmetries. The first example of such a symmetry was emphasized by Ehlers in the context of four-dimensional pure gravity [?]. Assuming the existence of one Killing vector and compactifying the metric on the associated circle, one finds a three-dimensional theory that has a global symmetry under the Lie algebra $\mathfrak{sl}(2)$ which is bigger than the expected $\mathfrak{u}(1)$. The appearance of such hidden symmetries could then be generalized to other gravity and supergravity theories in various dimensions. As finite-dimensional simple Lie algebras can be completely classified, one can also completely classify the theories that possess a hidden symmetry under these algebras [?]. A famous example is the E_8 symmetry of $D = 11$ supergravity in the presence of eight commuting Killing vectors.

Geroch first noticed [?] the appearance of an infinite-dimensional symmetry when two commuting Killing vectors are assumed in four-dimensional gravity. This symmetry algebra was later identified as the affine extension of $\mathfrak{sl}(2)$, the simplest example of an infinite-dimensional Kac-Moody algebra [?]. More generally, one can see that any theory that possesses a hidden symmetry under a finite-dimensional Lie algebra $\mathfrak{g}$ in three dimensions can be seen to be symmetric under the affine extension of $\mathfrak{g}$ in two dimensions [?]. Moreover, the two-dimensional theory is integrable.

When one more commuting Killing vector is assumed, that is, when all fields are assumed to depend only on one parameter, these theories then show a yet larger infinite-dimensional symmetry algebra, which is called hyperbolic (See [?] and references therein).

1.1 Outline

The lecture notes are organized as follows. **Section 2** very shortly presents general and usually known definitions to get started. We then in **section 3** review in some details the important tools that appear in the study of finite-dimensional simple Lie algebras. In particular, their full classification is presented as well as their appearance in the context of hidden symmetries. **Section 4** explains how the theory can be generalized to describe a larger class of algebras, called Kac-Moody algebras, which can also be infinite-dimensional. The simplest infinite-dimensional Lie algebras, namely affine Lie algebras, are dicussed in details in **section 5**. Finally, in **section 6**, we discuss some more important aspects: we describe some representations of Kac-Moody algebras and define the Weyl group.

1.2 References

The references mainly used for finite-dimensional Lie algebras are the books [?] by J. Fuchs and C. Schweigert and [?] by Bäuerle and de Kerf. The former is a very nice introduction into the subject, especially intended for physicists while the latter is more formal and gives a detailed and very clear presentation of the classification of finite-dimensional simple Lie algebras. We also made used of the book [?] by Humphreys. For Kac-Moody algebras, we made large use of the book [?] by V. Kac which we recommend for the reader who wants to look up more details.

2 General definitions

To get started, this short section contains very general definitions.

2.1 Lie algebra, structure constants, adjoint representation

Definition 1. *An* ***algebra*** $\mathfrak{U}$ *is a vector space over a field* F *endowed with an additional binary operation* $\diamond : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ *on which the only requirement is that it is bilinear.*

Note that this definition is still very general. Therefore, in order for the definition to be of any interest, the product is generally required to have further properties (e.g. associativity, possessing a unit element, ...). A highly interesting example is given by Lie algebras.

Definition 2. *A* ***Lie algebra*** $\mathfrak{g}$ *is a vector space over a field* F *endowed with a* ***Lie bracket*** $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that*

1. $[ax + by, z] = a\,[x, z] + b\,[y, z]$ *for all* $x, y, z \in \mathfrak{g}$ *and* $a, b \in F$ *(bilinearity),*
2. $[x, x] = 0$ *for all* $x \in \mathfrak{g}$*,*
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in \mathfrak{g}$ *(Jacobi identity).*

The field F is typically choosen as $\mathbb{R}$ or $\mathbb{C}$. Note that the conditions 1. and 2. imply that the bracket is asymmetric:

$$[x, y] = -\,[y, x]\,. \tag{2.1}$$

In the case of matrix Lie algebras, the Lie bracket is often chosen to be the commutator $[x, y] = x.y - y.x$ where $x.y$ is the matrix product.

Let $\mathcal{B} = \{T^a\}$ be a basis of $\mathfrak{g}$ seen as a vector space. The elements T^a of $\mathcal{B}$ are called **generators** of $\mathfrak{g}$. Because of the bilinearity, the Lie bracket is determined once it it known on a basis $\mathcal{B}$. Therefore one can define the Lie bracket through the expansion of the bracket of two generators with respect to the basis:

$$\left[T^a, T^b\right] = f^{ab}_{\ \ c}\, T^c. \tag{2.2}$$

The expansion coefficients $f^{ab}_{\ \ c}$ are called the **structure constants** of the Lie algebra $\mathfrak{g}$. They are antisymmetric in ab and, because of the Jacobi identity, they fulfill the following condition:

$$f^{ab}_{\ \ c}\, f^{cd}_{\ \ e} + f^{da}_{\ \ c}\, f^{cb}_{\ \ e} + f^{bd}_{\ \ c}\, f^{ca}_{\ \ e} = 0. \tag{2.3}$$

These are all very abstract definitions. In practice, we want to make contact with a physical system. In other words, we want the elements of the algebra to act on some space V, that would be the space of physical states for example. More precisely, by acting on V one means that for all $x \in \mathfrak{g}$, there is an associated map

$$R(x) : V \to V. \tag{2.4}$$

It is of particular interest to consider the case when V is a vector space over F and the $R(x)$ are linear mappings. The space of all linear mappings from V to V is itself a vector space over F, denoted $\mathfrak{gl}(V)$. If dim $V = n$ is finite, it is described by $n \times n$ matrices with entries in F. Endowed with the composition $\circ$ and defining the Lie bracket as the commutator of compositions it acquires the structure of a Lie algebra. We want the representation R to preserve the vector space structure and the the Lie algebra structure of $\mathfrak{g}$. We therefore need to impose that it is a homomorphism, which means that it is a linear and reproduces the Lie bracket,

$$[R(x), R(y)] = R([x, y]). \tag{2.5}$$

One then uses the following definition of a representation:

Definition 3. *A **representation** R of a Lie algebra $\mathfrak{g}$ on a **representation space** V is given by a homomorphism of Lie algebras $R : \mathfrak{g} \to \mathfrak{gl}(V)$. If R is injective, then the representation is said to be **faithful**.*

Obviously, the trivial representation, that associates the zero map to any element of the algebra, exists for any Lie algebra. There is also a non-trivial representation that exists for any Lie algebra, the **adjoint representation**, that acts on the vector space $\mathfrak{g}$ itself:

$$R_{\text{ad}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : x \to \text{ad } x, \tag{2.6}$$

with the map ad x defined as

$$(\text{ad } x)(y) = [x, y] \,. \tag{2.7}$$

Other ways to construct representations will be seen in section 6.

2.2 Ideal, abelian/simple/semisimple algebras

Let us first introduce a notation for subsets $\mathfrak{h}$, $\mathfrak{l}$ of a Lie algebra:

$$[\mathfrak{h}, \mathfrak{l}] \equiv \text{span}_F \{[x, y] \,|x \in \mathfrak{h}, y \in \mathfrak{l}\} \tag{2.8}$$

Using this notation one can easily make the following definitions:

Definition 4. *A (Lie) **subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subset of $\mathfrak{g}$ that is also a Lie algebra: $\mathfrak{h} \subseteq \mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.*

Definition 5. *(i) A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an **ideal** iff $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.*
*(ii) A **proper ideal** is an ideal that is neither equal to $\{0\}$ nor to $\mathfrak{g}$ itself (which are two generic ideals of $\mathfrak{g}$).*

Definition 6. *(i) A Lie algebra $\mathfrak{g}$ is said to be **abelian** iff $[\mathfrak{g}, \mathfrak{g}] = \{0\}$.*
*(ii) A Lie algebra $\mathfrak{g}$ is said to be **simple** iff it is not abelian and contains no proper ideal.*
*(iii) A direct sum of simple Lie algebras is called **semisimple**. A direct sum of simple and abelian Lie algebras is called **reductive**.*

Note that any one-dimensional Lie algebra is abelian. Up to an isomorphism, there exists only one one-dimensional Lie algebra, commonly noted $\mathfrak{u}(1)$. Moreover, any d-dimensional abelian Lie algebra $\mathfrak{g}_{Ab}$ is isomorphic to the direct sum of d one-dimensional abelian Lie algebras

$$\mathfrak{g}_{Ab} = \bigoplus_{i=1}^{d} \mathfrak{u}(1). \tag{2.9}$$

Therefore, the non-trivial part of the classification of reductive algebras is the classification of simple Lie algebras. It will be the treated in the next section.

3 Simple finite-dimensional Lie algebras

In this section, we restrict to the case of semisimple finite-dimensional Lie algebras. We will see how to characterize and classify them.

3.1 The Cartan subalgebra

A first fundamental structure of Lie algebras is the Cartan subalgebra, which is a maximal abelian subalgebra. A more precise characterisation requires a few other definitions first.

Definition 7. *An element x of a Lie algebra $\mathfrak{g}$ is called a* ***semisimple element*** *of $\mathfrak{g}$ if it has the property that the map $\operatorname{ad} x$ is diagonizable, that is, if there is a choice of basis $B = \{T^a\}$ such that $[x, T^a]$ is proportional to T^a for any element of B.*

Definition 8. *A field F is* ***algebraically closed*** *if any algebraic equation with coefficients in F has a solution in F.*

For example $\mathbb{R}$ is not algebraically closed but $\mathbb{C}$ is.

Theorem 1. *Every semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field F contains semisimple elements x.*

Proof. When comparing the general relation $[x, T^a] = M^a{}_b T^b$ to the equation for a semisimple element $[x, T^a] = \lambda T^a$, one sees that λ must be a solution of the secular or characteristic equation $\det\,(M - \lambda \mathbf{1}) = 0$. This equation is ensured to have a solution if F is algebraically closed. □

From now on we will assume that $F = \mathbb{C}$, so that we consider complex Lie algebras. The study of real Lie algebras, for which $F = \mathbb{R}$ is considerably harder.

One can now define a key concept in the study and classification of semisimple complex Lie algebras:

Definition 9. *A* ***Cartan subalgebra*** *$\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra consisiting entirely of semisimple elements.*

By maximal, we mean that it is not included in a bigger abelian subalgebra with the same property. This subalgebra is constructed by choosing, among the semisimple elements, a maximal set of linearly independent elements, denoted by H^i, which possess zero Lie brackets among themselves,

$$\left[H^i, H^j\right] = 0 \qquad \text{for } i, j = 1, 2, \ldots, r. \tag{3.1}$$

These elements H^i form a basis of a Cartan subalgebra $\mathfrak{h}$.

A semisimple Lie algebra can possess many different Cartan subalgebras. However, they are all related by automorphisms of $\mathfrak{g}$, so that the freedom in choosing a Cartan subalgebra does not lead to any arbitrariness in the description of semisimple Lie algebras. Moreover, all Cartan subalgebras have the same dimension r, that is accordingly a property of the Lie algebra $\mathfrak{g}$ and is called its **rank**.

3.2 Weights of a representation and roots

Let R be a representation of $\mathfrak{g}$ and h an element of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let us assume that $R_h \equiv R(h)$ has an eigenvalue $\mu(h)$ for the eigenvector $f_{\mu,A}$, where the index A stands for a possible degeneracy of the eigenvalue $\mu(h)$. Thus one has

$$R_h(f_{\mu,A}) = \mu(h) f_{\mu,A}. \tag{3.2}$$

The eigenvalue $\mu(h)$ is a complex number that depends linearly on h, and hence μ is a linear function $\mathfrak{h} \to \mathbb{C}$. By definition, the space of linear maps from a vector space V to the base field F is called the dual vector space $V^\star$ of V. Thus, μ is an element of the vector space $\mathfrak{h}^\star$ dual to $\mathfrak{h}$. It is called a **weight** of the representation R.

As the generators H^i of a Cartan subalgebra $\mathfrak{h}$ have zero Lie brackets among eachother, the adjoint maps of all elements of the Cartan subalgebra are simultaneously diagonalizable.

As a consequence, there exists a basis of generators of $\mathfrak{g}$ that are all eigenvectors of the maps ad h for all $h \in \mathfrak{h}$. In other words, $\mathfrak{g}$ is spanned by elements y that satisfy

$$[h, y] = \alpha_y(h)y. \tag{3.3}$$

The functions $\alpha_y(h)$ are the weights of the adjoint representation of the Cartan subalgebra and if they do not vanish they are called the **roots** of the Lie algebra $\mathfrak{g}$. The set of all roots of a Lie algebra is called the root system and is here denoted Δ. It is included in $\mathfrak{h}^\star$ (and has the same dimension r). Moreover, on can prove that vanishing weights correspond to elements of the Cartan subalgebra: $[h, x] = 0$ for all $h \in \mathfrak{h} \Rightarrow x \in \mathfrak{h}$. This is equivalent to saying that the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{h}$ itself. (The proof can for example be found in section 8.2 of [**?**].)

3.3 Cartan-Weyl basis

Because all generators of $\mathfrak{g}$ satisfy the relation (3.3) and because all vanishing weights correspond to elements of the Cartan subalgebra, the Lie algebra can be written as a direct sum of vector spaces according to

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} |\, [h, x] = \alpha(h)x \ \ \forall\, h \in \mathfrak{h}\}. \tag{3.4}$$

The dimension of $\mathfrak{g}_\alpha$ is called the **multiplicity** of the root α. In the case of simple finite-dimensional Lie algebras, one can show that the multiplicity of any root is one. As a consequence, each root α determines, up to a normalization, one basis element of the algebra, here denoted E^α, sometimes called a step operator. A full basis of generators of the Lie algebra $\mathfrak{g}$ is accordingly given by

$$B_{CW} = \{H^i \mid i = 1, \ldots, r\} \cup \{E^\alpha \mid \alpha \in \Delta\}, \tag{3.5}$$

with the commutation relations

$$\left[H^i, H^j\right] = 0, \qquad\qquad \left[H^i, E^\alpha\right] = \alpha(H^i)E^\alpha \ \text{ for } \ \alpha \in \Delta, \tag{3.6}$$

where $\alpha(H^i)$ is non-vanishing for at least one value of i. A basis of this form is called a **Cartan-Weyl basis** of $\mathfrak{g}$. Note that if $\left[E^\alpha, E^\beta\right] \neq 0$, then $\left[E^\alpha, E^\beta\right] \propto E^{\alpha+\beta}$.

3.4 The Killing form

To analyse further the root system Δ, one will need to define an inner product on Δ. This can be done by first defining an analogous structure on its dual space $\mathfrak{h}$, that can in turn be otained as the restriction of a bilinear form on the whole algebra $\mathfrak{g}$. It must possess the properties of symmetry, bilinearity and invariance (or associativity). The latter condition means that, for all $x, y, z \in \mathfrak{g}$,

$$\kappa([x, y]\,, z) = \kappa(x, [y, z]). \tag{3.7}$$

For simple finite-dimensional Lie algebras, one can see that, up to an overall factor, there is a unique invariant symmetric bilinear form, given by

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}\ x \circ \mathrm{ad}\ y) \tag{3.8}$$

where $\circ$ denotes the composition of maps and tr is the trace of linear maps. This bilinear form is called the **Killing form**.

For an arbitrary finite Lie algebra, the Killing form κ can be degenerate, i.e. there may exist elements $x \neq 0$ in $\mathfrak{g}$ such that $\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$. It that case, κ will not lead to proper inner product in $\mathcal{H}^\star$. However, Cartan proved the important result that a finite-dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if κ is non-degenerate. Moreover, te restriction of κ on the Cartan subalgebra $\mathfrak{h}$ is also non-degenerate.

Using the Killing form, one can define a vector space isomorphism

$$\nu : h \in \mathfrak{h} \to \nu_h \in \mathfrak{h}^\star \tag{3.9}$$

between $\mathfrak{h}$ and its dual $\mathfrak{h}^\star$ in the following way:

$$\nu_h(h') := \kappa(h, h'). \tag{3.10}$$

The element of $\mathfrak{h}$ mapped to the root α by the isomorphism ν is called the **star vector** associated to α and is noted t_α:

$$t_\alpha := \nu^{-1}(\alpha) \in \mathfrak{h}. \tag{3.11}$$

It satisfies

$$\alpha(h) = \nu_{t_\alpha}(h) = \kappa(t_\alpha, h). \tag{3.12}$$

This will allow us to define a Euclidean inner product in section 3.6.

3.5 Root chains and properties of roots

We will see here a series of theorems about roots of semisimple Lie algebras, providing a first glance at the important restriction imposed for a element of $\mathfrak{h}^\star$ to be a root. These theorems will be crucial for the classification.

Theorem 2. *1. The only multiples of $\alpha \in \Delta$ that are roots are $\pm\alpha$.*

2. *Let $\alpha \in \Delta$ and E^α an associated step operator. Then there exists an element F^α that is a step operator associated to $-\alpha$ such that the set $\{E^\alpha, F^\alpha, H^\alpha\}$, where $H^\alpha := [E^\alpha, F^\alpha]$, spans a three-dimensional simple Lie algebra.*

3. *The vector H^α satisfies*

$$H^\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \tag{3.13}$$

For a proof, see for example the section 6.4 of [**?**]. Concerning parts 2. and 3. of the theorem, without actually proving them, let us make some comments. One easily sees that $\{E^\alpha, F^\alpha, H^\alpha\}$ span an algebra. Indeed,

$$[H^\alpha, E^\alpha] = \alpha(H^\alpha)E^\alpha, \tag{3.14}$$

$$[H^\alpha, F^\alpha] = -\alpha(H^\alpha)F^\alpha, \tag{3.15}$$

$$[E^\alpha, F^\alpha] = H^\alpha. \tag{3.16}$$

Moreover, with the third part of the theorem and using equation (3.12), one sees that $\alpha(H^\alpha) = 2$ and thus one recovers the usual commutation relations of $\mathfrak{sl}(2, \mathbb{C})$, the Lie algebra of complex 2×2 matrices, with the matrix commutator as Lie bracket. The generator E^α plays the role of the positive step operator and F^α the negative one.

Theorem 3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let α and $\beta \neq \pm\alpha$ be roots and let M be the set of integers $\{t\}$ such that $\beta + t\alpha$ is a root, i.e.*

$$M := \{t \in \mathbb{Z} | \beta + t\alpha \in \Delta\} \tag{3.17}$$

1. *Then M is a closed interval $[-p,q]$ of integers*

$$M = [-p,q] \cap \mathbb{Z} \tag{3.18}$$

with p and q non-negative integers.

The sequence of roots

$$\beta - p\alpha, \ldots, \beta + q\alpha \tag{3.19}$$

*is called the **α-chain through** β.*

2. *Moreover on has $p - q = \beta(H_\alpha)$. In particular, $\beta(H_\alpha)$ is an integer, called a **Cartan integer**:*

$$\beta(H^\alpha) \in \mathbb{Z}. \tag{3.20}$$

This theorem will be proven in section 6.1. Note that $\beta(H^\alpha) = p - q \in [-p,q] \cap \mathbb{Z} = M$, which implies that

$$\beta - \beta(H^\alpha)\alpha \in \Delta. \tag{3.21}$$

3.6 Euclidean structure on the roots

Using the theorems 1 and 2 in the preceding section, one can define a Euclidean structure on the roots. Let us first define a symmetric bilinear form on $\mathfrak{h}^\star$:

Definition 10. *For α and $\beta \in \mathfrak{h}^\star$, one defines*

$$(\alpha|\beta) := \kappa(\nu^{-1}(\alpha), \nu^{-1}(\beta)) = \kappa(t_\alpha, t_\beta). \tag{3.22}$$

Consider now the real vector space E that is the $\mathbb{R}$-span of Δ:

$$E := \Big\{ \sum_{\alpha\in\Delta} r_\alpha \alpha | r_\alpha \in \mathbb{R} \Big\} \tag{3.23}$$

One can prove that the restriction of the bilinear form $(\cdot|\cdot)$ to the subspace E of $\mathfrak{h}^\star$ is positive-definite and thus non-degenerate. It accordingly corresponds to a Euclidean inner product on E. One can then define the norm $||\alpha||$ of a root α and the angle $\phi_{\alpha\beta}$ between two roots α and β in the usual way:

$$||\alpha|| = \sqrt{(\alpha|\alpha)}, \tag{3.24}$$
$$(\alpha|\beta) = ||\alpha||\ ||\beta|| \cos\phi_{\alpha\beta}. \tag{3.25}$$

Moreover, the Cartan integers can be rewritten using the inner product:

$$\beta(H^\alpha) = \frac{2(\alpha|\beta)}{(\alpha|\alpha)} \in \mathbb{Z}. \tag{3.26}$$

Note also that, as a consequence of (3.21) and (3.26), if $\alpha, \beta \in \Delta$, then

$$\beta - \frac{2(\alpha|\beta)}{(\alpha|\alpha)}\alpha \in \Delta. \tag{3.27}$$

This new root is the image of β under a reflection in the hyperplane perpendicular to α. To see this, note that the new root can be rewritten as $\beta - 2||\beta||cos\,\phi_{\alpha\beta}\mathbf{1}_\alpha$, where $\mathbf{1}_\alpha = \alpha/||\alpha||$. This corresponds to a symmetry of the root system and is called a **Weyl reflection**.

3.7 Positive roots and the triangular decomposition

The presence of a euclidean structure on the root system enables us to develop a geometrical picture of the roots. In particular one can define positive and negative roots and introduce a relation of order. A common way to do so is the so-called **lexicographical ordering**. Consider a basis $\{\alpha_i, i = 1, \ldots, r\}$ of the root system Δ, such that any root is given by its components a_i in this basis:

$$\alpha = \sum_{i=1}^{r} a_i \alpha_i.$$

Definition 11. *(i) α is* ***positive*** *if its first non vanishing component is positive and one then writes $\alpha > 0$.*

(ii) α is ***negative*** *if its first non vanishing component is negative and one then writes $\alpha < 0$.*

(iii) α is ***greater*** *than β (or β is* ***smaller*** *than α) if $\alpha - \beta > 0$. One then writes $\alpha > \beta$.*

Note that a root is always either positive or negative. The set of positive (negative) roots is noted Δ_+ (Δ_-). Moreover, as a consequence of the first property given in Theorem 2, one has a root in Δ_- for each root in Δ_+, and hence we can write

$$\{E^\alpha | \alpha \in \Delta\} = \{E^\alpha | \alpha \in \Delta_+\} \cup \{E^{-\alpha} | \alpha \in \Delta_+\} \tag{3.28}$$

The step operator associated to a positive root is also called a **raising operator**, while the ones associated to negative roots, $E^{-\alpha}$, also noted F^α, are called **lowering operator**. Note that the way you divide the roots in positive and negative ones depends on the choice of a criterium, and in the choice made here, it still depends on the choice of a basis. In any case, it also follows from the property above that the number of elements in Δ_+ (or in Δ_-) is equal to $\frac{d-r}{2}$ if the Lie algebra is of dimension d and rank r. In particular, it implies that the difference $d - r$ between the dimension and the rank of $\mathfrak{g}$ is always even.

In addition, it can be seen easily that the subspace of $\mathfrak{g}$ that is spanned by the step operators for positive (negative) roots is in fact a subalgebra. It will be denoted here as $\mathfrak{n}_+$ ($\mathfrak{n}_-$) and is called a nilpotent subalgebra of $\mathfrak{g}$. According to the decompositions (3.4) and (3.28), $\mathfrak{g}$ can be written as the following direct sum of vector spaces:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \tag{3.29}$$

This is called the **triangular** or **Cartan decomposition**.

3.8 Restrictions on angles and norms

As already stated, once given the inner product $(\cdot|\cdot)$, one can define the norm $||\alpha||$ of a root α and the angle $\phi_{\alpha\beta}$ between two roots α and β in the usual way:

$$||\alpha|| = \sqrt{(\alpha|\alpha)}, \tag{3.30}$$
$$(\alpha|\beta) = ||\alpha||\,||\beta|| \cos\phi_{\alpha\beta}. \tag{3.31}$$

A first step towards the full classification of finite dimensional semi-simple Lie algebras is to see the norm and angles of roots can only take certain values. To obtain the restrictions, consider two Cartan integers M and N defined from two roots α and β with $\beta \neq \pm\alpha$ as

$$M := \frac{2(\alpha|\beta)}{(\alpha|\alpha)}, \qquad N := \frac{2(\alpha|\beta)}{(\beta|\beta)}. \tag{3.32}$$

Notice that M and N have the same sign, so that their product is a natural number $MN \in \mathbb{N}$. Moreover

$$MN = 4 \cos^2\phi_{\alpha\beta}. \tag{3.33}$$

As a consequence,

$$0 \leq MN < 4. \tag{3.34}$$

The second inequality is strict because the equality woul correspond to $\beta = \pm\alpha$. One therefore has that a Cartan integer can only take the values $0, \pm1, \pm2, \pm3$. The angle is also constrained to the values such that $\cos^2\phi_{\alpha\beta} \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. The ratio between the norms of β and α is given by the formula

$$\frac{||\beta||}{||\alpha||} = \sqrt{\frac{M}{N}}. \tag{3.35}$$

With these remarks, one can list all possible values for angles and norm ratio between two roots α and β. The results are given in Table 1.

Table 1: The angles and ratios of norm for pairs of roots.

N	M	$cos^2\phi_{\alpha\beta}$	$cos\,\phi_{\alpha\beta}$	$\phi_{\alpha\beta}$	$\frac{\lVert\beta\rVert}{\lVert\alpha\rVert}$
0	0	0	0	$\pi/2$	undetermined
1	1	1/4	1/2	$\pi/3$	1
-1	-1	1/4	$-1/2$	$2\pi/3$	1
1	2	1/2	$\sqrt{2}/2$	$\pi/4$	$\sqrt{2}$
-1	-2	1/2	$-\sqrt{2}/2$	$3\pi/4$	$\sqrt{2}$
1	3	3/4	$\sqrt{3}/2$	$\pi/6$	$\sqrt{3}$
-1	-3	3/4	$-\sqrt{3}/2$	$5\pi/6$	$\sqrt{3}$

3.9 Simple roots and their properties

Definition 12. *The* ***simple roots*** *are the positive roots that cannot be written as a linear combination of other positive roots.*

The simple roots are the ones that are closest to the hyperplane that separates the positive and the negative roots. There are r simple roots for an algebra of rank r and they form a basis of the root system. They moreover have the following properties:

Theorem 4. *Let* Π *be the set of simple roots in a root system* Δ.

1. $\alpha_i, \alpha_j \in \Pi \Rightarrow \alpha_i - \alpha_j \notin \Delta$,
2. $\alpha_i, \alpha_j \in \Pi \Rightarrow (\alpha_i|\alpha_j) \leq 0 \Rightarrow \phi_{ij} \equiv \phi_{\alpha_i\alpha_j}$ *is a right or obtuse angle,*
3. *The coefficients of a positive (negative) root in the basis of the simple roots are all non-negative (non-positive).*

Proof. As an illustration, we only prove here the first two points.

1. Let us assume, without loss of generality, that $\alpha_i > \alpha_j$. Then, if $\alpha_i - \alpha_j \in \Delta$, it is a positive root and α_i can be written as the sum of two positive roots $\alpha_i = (\alpha_i - \alpha_j) + \alpha_j$. This contradicts the assumption that α_i is a simple root.

2. Consider the α_j-chain through α_i. As $\alpha_i - \alpha_j$ is not a root, one has $p = 0$. As a consequence, q is such that

$$-q = \alpha_i(H^{\alpha_j}) = \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}. \tag{3.36}$$

It follows that $(\alpha_i|\alpha_j) \leq 0$.

□

3.10 Cartan matrix and Dynkin diagram

From the simple roots, one can determine all the roots of the algebra. The whole algebra is encoded in the **Cartan matrix** A, whose elements are Cartan integers of the simple roots as follows:

$$A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}. \tag{3.37}$$

All the properties of roots and in particular simple roots detailed in the previous chapters lead to a series of properties fulfilled by the Cartan matrix, and that will enable us to classify all finite semisimple Lie algebras in the following part.

Theorem 5. *Let A be the Cartan matrix of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$. Then*

1. *A is symmetrizable, that is, there exists a diagonal matrix D such that DA is symmetric. Choose for example $D_{ii} = 1/(\alpha_i|\alpha_i)$. The symmetrized matrix is positive-definite, because the bilinear form $(\cdot|\cdot)$ is.*

2. *$A_{ii} = 2$,*

3. *$A_{ij} \in \{0, -1, -2, -3\}$ $(i \neq j)$*

4. *$A_{ij} = 0 \Leftrightarrow A_{ji} = 0$*

5. *$A_{ij} = -2 \Rightarrow A_{ji} = -1$*

6. *$A_{ij} = -3 \Rightarrow A_{ji} = -1$*

To a Cartan matrix is associated a **Dynkin diagram**, consisting of vertices representing the simple roots and (oriented) lines connecting them. The Dynkin diagram of an algebra $\mathfrak{g}$ of rank r is constructed using the following rules:

1. Draw r vertices, one for each simple root α_i,

2. Connect the vertices i and j with a number of lines equal to $\max\{|A_{ij}|, |A_{ji}|\}$, or equivalently[1] to the product $A_{ij}A_{ji}$,

3. If $|A_{ij}| > |A_{ji}|$, draw an arrow going from vertex i to vertex j (that is from the biggest to the smallest root).

The non-oriented diagram obtained by taking only the two first rules into account is called a **Coxeter diagram**.

[1] Note that the equivalence between the two conventions for the number of lines between two vertices is only true for finite-dimensional Lie algebras. The first one is more practical for the infinite-dimensional generalizations.

Figure 1: *The two diagrams of schemes with three vertices.*

3.11 Classification

In order to classify all simple Lie algebras, we need to classify all systems of simple roots. In a more general setting, we will rather classify schemes:

Definition 13. *A* ***scheme*** *S is a system of r linearly independent non-vanishing vectors $\{\alpha_1, \dots, \alpha_r\}$ in an r-dimensional euclidean vector space E such that*

$$A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)} \tag{3.38}$$

is a non-positive integer for $i \neq j$.

Such a system is called a **scheme** and denoted S. Two vectors α_1, α_2 in S are said **linked** if $(\alpha_i|\alpha_j) \neq 0$. The number r of vectors in S is called the **rank** of the scheme S. Clearly, a system of simple roots is a scheme and all schemes correspond to a system of simple roots.

One can see that as consequences of this definition, $A_{ij}A_{ji} \in \{0, 1, 2, 3\}$ and the allowed angles between two vectors of a scheme are $\pi/2$, $2\pi/3$, $3\pi/4$ and $5\pi/6$.

Moreover, and it will show crucial for the classification, a subset of a scheme is a scheme, and is accordingly called a **subscheme**.

One associates to a scheme a Coxeter diagram, just as defined for simple roots. Such a diagram is connected iff the scheme is indecomposable, i.e. one cannot write S as the union of two orthogonal subschemes. We will restrict to that case in the following. We will do the classification first separately for ranks $r = 1, 2, 3$ and then for all higher ranks.

$r = 1$: There is only one vertex and the Coxeter diagram is trivial.

$r = 2$: There are two vertices that can be linked by 1, 2 or 3 lines.

$r = 3$: There are only two possibilities, given in figure 1. To see this, first note that the sum of the angles between three linearly independent vectors is strictly smaller than 2π. (It is equal to 2π when the three vectors are coplanar.) One thus has

$$\phi_{12} + \phi_{23} + \phi_{31} < 2\pi. \tag{3.39}$$

Having in mind the allowed angles, one already sees that at least one of the angles must be $\pi/2$. In fact, exactly one must be $\pi/2$ otherwise the scheme is decomposable. There are then two possiblilities for the two other angles: $2\pi/3$ and $2\pi/3$ or $2\pi/3$ and $3\pi/4$.

$r > 3$: One first thing can be deduced from the previous case: the Coxeter diagram of a scheme of rank $r \geq 3$ cannot contain triple bounds.

Theorem 6. *Let S be a scheme of rank r. Then the number p of links in S is smaller than r:*

$$p < r. \tag{3.40}$$

Proof. Let $S = \{\alpha_1, \ldots, \alpha_r\}$ and define χ as

$$\chi = \sum_{i=1}^{r} \frac{\alpha_i}{\sqrt{(\alpha_i|\alpha_i)}} \tag{3.41}$$

As the a_i's are linearly independent, $\chi \neq 0$. Therefore

$$0 < (\chi|\chi) = r + 2\sum_{i<j} cos\ \phi_{ij}, \tag{3.42}$$

which means that

$$r > -2\sum_{i<j} cos\ \phi_{ij}. \tag{3.43}$$

The number p of links is equal to the number of times that $cos\ \phi_{ij} \neq 0$. But $cos\ \phi_{ij} \neq 0$ implies $-2cos\ \phi_{ij} \geq 1$ (cf. allowed angles). The right hand side of (3.43) is therefore bigger than p, and the theorem is proven. □

A consequence of this theorem is the following:

Theorem 7. *Schemes do not contain closed circuits (=loops). In other words, they are tree diagrams.*

Theorem 8. *The number of lines emanating from a vertex of a scheme is at most equal to three.*

Proof. Let α_i be an element of S and $\{\beta_1, \ldots, \beta_m\}$ the other elements of S that are linked to α_i, that is, such that $(\alpha_i|\beta_j) \neq 0$. The number N of lines emanating from α_i is given by

$$N = \sum_{j=1}^{m} n_{ij} \tag{3.44}$$

where

$$n_{ij} = 4\frac{(\alpha_i|\beta_j)(\beta_j|\alpha_i)}{(\beta_j|\beta_j)(\alpha_i|\alpha_i)} = 4\frac{(\alpha_i|\gamma_j)^2}{(\alpha_i|\alpha_i)}, \tag{3.45}$$

where $\gamma_j = \beta_j/\|\beta_j\|$. The γ_j must be all orthogonal to each other. Indeed, if there exists $(\gamma_j|\gamma_k) \neq 0$ for some $j \neq k$, it means that β_j and β_k are linked and as a consequence $\{\alpha_i, \beta_j, \beta_k\}$ form a loop. Let us introduce an extra unit vector γ_0 that is orthogonal to all γ_j (or equivalently to all β_j) but not to α_i. (Such a vector exists because $\alpha_i, \beta_1, \ldots, \beta_m$ are linearly independent.) One has

$$(\alpha_i|\gamma_0) \neq 0. \tag{3.46}$$

The set $\{\gamma_0, \gamma_1, \ldots, \gamma_m\}$ is an orthonormal set. Hence

$$\alpha_i = \sum_{j=0}^{m} (\alpha_i|\gamma_j)\gamma_j, \tag{3.47}$$

and then

$$(\alpha_i|\alpha_i) = \sum_{j=0}^{m} (\alpha_i|\gamma_j)^2. \tag{3.48}$$

Going back to N, one finally gets

$$N = 4\sum_{j=0}^{m} \frac{(\alpha_i|\gamma_j)^2}{(\alpha_i|\alpha_i)} - 4\frac{(\alpha_i|\gamma_0)^2}{(\alpha_i|\alpha_i)} = 4 - 4\frac{(\alpha_i|\gamma_0)^2}{(\alpha_i|\alpha_i)} < 4. \tag{3.49}$$

□

As a consequence of this theorem, a vertex of the Coxeter diagram of a scheme can have either up to three single links or one double link and possibly another single link. The case of a vertex with three single links is called a **node**.

Definition 14. *One calls a* ***chain*** *a (sub)scheme $\{\delta_1, \ldots, \delta_m | m > 2\}$ such that all the pairs $\{\delta_i, \delta_{i+1}\}$ are linked. The chain is* ***homogeneous*** *if the links are all single.*

Theorem 9. *Let S be a scheme containing a chain $\{\delta_1, \ldots, \delta_m\}$. Then the system obtained by replacing the chain $\{\delta_1, \ldots, \delta_m\}$ by a single vector $\delta = \sum_{i=1}^{m} \delta_i$ is also a scheme. From the point of view of the Coxeter diagram, the new scheme is obtained by shrinking the chain to one vertex.*

Proof. We have to prove two things : first that the vectors of the new scheme are linearly independent and second that the matrix elements A_{ij} are non-positive integers for $i \neq j$ in the new scheme.

Let us note $S' = S \backslash \{\delta_1, \ldots, \delta_m\}$. If $S = S' \cup \{\delta_1, \ldots, \delta_m\}$ is linearly independent, it is clear that $S' \cup \{\delta\}$ is linearly independent too.

For the second part, we only need to check the matrix elements relating vectors of S' to δ. We therefore consider $2\frac{(\delta|\alpha_i)}{(\alpha_i|\alpha_i)}$ and $2\frac{(\delta|\alpha_i)}{(\delta|\delta)}$ where $\alpha_i \in S'$. First note that α_i can only be linked to one of the vertices of the chain, otherwise there exists a loop in the diagram. Consequently, $(\delta|\alpha_i) = (\delta_k|\alpha_i)$ for some k.

The first element we consider is

$$2\frac{(\delta|\alpha_i)}{(\alpha_i|\alpha_i)} = 2\frac{(\delta_k|\alpha_i)}{(\alpha_i|\alpha_i)}. \tag{3.50}$$

It is non-positive because S is a scheme.

For the second matrix element, we need to compute $(\delta|\delta)$.

$$\begin{aligned}
(\delta|\delta) &= (\delta_m|\delta_m) + \sum_{j=1}^{m-1} \{(\delta_j|\delta_j) + 2(\delta_j|\delta_{j+1})\} && (3.51)\\
&= (\delta_m|\delta_m) + \sum_{j=1}^{m-1} \{(\delta_j|\delta_j) - (\delta_j|\delta_j)\} && (3.52)\\
&= (\delta_k|\delta_k). && (3.53)
\end{aligned}$$

From (3.51) to (3.52), one uses that

$$\frac{2(\delta_j|\delta_{j+1})}{(\delta_j|\delta_j)} = -1.$$

From (3.52) to (3.53), one uses that in a chain, all vertices have the same norm. Using what precedes, one then finds that the second matrix element to check is

$$2\frac{(\delta|\alpha_i)}{(\delta|\delta)} = 2\frac{(\delta_k|\alpha_i)}{(\delta_i|\delta_i)}, \tag{3.54}$$

which is non positive because S is a scheme. □

Theorems 8 and 9 directly imply the following restriction:

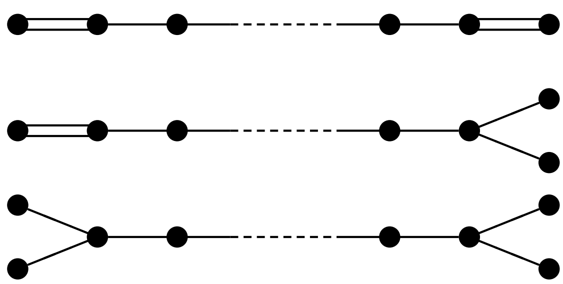


Figure 2: *These three diagrams cannot be subdiagrams of schemes because, up to the shrinking of a chain to a vertex, they are equivalent to diagrams with four vertices emanating from a vertex.*

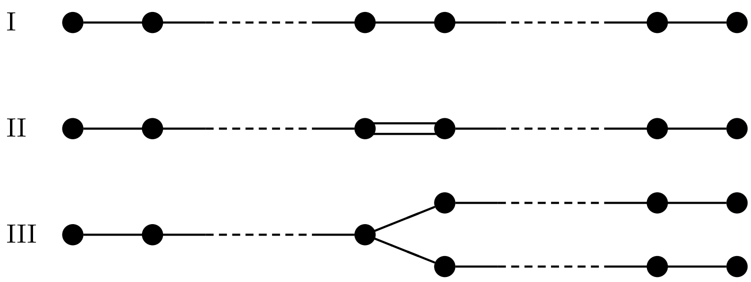


Figure 3: *These classes of diagrams of indecomposable schemes.*

Theorem 10. *The Coxeter diagram of a scheme does not contain a subdiagram of the type depicted in figure 2.*

Let us summarize the current state of the classification. Any Coxeter diagram of an indecomposable scheme S is a tree diagram with one the following alternatives:

1. It has no double link and no node. Then it has only single links (see class I in figure 3),
2. It has one double link, then it has no node (see class II in figure 3).
3. It has one node, then it has no double link (see class III in figure 3).

To all diagram of class I with k vertices corresponds a scheme, whose A matrix is given by $A_{ii} = 2$ and $A_{ij} = -1$ for $i \neq j$. The corresponding Lie algebra is denoted A_k. (cf. figure 6)

Only some subclass of class II corresponds to schemes. To see that, let us first note vertices $\alpha_1, \dots, \alpha_p$ on the left of the double link and $\alpha_{p+1} = \beta_q, \dots \alpha_{p+q} = \beta_1$ on the right of the double link (cf. figure 4).

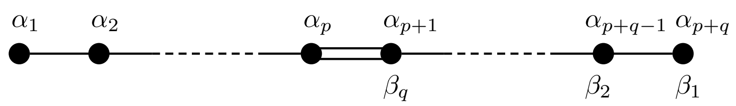


Figure 4: *Labelling of vertices of a class II diagram.*

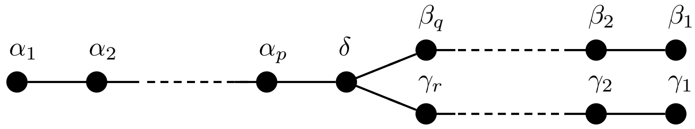


Figure 5: *Labelling of vertices of a class III diagram.*

One has $A_{p,p+1}A_{p+1,p} = 2$. Let us assume, without loss of generality that $A_{p,p+1} = -1$ and $A_{p+1,p} = -2$, implying that

$$(\alpha_{p+1}|\alpha_{p+1}) = (\beta_q|\beta_q) = 2(\alpha_p|\alpha_p),$$

and let us set

$$(\alpha_i|\alpha_i) = a, \quad i = 1, \ldots, p.$$

Let us moreover define

$$\alpha = \sum_i^p i\alpha_i \quad \beta = \sum_j^q j\beta_j.$$

Since α an β are linearly independent, we have

$$(\alpha|\beta)^2 < (\alpha|\alpha)(\beta|\beta).$$

By calculating each factor in this inequality, one can show that it is equivalent to

$$(p-1)(q-1) < 2.$$

As a consequence there are only three admissible subclasses (cf figure 6)

- $p = 1$ and $q = 1, 2, \ldots$, the corresponding Lie algebras being B_{q+1},
- $q = 1$ and $p = 1, 3, \ldots$, the corresponding Lie algebras being C_{q+1},
- $q = 2$ and $p = 2$, the corresponding Lie algebra beging F_4.

In class III, calling δ the vertex at the center of the node and $\alpha_1, \ldots \alpha_p$, $\beta_1, \ldots, \beta_q$ and $\gamma_1, \ldots, \gamma_r$ the three chains of vertices around it (cf figure 5), defining then α and β as before and γ in a similar fashion, the important inequality in this case will be that

$$cos^2\widehat{\alpha\delta} + cos^2\widehat{\beta\delta} + cos^2\widehat{\gamma\delta} < 1,$$

where $\widehat{\alpha\delta}$ is the angle between α and δ. This inegality can be deduced from the fact that α, β and γ are orthogonal and δ does not belong to their linear span.

At the end of the day, one finds that only the following possibilities exist (cf figure 6)

- $r = 1$, $q = 1$ and $p = 1, 2, \ldots$, that correspond to the algebras D_{p+3},
- $r = 1$, $q = 2$ and $p = 2, 3, 4$, that correspond to the algebras E_{p+4}.

3.12 Root diagrams

For algebras with rank $r = 1$ or 2, the roots can be represented by a diagram on a plane. For $r = 1$, there is only one simple root, α, and the only other possible root is $-\alpha$. It corresponds to A_1 and the root diagram is presented in figure 7.

For $r = 2$, there are two simple roots, α_1 and α_2. The angle between them can only take four values:

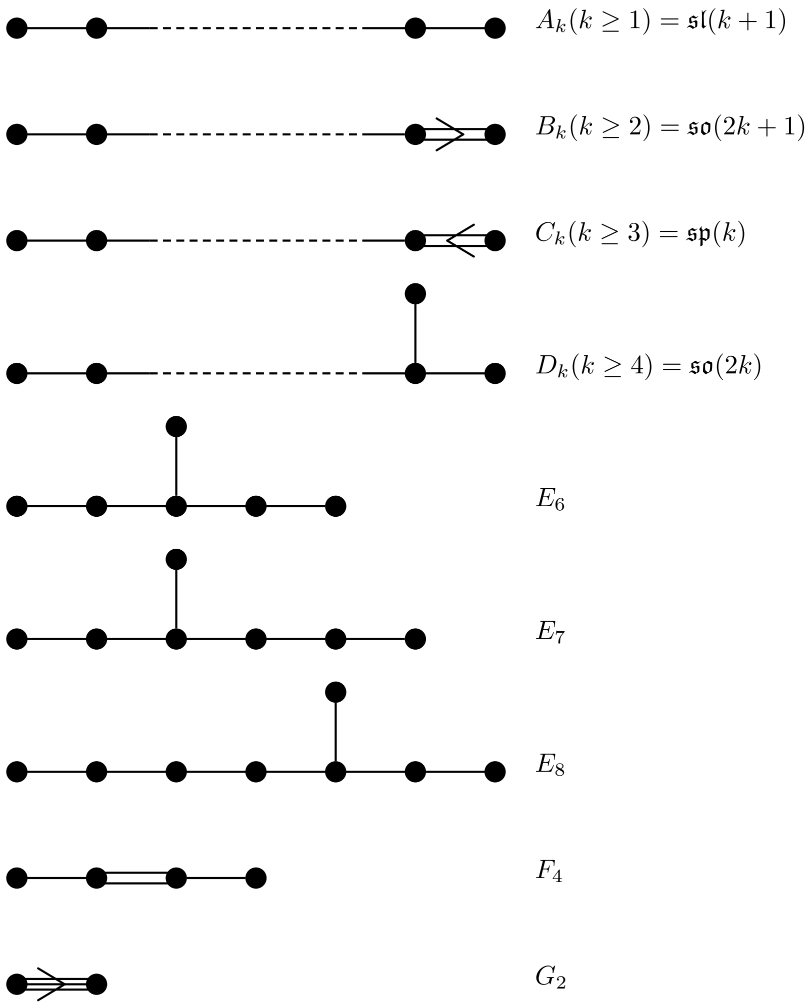


Figure 6: *Dynkin diagrams of simple Lie algebras.*

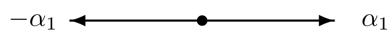


Figure 7: *Root diagram of A_1.*

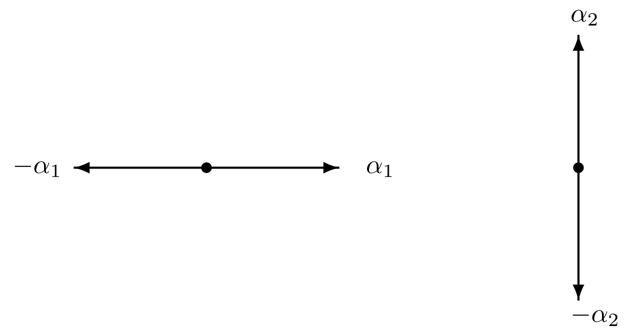


Figure 8: *Root diagram of* $A_1 \oplus A_1$.

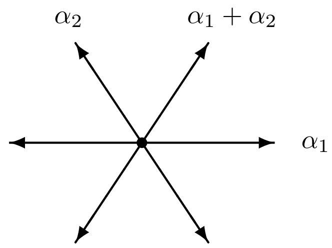


Figure 9: *Root diagram of* A_2.

1. $\phi_{\alpha_1\alpha_2} = \pi/2 \Rightarrow$ ratio of norms undetermined,
2. $\phi_{\alpha_1\alpha_2} = 2\pi/3 \Rightarrow$ roots of equal lengths,
3. $\phi_{\alpha_1\alpha_2} = 3\pi/4 \Rightarrow$ two different lengths,
4. $\phi_{\alpha_1\alpha_2} = 5\pi/6 \Rightarrow$ two different lengths.

By use of Weyl reflections, all other roots can then be found. The cooresponding algebras are respectively $A_1 \oplus A_1$, A_2, B_2 and G_2. The roots diagram are in figure 8 to 11.

3.13 Application: Hidden symmetries

In the presence of n commuting Killing vectors, or in other words, when compactified on a n-torus, a D-dimensional theory can be seen as a $D - n$ theory that will contain a certain number of scalar fields. It is a remarkable fact that in several cases, these scalar fields form a non-linear sigma model that is symmetric under a simple finite-dimensional algebra. In particular, if $n = D - 3$, the reduced theory is in three dimensions and all p-form fields can be dualised to scalars. The three-dimensional theory is then only three-dimensional gravity plus scalars forming a non-linear sigma model with the target space being G/K where G is the group obtained by exponentiation of a simple finite-dimensional Lie algebra $\mathfrak{g}$ and K its maximal compact subgroup. In this section, we only consider the case when the reduction is performed on spatial coordinates. The definition of K is slightly more complicated in other cases.

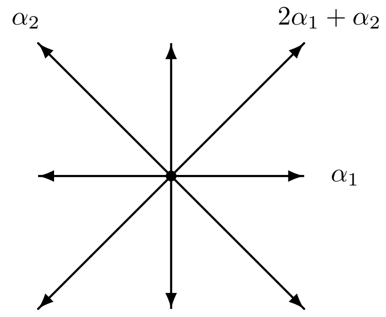


Figure 10: *Root diagram of* B_2.

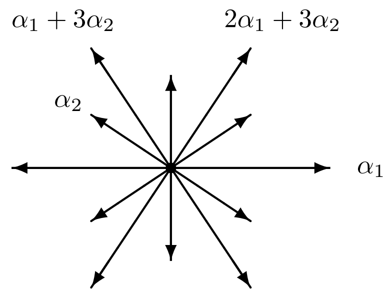


Figure 11: *Root diagram of* G_2.

Let us first review the construction [?] of a non-linear sigma model on the target space G/K. We will consider physical examples afterwards. Remember that a d-dimensional rank r Lie algebra $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ . \tag{3.55}$$

where $\mathfrak{n}_-$ is generated by the F^i generators, $\mathfrak{n}_+$ by the E^i and $\mathfrak{h}$ is spanned by the Cartan generators H^a ($i = 1, \dots, \frac{d-r}{2}$; $a = 1, \dots, r$). Let us now define the generators $K_i = E^i - F^i$, that generate the algebra $\mathfrak{k}$. The Lie algebra $\mathfrak{g}$ can now also be decomposed as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h} \oplus \mathfrak{n}_+ . \tag{3.56}$$

This is called the Iwasawa decomposition. It has a corresponding decomposition at the level of the group

$$G = K \times H \times N_+ . \tag{3.57}$$

This means that any element in the group G can be written as the product of elements in K, H and N_+.

Now, the coset space G/K, that if we want to be accurate should rather be written $K\backslash G$, is the space of left equivalence classes of G under K. That is, one identifies two elements of G if they differ by an element of K:

$$g, g' \in G : \; g \sim g' \Leftrightarrow g' = g_k \, g \text{ for } g_k \in K . \tag{3.58}$$

Each class will be represented by an element, that one can choose as

$$\mathcal{V} = g_0 \; g_+ \tag{3.59}$$

where $g_0 \in H$ and $g_+ \in K$. In terms of algebra elements one therefore has

$$\mathcal{V} = e^{\sum_a A_a(x) H^a} e^{\sum_i B_i(x) E^i} . \tag{3.60}$$

where $A_a(x)$ and $B_a(x)$ are some functions of the base space coordinates x. To construct the corresponding Lagrangian, consider first the algebra element $\mathcal{V}^{-1} d\mathcal{V}$ and decompose it into a part in $\mathfrak{k}$ and the rest:

$$\mathcal{V}^{-1} d\mathcal{V} = \mathcal{Q} + \mathcal{P} \tag{3.61}$$

where $\mathcal{Q} \in \mathfrak{k}$ and $\mathcal{P} \in \mathfrak{g} \ominus \mathfrak{k}$ is the component of $\mathcal{V}^{-1} d\mathcal{V}$ that is parallel to the coset space. One can now write a Lagrangian that is manifestly invariant under global G transformation and local K transformation as

$$\mathcal{L}_{coset} = -\frac{1}{2} \mathrm{Tr}(\star \mathcal{P} \wedge \mathcal{P}) . \tag{3.62}$$

Let us now consider some example from physics and see how they fit in this picture.

Pure 5D gravity The field content in five dimensions is of course just a metric $g^{(5)}_{\mu\nu}$. After reduction to three dimensions, one is left with a three dimensional metric g_{mn}, two one-forms $V^{(1)}_m$ and $V^{(2)}_m$ and three scalars: two dilatons ϕ_1, ϕ_2 and one axion χ_1. Schematically[2], they are obtained from the five-dimensional metric as follows

$$g^{(5)}_{\mu\nu} = \left(\begin{array}{c|c|c} g_{mn} & V^{(1)}_m & V^{(2)}_m \\ \hline V^{(1)}_n & \phi_2 & \chi_1 \\ \hline V^{(2)}_n & \chi_1 & \phi_1 \end{array} \right) \tag{3.63}$$

[2]In practice, a less naive reduction ansatz is needed in order make the underlying symmetry transparent. The lectures on Kaluza-Klein theory by Pope [?] give a clear introduction to dimensional reductions.

The two one-forms can be dualised into scalars χ_2 and χ_3 through an expression of the form

$$d_m V_n = \epsilon_{mn}{}^p d_p \chi. \tag{3.64}$$

As a consequence one is left with a metric, two dilatons and three axions. The identification with a non linear sigma model is done by associating the dilatons to Cartan generators and the axions to positive step operators, or positive roots. As a consequence, in this case, one need an algebra that has rank two and dimension $d = 2 + 2.3 = 8$. Such an algebra is given by $A_2 = sl(3)$. Its compact subalgebra is $so(3)$.

Now, more precisely, in order to prove that the scalars form a non linear sigma model, one must choose as a coset representative

$$\mathcal{V} = e^{\sum_a \phi_a(x) H^a} e^{\sum_i \chi_i(x) E^i}, \tag{3.65}$$

then compute the corresponding Lagrangian and compare it to the physical Lagrangian obtained after dimensional reduction and dualisation of the vectors. This process requires a series of extra subtleties that are outside the scope of these lectures.

5D minimal supergravity 5D minimal supergravity describes 5D gravity in presence of matter fields. Besides a five dimensional metric, it contains a one-form $A^{(5)}_\mu$, that decomposes under reduction to $3D$ into a one-form A_m and two axions χ_4, χ_5:

$$A^{(5)}_\mu = \left(\quad A_m \quad \mid \chi_4 \mid \chi_5 \right) \tag{3.66}$$

Again, in three dimensions, the one-form A_m is dual to a scalar χ_6. With the fields obtained from the metric, one is accordingly left with a scalar content made of two dilatons and six axions. An algebra of rank 2 and dimensions 14 exists, that is G_2 that has $so(4)$ as a compact subalgebra [?]. One can show that the corresponding sigma model indeed reproduces the scalar part of the Lagrangian of 5D minimal supergravity after reduction to 3D.

4 Generalisation to Kac-Moody algebras

4.1 Generalisation of the Cartan matrix

As we have seen, the classification of simple finite-dimensional Lie algebras is based on the assignment of a square integer Cartan matrix to any such Lie algebra, which is unique up to permutations of the index set labeling rows and columns.

We recall that the requirement that the Lie algebra be simple and finite-dimensional implies the following conditions on the corresponding Cartan matrix A:

- The diagonal entries are all equal to 2.
- The off-diagonal entries are all non-positive, with $A_{ij} = 0$ if and only if $A_{ji} = 0$.
- The Cartan matrix is indecomposable.
- The Cartan matrix is symmetrizable, and the symmetrized matrix is positive-definite.

By relaxing the last condition one obtains a much larger class of Lie algebras, called Kac-Moody algebras. We will henceforth talk about Cartan matrices in this generalized meaning. By definition, each of them gives rise to a Kac-Moody algebra in a unique way that we will describe. If two Cartan matrices give rise to two isomorphic Kac-Moody algebras, then they differ only by a permutation of the index set labeling rows and columns. We will now review the classification of Cartan matrices, following [?]. Since there is a one-to-one correspondence between Cartan matrices and Kac-Moody algebras this will then correspond to a classification of Kac-Moody algebras. We will then explain how a Kac-Moody algebra is constructed from its Cartan matrix A if $\det A \neq 0$.

For any column matrix a, we write $a > 0$ if all entries are positive, and $a < 0$ if all entries are negative. We now define an $r \times r$ Cartan matrix A to be

- **finite** *if* $Ab > 0$*,*
- **affine** *if* $Ab = 0$*,*
- **indefinite** *if* $Ab < 0$

for some $r \times 1$ matrix $b > 0$. One and only one of these three assertions is valid for any A, and in the affine case, b is uniquely defined up to normalization. Affine Cartan matrices can also be characterized in the following way.

- *A is affine if and only if* $\det A = 0$ *and deletion of any row and the corresponding column gives a direct sum of finite Cartan matrices.*

As we will describe in section 4.2, any Cartan matrix defines uniquely a Lie algebra, and all Lie algebras that can be obtained in this way are called Kac-Moody algebras. Thus we can say that a Kac-Moody algebra is **finite**, **affine** or **indefinite** if the same holds for its Cartan matrix. Finite Kac-Moody algebras are then nothing but simple finite-dimensional Lie algebras, and their construction gives us back the classification in section 3.11. Also the affine Kac-Moody algebras are well understood, as certain extensions of finite algebras (see section 5). On the other hand, the indefinite Kac-Moody algebras are not fully classified nor well understood. We need to impose further conditions in order to study them along with the finite and affine algebras. In what follows we will always require an indefinite Cartan matrix A to be symmetrizable, a requirement that is already satisfied by finite and affine Cartan matrices. One can then show that

- *A is finite if and only if A is symmetrizable and the symmetrized matrix has signature* $(+\cdots++)$*,*
- *A is affine if and only if A is symmetrizable and the symmetrized matrix has signature* $(+\cdots+\,0\,)$*.*

Analogously, we define A to be **Lorentzian** if A is symmetrizable and the symmetrized matrix has signature $(+\cdots+-)$. (With signature we mean the number of positive, negative or zero eigenvalues. Their order does not matter.) Clearly, the Lorentzian algebras form a subclass of the class of indefinite algebras, but we can restrict it even further. Similarly to the characterization of the affine case above, we define **hyperbolic** Cartan matrices in the following way.

- *A is hyperbolic if and only if* $\det A < 0$ *and deletion of any row and the corresponding column gives a direct sum of affine or finite matrices.*

It can be shown that any hyperbolic Cartan matrix is Lorentzian. We say that a Kac-Moody algebra is **Lorentzian** or **hyperbolic** if the same holds for its Cartan matrix.

In the finite case we had $1 \leq A_{ij}A_{ji} \leq 3$ for the nonzero off-diagonal entries (no summation). In the general case there is no such restriction. However, in the cases that we consider in these lectures, we have $1 \leq A_{ij}A_{ji} \leq 4$. Thus we can still unambiguously specify the algebra with a Dynkin diagram where the number of lines between the vertices i and j is $\max\{|A_{ij}|, |A_{ji}|\}$, with an arrow pointing from i to j if $|A_{ij}| > |A_{ji}|$. The difference compared to the finite case is that we now also can have $A_{ij} = A_{ji} = -2$, but then we simply draw two lines without any arrow.

Since there is a one-to-one correspondence between Cartan matrices and Dynkin diagrams, we can talk about finite, affine and indefinite Dynkin diagrams. The characterizations above of affine and hyperbolic matrices can now be formulated as

- *A is affine if* $\det A = 0$ *and deletion of any vertex leads to finite diagrams.*
- *A is hyperbolic if* $\det A < 0$ *and deletion of any vertex leads to affine or finite diagrams.*

Permutation of rows and (the corresponding) columns in A corresponds to relabeling the vertices in the Dynkin diagram. An indecomposable Cartan matrix corresponds to a connected Dynkin diagram.

4.2 The Chevalley-Serre relations

We will now describe how a Lie algebra can be constructed from a given Cartan matrix A or, equivalently, from its Dynkin diagram. The Lie algebra $\mathfrak{g}'$ obtained in this way is called the **derived Kac-Moody algebra** of A. The Kac-Moody algebra $\mathfrak{g}$ of A is then defined as a certain extension of $\mathfrak{g}'$ in the case when A is affine. We will explain this in section 5. If A is finite or indefinite, then $\mathfrak{g}$ coincides with $\mathfrak{g}'$.

In the construction of the Lie algebra $\mathfrak{g}'$ from its Cartan matrix A, one starts with the fact that a simple finite-dimensional Lie algebra is generated by the raising and lowering operators corresponding to the simple roots α_i. These are called **Chevalley generators** and we will henceforth write

$$e_i \equiv E^{\alpha_i}, \qquad f_i \equiv F^{\alpha_i}. \tag{4.1}$$

The corresponding Cartan element $H^{\alpha_i} = [E^{\alpha_i}, F^{\alpha_i}]$ will be called **Cartan generator** and denoted

$$h_i \equiv H^{\alpha_i}. \tag{4.2}$$

(The Cartan generators should not be confused with the tentative basis elements H^i of the Cartan subalgebra.) It follows from the definition of the Cartan-Weyl basis (applied to the simple roots) and the definition of the Cartan matrix, that the Chevalley and Cartan generators satisfy the **Chevalley relations** (no summation)

$$\begin{aligned} &[e_i, f_j] = \delta_{ij} h_j, &&[h_i, e_j] = \alpha_j(h_i) e_j = 2\frac{(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} e_j = A_{ji} e_j, \\ &[h_i, h_j] = 0, &&[h_i, f_j] = -\alpha_j(h_i) e_j = -2\frac{(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} e_j = -A_{ji} e_j \end{aligned} \tag{4.3}$$

If A is of order r, the derived Kac-Moody algebra $\mathfrak{g}$ is now *defined* as the Lie algebra generated by $3r$ elements e_i, f_i, h_i modulo the Chevalley relations and the additional Serre relations (no summation)

$$(\text{ad } e_i)^{1-A_{ji}} e_j = 0, \qquad (\text{ad } f_i)^{1-A_{ji}} f_j = 0. \tag{4.4}$$

It follows from the Chevalley relations (4.3) that $\mathfrak{g}$ is spanned by the Cartan generators and the set of multiple commutators

$$[\cdots[[e_{i_1}, e_{i_2}], e_{i_3}], \ldots, e_{i_k}], \qquad [\cdots[[f_{i_1}, f_{i_2}], f_{i_3}], \ldots, f_{i_k}], \tag{4.5}$$

for all $k \geq 1$, which is restricted by the Serre relations (4.4).

The number r is called the **rank** of the (derived) Kac-Moody algebra. In the finite and indefinite case it coincides with the rank of the Cartan matrix, but if A is an affine Cartan matrix then it has matrix rank $r - 1$.

4.3 The Killing form revisited

In section 3.4 we defined the symmetric and invariant Killing form of a finite-dimensional Lie algebra as $\kappa(x,\, y) = \text{tr}\,(\text{ad}\, x \circ \text{ad}\, y)$. But if $\mathfrak{g}$ is simple and finite-dimesional then κ can equivalently, up to an overall factor, be defined by

$$\kappa(e_i,\, f_j) = D_{ij}, \qquad \kappa(h_i,\, h_j) = (DA)_{ij}, \tag{4.6}$$

for all $i,\, j = 1,\, 2,\, \ldots,\, r$, where D is a diagonal matrix (unique up to an overall factor) such that DA is symmetric. In all other cases the Killing form is defined to be zero. It can then be extended to the full Cartan-Weyl basis by the symmetry and invariance properties, together with the Chevalley relations. (Actually, already the first equation above is enough.) The Killing form will then be symmetric and invariant by construction, but also non-degenerate. Moreover, these properties define the Killing form uniquely up to automorphisms and an overall normalization. In section 5 we will see how the Killing form is defined for affine Kac-Moody algebras.

4.4 Roots of Kac-Moody algebras

One can define roots in the same way as for finite Kac-Moody algebras. But there are two important differences:

- The root space, dual to the Cartan subalgebra with respect to the Killing form, is in general not Euclidean. In particular, there can be roots with zero (**lightlike**) or negative (**timelike**) norm.
- The eigenspaces corresponding to the roots (unfortunately also called root spaces) are in general not one-dimensional, so there can be many linear independent root vectors corresponding to one root. Then the root is said to have **multiplicity** greater than one.

The first fact is a simple consequence of allowing for Cartan matrices that are not positive-definite – it follows from the definition of the bilinear form that the signature of the root space or the Cartan subalgebra is the same as of the Cartan matrix. The second fact is much less obvious, but it follows from the Serre relations. It is also a non-trivial consequence of the Serre relations that an affine or indefinite Kac-Moody algebra is infinite-dimensional.

4.5 The Chevalley involution

An **involution** of an algebra is an automorphism that squares to the identity map. Different Kac-Moody algebras admit different involutions, but it follows from the Chevalley relations that there is always an involution given by

$$\omega(h_i) = -h_i, \qquad \omega(e_i) = -f_i, \qquad \omega(f_i) = -e_i \tag{4.7}$$

for the Chevalley generators. It can then be extended to the whole algebra by the homomorphism property. This involution ω is called the **Chevalley involution**.

For any automorphism ω of an algebra, the set of elements that are invariant under ω constitutes a subalgebra. In the case where ω is the Chevalley involution of a Kac-Moody algebra $\mathfrak{g}$, the subalgebra $\mathfrak{k}$ is called the **maximal compact subalgebra**. All the elements in this subalgebra have the form $x + \omega(x)$ for all $x \in \mathfrak{g}$. As we have already seen in section 3.13, this subalgebra (and the corresponding **maximal compact subgroup**) is very important in applications where Kac-Moody algebras appear as hidden symmetries. It follows from the definition of the Killing form in the previous subsection that $\mathfrak{k}$ is always negative-definite or positive-definite, since the entries in the diagonal matrix D all have the same sign. Usually they are taken to be positive, so that the Killing form is negative-definite on $\mathfrak{k}$. Its orthogonal complement $\mathfrak{p}$ is the direct sum of the Cartan subalgebra and a positive-definite subspace. The elements in $\mathfrak{p}$ have the form $x - \omega(x)$ for all $x \in \mathfrak{g}$.

4.6 Examples

We end this section with two examples of a Kac-Moody algebra. The simplest example is A_1, for which the Dynkin diagram consists of only one vertex, and the Cartan matrix of only one entry. We have in this chapter seen how any Kac-Moody algebra is built up from copies of A_1, corresponding to the different nodes in the Dynkin diagram. The lines and arrows between the nodes tell us how these A_1 subalgebras 'interact' with each other to build up the full Kac-Moody algebra.

For A_1 there is only one single root, and this is not enough to build any more positive roots. Thus A_1 is three-dimensional with the Chevalley and Cartan generators corresponding to the only simple root as basis elements. The full set of commutation relations for A_1 is given by the Chevalley relations corresponding to the only single root:

$$[h,\, e] = 2e, \qquad [h,\, f] = -2f, \qquad [e,\, f] = h \tag{4.8}$$

(We have dropped the subscript $i = 1, 2, \ldots, r$ since $r = 1$ in this case.)

We turn to our second example of a Kac-Moody algebra, A_2, for which the Dynkin diagram consists of two vertices connected with a single line. Thus the two off-diagonal entries in the Cartan matrix are both equal to -1. Each of the nodes corresponds to an A_1 subalgebra, but apart from these two A_1 subalgebras there are also the basis elements $[e_1,\, e_2]$ and $[f_1,\, f_2]$. On the other hand,

$$[e_1,\, [e_1,\, e_2]] = [e_2,\, [e_2,\, e_1]] = [f_1,\, [f_1,\, f_2]] = [f_2,\, [f_2,\, f_1]] = 0 \tag{4.9}$$

by the Serre relations, so we cannot construct any more basis elements. Thus A_2 is 8-dimensional. The elements $e_\theta = [e_1,\, e_2]$ and $f_\theta = -[f_1,\, f_2]$ form a third A_1 subalgebra together with the Cartan element $h_\theta = h_1 + h_2$ since

$$[h_\theta,\, e_\theta] = 2e_\theta, \qquad [h_\theta,\, f_\theta] = -2f_\theta, \qquad [e_\theta,\, f_\theta] = h_\theta. \tag{4.10}$$

This A_1 subalgebra corresponds to the third positive root θ, which is the sum of the two simple roots: $\theta = \alpha_1 + \alpha_2$.

5 Affine algebras

5.1 Definition and Cartan matrix

As already explained in section 4.1, an important and quite well known class of Kac-Moody algebras is given by affine Lie algebras, defined as follows:

Definition 15. *An* ***affine Lie*** *algebra is a Lie algebra constructed out of a so-called affine Cartan matrix A, that fulfills the following conditions*

- $A_{ii} = 2$,
- *The off diagonal elements are non positive integers and $A_{ij} = 0$ iff $A_{ji} = 0$,*
- *A is symmetrizable,*
- $\det A = 0$ *and the deletion of any row and the corresponding column gives a direct sum of finite Cartan matrices.*

The matrix A is thus positive semidefinite with only one vanishing eigenvalue. This condition can be replaced by asking that there exists an invertible diagonal matrix D such that DA is symmetric and positive semidefinite. (The requirement already implies that DA can have at most one zero eigenvalue.)

In the context of Kac-Moody algebras, the rank of an algebra is more generally defined as the dimension of the Cartan matrix, that, in the case of affine algebras, does not correspond to the dimension of the Cartan subalgebra, as we will see in the following. Note also that it obviously does not correspond to the rank of the Cartan matrix either.

This way of relaxing the definite positiveness condition on the Cartan matrix happens to be nearly as restrictive as the original requiremement, as we will see immediately that the affine algebras can be completely classified and listed on one page.

5.2 Classification of affine Lie algebras

An affine $r \times r$ Cartan matrix has rank $R = r - 1$. Once the classification of finite Cartan matrices (associated to simple Lie algebras) is known up to some rank, the classification of affine Cartan matrices of same rank is straightforward.

For $R = 1$ (so that $r = 2$) the problem is very simple: determine the possible non diagonal entries (non negative and with symmetric zero's) solving

$$\det A \equiv A^{11}A^{22} - A^{21}A^{12} = 4 - A^{21}A^{12} = 0.$$

One finds two rank $r = 2$ affine algebras whose Cartan matrices are given by

$$A(A_1^{(1)}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad A(A_1^{(2)}) = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}. \tag{5.1}$$

The notation for the affine algebras is of the form $X^{(n)}$ where X stands for the finite Lie algebra obtained if one removes one row and the corresponding column (any choice in this case, but we have to be more specific at higher ranks). The superscript index (n) characterizes the two different cases: the $A_1^{(1)}$ is an untwisted affine algebra and $A_1^{(2)}$ is a twisted one. These two notions will be defined in sections 5.5 and 5.6, where the notation (associated finite algebra and superscript) will then make complete sense.

For $R > 1$, it is interesting to first notice the following: when one removes a row and the corresponding column from an affine Cartan matrix, one obtains a finite Cartan matrix. Again if one removes a row and the corresponding column from a finite Cartan matrix, one obtains a finite Cartan matrix again. As a consequence, the 2×2 matrix obtained after removing $R - 2$ rows and the corresponding columns from an affine Cartan matrix must be one of the rank two finite Cartan matrices, that correspond to the algebras $A_1 \oplus A_1$, A_2, B_2 and G_2. Therefore, the elements A^{ij} with $i \neq j$ of an affine Cartan matrix can only take the values $0, -1, -2, -3$ with the extra conditions that $A^{ij}A^{ji} \leq 3$ and $A^{ij} = 0 \iff A^{ji} = 0$. Using these and the vanishing of the global determinant, finding all affine algebras is only a question of combinatorics.

Example Let us consider as an example the case of an affine algebra of rank 3 and containing as a 2×2 submatrix the Cartan matrix of G_2. Up to a renumbering of the rows and columns the affine matrix reads

$$\begin{pmatrix} 2 & a & b \\ c & 2 & -3 \\ d & -1 & 2 \end{pmatrix} \tag{5.2}$$

where we have to determine the numbers a, b, c, d fulfilling the above conditions. The condition on the determinant reads

$$\det A = 2 - a(2c + 3d) + b(-c - 2d) = 0.$$

Let us then consider the possible values case by case:

1. $a = 0 \Rightarrow c = 0 \Rightarrow \det A = 2 - 2bd \Rightarrow b = -1$ and $d = -1$.

2. $a = -1$

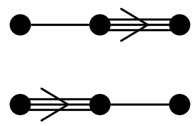


Figure 12: *Dynkin diagrams of* $G_2^{(1)}$ *and* $G_2^{(3)}$.

- $c = -1 \Rightarrow \det A = 3d + b + 2bd \Rightarrow b = d = 0.$
- $c = -2 \Rightarrow \det A = -2 + 3d + 2b - 2bd \leq -2 \Rightarrow$ no solution.
- $c = -3 \Rightarrow \det A = -4 + 3d + 3b - 2bd \leq -4 \Rightarrow$ no solution.

3. $a = -2 \Rightarrow c = -1 \Rightarrow \det A = -2 + 6d + b - 2bd \leq -2 \Rightarrow$ no solution.
4. $a = -3 \Rightarrow c = -1 \Rightarrow \det A = -4 + 9d + b - 2bd \leq -4 \Rightarrow$ no solution.

The only two solutions are thus

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix} \tag{5.3}$$

that correspond to algebra respectively noted $G_2^{(1)}$ (untwisted) and $G_2^{(3)}$ (twisted), whose Dynkin diagrams are given in figure 12.

The whole list of affine algebras can be found for example in [**?**].

5.3 Infinite-dimensional algebras

In this section, we will illustrate the fact that affine Lie algebras are infinite-dimensional by sketching the construction of all the roots of $A_1^{(1)}$, that is the simplest case.

The affine algebra $A_1^{(1)}$ has two simple roots, α_0 and α_1, associated to the two triplets h_0, e_0, f_0 and h_1, e_1, f_1. The Chevalley-Serre relations tell us that

$$[e_0, [e_0, [e_0, e_1]]] = 0 \tag{5.4}$$

and the same for $0 \leftrightarrow 1$.

Let us denote the simple roots as $\alpha_0 = \delta - \alpha$ and $\alpha_1 = \alpha$.

- $e_2 = [e_0, e_1] \neq 0 \Rightarrow \delta$ is a root,
- $e_3 = [e_0, [e_0, e_1]] \neq 0 \Rightarrow 2\delta - \alpha$ is a root,
- $e_4 = [e_1, [e_0, e_1]] \neq 0 \Rightarrow \delta + \alpha$ is a root,
- $[e_0, e_3] = [e_0, [e_0, [e_0, e_1]]] = 0 \Rightarrow 3\delta - 2\alpha$ is not a root,
- $[e_1, e_4] = [e_1, [e_1, [e_1, e_0]]] = 0 \Rightarrow \delta + 2\alpha$ is not a root,
- $e_5 = [e_1, e_3] \neq 0 \Rightarrow 2\delta$ is a root,
- $e_6 = [e_1, e_5] \neq 0 \Rightarrow 2\delta + \alpha$ is a root,
- $\ldots$

Iterating this procedure, one finds that the root system of $A_1^{(1)}$ is given by the infinite set

$$\Delta = \{n\delta \pm \alpha \mid n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z}\backslash\{0\}\}. \tag{5.5}$$

Note in particular that in contrast with semisimple finite algebras, any multiple of a root can be a root for affine algebras.

5.4 Coxeter labels and central element

Definition 16. *The **Coxeter labels** a_i and **dual Coxeter labels** $a_i^\vee$, where $i = 1, \dots, r$ are defined as the components of the left and right eigenvectors of the affine Cartan matrix A, that is,*

$$\sum_{j=1}^{r} a_j A^{ji} = 0 = \sum_{j=1}^{r} A^{ij} a_j^\vee. \tag{5.6}$$

Moreover, one requires the following normalisation condition:

$$\min\{a_i \mid i = 1 \dots r\} = 1 = \min\{a_i^\vee \mid i = 1 \dots r\} \tag{5.7}$$

Now, consider the element

$$K = \sum_{i=1}^{r} a_i^\vee h_i \tag{5.8}$$

of an affine Lie algebra $\mathfrak{g}$ where h_i are the Cartan generators of $\mathfrak{g}$ in the Chevalley-Serre basis and $a_i^\vee$ are the dual Coxeter labels. First, as K is itself in the Cartan subalgebra of $\mathfrak{g}$, one has $[K, h_i] = 0$ for $i = 1, \dots, r$. Second, thanks to the definition of the dual Coxeter labels, the Lie bracket of K with the simple step operators vanishes too:

$$[K, e_i] = \sum_{j=1}^{r} a_j^\vee A^{ij} e_i = 0 \quad , \quad [K, f_i] = -\sum_{j=1}^{r} a_j^\vee A^{ij} f_i = 0. \tag{5.9}$$

By iteration, K vanishes with all generators of $\mathfrak{g}$. Thus K is a central element of $\mathfrak{g}$. Because affine Cartan matrices possess only one vanishing eigenvalue, the center of $\mathfrak{g}$ is one-dimensional, that is, all central elements are scalar multiples of K.

A consequence of the existence of a central element is that any invariant (symmetric) bilinear form is degenerate on the affine algebra $\mathfrak{g}$. Indeed, from the invariance condition, one has, for any $x, y \in \mathfrak{g}$

$$\kappa([x, y], K) = \kappa(x, [y, K]) = 0 \tag{5.10}$$

as $[y, K] = 0$ for all $y \in \mathfrak{g}$. Moreover, the algebra $\mathfrak{g}$ constructed above is the derived algebra, that is $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. In other words, any element z in $\mathfrak{g}$ can be written as a Lie bracket $[x, y]$ of two elements. We have therefore proved that for all $z \in \mathfrak{g}$, $\kappa(z, K) = 0$ and hence κ is degenerate.

This can be remedied by adding an extra generator D, called a derivation, that cannot be written as the commutator of two elements. This will be further explained in the next section.

5.5 Untwisted affine algebras as the central extension of a loop algebra

In the case of untwisted affine algebras $X^{(1)}$, the construction above is reproduced by the central extension of the loop algebra corresponding to the finite underlying algebra X. This construction defines the notion of an untwisted algebra.

The loop algebra. The loop algebra associated to a Lie algebra $\mathfrak{g}_0$ is the space of analytic (single valued) mappings from the circle S^1 to $\mathfrak{g}_0$. Let $\{T^a | a = 1, \dots, d\}$ be a basis of the d-dimensional Lie algebra $\mathfrak{g}_0$, with the structure constants $f^{bc}{}_a$ and t a complex coordinate on S^1, then a basis of the loop algebra $\mathfrak{g}_L$ associated to $\mathfrak{g}_0$ is given by

$$\{T^a_m | a = 1, \dots, d; m \in \mathbb{Z}\}$$

where

$$T^a_{\ m} = T^a \otimes t^m. \tag{5.11}$$

So we can write the loop algebra as

$$\mathfrak{g}_L = \mathbb{C}(t, t^{-1}) \otimes \mathfrak{g}_0,$$

where $\mathbb{C}(t, t^{-1})$ denotes the algebra of Laurent polynomials in t (elements of the form $\sum_{k\in\mathbb{Z}} c_k t^k$ where only a finite number of the complex parameters c_k are non zero). The commutators read

$$\left[T^a_{\ m}, T^b_{\ n}\right] = f^{ab}_{\ \ c} T^c_{\ m+n}.$$

Note that the zero modes T^a_0 span the finite Lie algebra $\mathfrak{g}_0$.

The central extension. Hovever, the loop algebra does not possess a central extension, while the affine algebra does, as explained in the previous subsection. One can see that there is a unique non trivial central extension $\mathfrak{g}_C$ of $\mathfrak{g}_L$. It is obtained by adding a generator K such that

$$\mathfrak{g}_C = \mathfrak{g}_L \oplus \mathbb{C}K.$$

The commutators now read

$$[K, T^a_{\ m}] = 0, \qquad \left[T^a_{\ m}, T^b_{\ n}\right] = f^{ab}_{\ \ c} T^c_{\ m+n} + m\delta_{m+n,0}\kappa^{ab}K.$$

This algebra $\mathfrak{g}_C$ is the derived algebra, isomorphic to the one obtained in the previous section by constructing all possible generators as prescribed by the Cartan matrix.

The derivation. As was explained in the previous section, the derived algebra has a degenerate Killing form, whichever way it is defined. A last generator must accordingly be added to the centrally extended loop algebra $\mathfrak{g}_C$ in order to make the Killing form non-degenerate. For this, introduce a generator D such that the full well defined affine algebra $\mathfrak{g}$ reads

$$\mathfrak{g} = \mathfrak{g}_C \oplus \mathbb{C}D = \big(\mathbb{C}(t, t^{-1}) \otimes \mathfrak{g}_0\big) \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

The commutators involving the new generator D read

$$[D, T^a_{\ m}] = mT^a_{\ m}, \quad [D, K] = 0,$$

while the other generators remain unchanged. Thus the generator D measures the mode number m of the generator $T^a_{\ m}$, or in other words the degree with respect to the gradation (5.11). Note that D never appears on the right hand side of any commutator, such that

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_C.$$

The generator D is called a **derivation**. One can now construct the Killing form of this algebra and see that it is non degenerate.

5.6 Twisted affine algebras

One can construct the twisted affine algebras along the same lines. The only modification is that one abandon the requirement that the maps are single valued, but one rather consider maps towards N-fold covering of the circle. Symmetry arguments then show that only $N = 2$ and $N = 3$ are relevant, and they correspond to the superscript notation in table 9.

6 Representations and the Weyl group

In many physical applications, Lie algebras appear through their representations. But also if one is only interested in the Lie algebras themselves it can be useful to first be more general and study arbitrary representations, and then restrict to the adjoint representation. As we saw in section 2.1, this is the representation where the Lie algebra simply acts on itself through the Lie bracket.

6.1 Representations of $\mathfrak{sl}(2)$

Recall that A_1 is the Kac-Moody algebra for which the Dynkin diagram only consists of a single vertex. Since A_1 constitute the building blocks of any Kac-Moody algebra, understanding its representations is fundamental for the representation theory of general Kac-Moody algebras.

The *defining* representation of the algebra A_1 is the one that allows us to identify it with the matrix Lie algebra $\mathfrak{sl}(2)$, consisting of complex traceless 2×2 matrices. (We will henceforth often write $\mathfrak{sl}(2)$ instead of A_1.) This is usually done in the following way:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.1}$$

Accordingly, the representation is two-dimensional, and it is the smallest nontrivial representation of $\mathfrak{sl}(2)$. It is easy to check that the commutation relations (4.8) are indeed satisfied with the commutator as Lie bracket. By induction we get the relations

$$[h,\, e^k] = 2ke^k, \qquad [e,\, f^k] = -k(k-1)f^{k-1} + kf^{k-1}h, \tag{6.2}$$

$$[h,\, f^k] = -2kf^k, \qquad [f,\, e^k] = -k(k-1)e^{k-1} + ke^{k-1}h, \tag{6.3}$$

(in the associative matrix algebra) that we will use below. Set

$$v_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6.4}$$

and otherwise $v_n = 0$. Then the defining representation (6.1) can also be written

$$h(v_m) = 2mv_m, \qquad e(v_m) = (\tfrac{3}{2} + m)v_{m+1}, \qquad f(v_m) = (\tfrac{3}{2} - m)v_{m-1} \tag{6.5}$$

where $m = -\frac{1}{2}, \frac{1}{2}$. This can be generalized to

$$h(v_m) = 2mv_m, \qquad e(v_m) = (\tfrac{d+1}{2} + m)v_{m+1}, \qquad f(v_m) = (\tfrac{d+1}{2} - m)v_{m-1} \tag{6.6}$$

for any integer $d \geq 1$, where

$$m \in \{-\tfrac{d-1}{2}, -\tfrac{d-1}{2} + 1, \ldots, \tfrac{d-1}{2} - 1, \tfrac{d-1}{2}\} \qquad (d \text{ elements}) \tag{6.7}$$

and $v_n = 0$ if n does not belong to this set. One can show by induction that this will always be a representation with the v_m as basis elements of the module. These eigenvalues are called **weights** of the $\mathfrak{sl}(2)$ representation. The representation (6.6) is irreducible since we can step from any weight v_m to any other weight v_n by acting successively with e (if $m < n$) or f (if $m > n$). This is the reason why the Chevalley generators e and f are called raising and lowering operators, respectively (or step operators with a common name) – they take us upwards and downwards between the weights.

Theorem 11. *Any finite-dimensional irreducible representation of* $\mathfrak{sl}(2)$ *has the form (6.6).*

Proof. We can always find at least one (complex) eigenvalue μ and eigenvector u of h. Then it follows from (6.2) that

$$h(e^s u) = ([h, e^s] + e^s h)u = 2se^s u + e^s hu = (\mu + 2s)e^s u, \tag{6.8}$$

for any $s \geq 0$. Thus $e^s u$ is either zero or another eigenvector of h, with the eigenvalue $\mu + 2s$. But since the representation is finite-dimensional, there must be an integer $p \geq 0$ such that $e^p u \neq 0$ but $e^{p+1}u = 0$. We denote this eigenvector by $e^p u$ and the corresponding eigenvalue, called the **highest weight** of the $\mathfrak{sl}(2)$ representation, by ν.

Now we can step downwards from the highest weight and obtain fv, f^2v, ... with eigenvalues $\nu - 2$, $\nu - 4$, ... It follows from (6.3) that

$$(ef)f^{s-1}v = [e, f^s]v = s(\nu + 1 - s)f^{s-1}v. \tag{6.9}$$

From this we conclude that all the eigenspaces that can be obtained from the highest weight v by applying the step operators are one-dimensional – each eigenvalue has multiplicity one. Furthermore, the eigenvectors span the whole module since the representation is irreducible. (This means that also the tentative weights $e^s u$ for $0 \leq s < p$ can be obtained in this way.) Again since the representation is finite-dimensional, there must be an integer $d \geq 2$ such that $f^{d-1}v \neq 0$ but $f^d v = 0$. But then, if we insert $s = d$ in (6.9) we see that we must have $\nu + 1 - d = 0$. Thus all eigenvalues (that *a priori* were complex numbers) are in fact integers, given by (6.7). □

6.2 Highest weight representations

As we have already seen, the concept of weights can be generalized to a general Kac-moody algebra $\mathfrak{g}$. Recall from section 2 that a **weight** of a $\mathfrak{g}$-module V is an element $\lambda \in \mathfrak{h}^\star$ (a linear map $\mathfrak{h} \to \mathbb{C}$) such that $h(v) = \lambda(h)v$ for some $v \in V$ and all $h \in \mathfrak{h}$. Thus v is an eigenvector of h with the eigenvalue $\lambda(h)$. Since the elements in $\mathfrak{h}$ commute, they have the same set of eigenvalues.

If there is a unique weight v such that $e_i(v) = 0$ for all $i = 1, 2, \ldots, r$, then v is called the **highest weight** of the **highest weight module** V (or of the corresponding **highest weight representation**). Theorem 11 can now be generalized as follows:

Theorem 12. *Any finite-dimensional irreducible representation of a Kac-Moody algebra $\mathfrak{g}$ is a highest weight representation.*

However, this theorem actually makes sense only when $\mathfrak{g}$ is finite, since there are no finite-dimensional representations of affine and indefinite Kac-Moody algebras. Such a representation would be unfaithful (that is, non-injective) as a linear map from an infinite-dimensional vector space to a finite-dimensional one. This means that $\mathfrak{g}$ cannot be simple (exercise). But an indefinite Kac-Moody algebra is always simple, and in the affine case, we would get an induced finite-dimensional representation of the loop algebra, which is also simple and infinite-dimensional. Thus we have:

Theorem 13. *Any finite-dimensional irreducible representation of a finite-dimensional simple Lie algebra (that is, a finite Kac-Moody algebra) is a highest weight representation.*

As for $\mathfrak{sl}(2)$, such a representation is characterized by its highest weight. But there is an important difference: in general the weights can appear with multiplicities greater than one, although the representation is irreducible. In other words, there can be linearly independent eigenvectors with the same eigenvalues. To illustrate this, we consider the simplest example of a Kac-Moody algebra except for $A_1 = \mathfrak{sl}(2)$, namely $A_2 = \mathfrak{sl}(3)$, for which the Dynkin diagram consists of two nodes connected by a single line.

Since $A_2 = \mathfrak{sl}(3)$ has rank 2, its Cartan subalgebra $\mathfrak{h}$ is spanned by two linearly independent elements h_1 and h_2. Accordingly the dual space $\mathfrak{h}^\star$ is also two-dimensional.

For $A_1 = \mathfrak{sl}(2)$, we saw that the weights in fact only took half-integer values along the real line. We can identify this line with the one-dimensional dual space of the Cartan subalgebra. In the general case, the weights belong to a discrete subspace of the dual space, which is called the **weight lattice**.

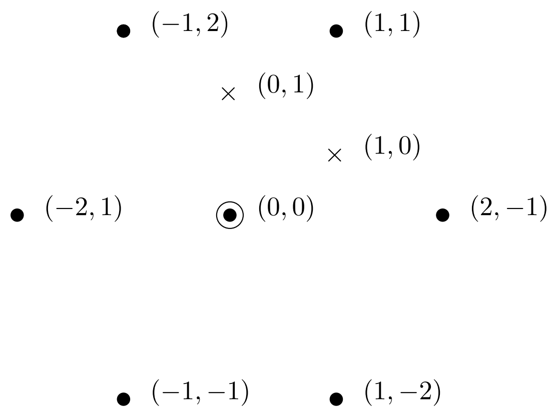
Recall that the Cartan subalgebra $\mathfrak{h}$ of a rank r has a basis consisting of the r Cartan generators h_i ($i = 1, 2, \ldots, r$) and that the dual basis of $\mathfrak{h}^\star$, with respect to the Killing form, consists of simple roots. If we instead take the dual basis with respect to a bilinear form such that the Cartan generators form an *orthonormal* basis, then we obtain the **fundamental weights** as basis elements of $\mathfrak{h}^\star$. This is a convenient choise of basis, since it turns out that any weight of a highest weight representation has integer components, and furthermore, the highest weight has positive components. (It is a bit misleading to call the basis vectors fundamental weights, since they are not always weights, for example not of the $\mathfrak{sl}(3)$ representation that we will consider next. But sometimes all vectors on the weight lattice are called weights.) The components of a weight in the basis of the fundamental weights are called **Dynkin labels** and are the eigenvalues of the Cartan generators. (For $\mathfrak{sl}(2)$ we simply identified the eigenvalues with the weights since there is only one Cartan generator of $\mathfrak{sl}(2)$.)

For $A_1 = \mathfrak{sl}(2)$ we consider now the representation where the highest weight v has the Dynkin labels $(1, 1)$. The eigenvalue is thus 1 for both Cartan generators,

$$h_1(v) = h_2(v) = v. \tag{6.10}$$

Now we can step downwards in two directions, by acting with either f_1 or f_2 on the highest weight v. The corresponding eigenvalue will then change from 1 to -1, and acting with the same lowering operator again will not give any weight. This follows from Theorem 11 if we consider the induced representations of the two $\mathfrak{sl}(2)$ subalgebras separately. But these two subalgebras 'talk to each other' through the line between the two nodes, or through the off-diagonal elements in the Cartan matrix. Therefore, acting on a weight (λ_1, λ_2) with f_1 will not only change the eigenvalue of h_1 from λ_1 to $\lambda_1 - 2$ but also the eigenvalue of h_2 from λ_2 to $\lambda_1 + 1$, and vice versa. This follows easily from the Chevalley relations (4.3). Thus we can arrive at the weight $(0, 0)$ in two ways, either by acting on v with first f_1 and then f_2, or the other way around. The question is then whether we will get the same eigenvector of h_1 and h_2 (up to normalization), or two linearly independent eigenvectors with the same eigenvalues.

To answer the question, suppose that $f_1 f_2 v = \alpha f_2 f_1 v$ for some (complex) number α. Then acting with e_1, we get $2 f_2 v = \alpha f_2 v$. But acting with e_2 we get $2\alpha f_1 v = f_1 v$, and thus a contradiction. We conclude that $f_1 f_2 v$ and $f_2 f_1 v$ are linearly independent, and that the weight $(0, 0)$ has multiplicity 2. Recall from section 3.10 that a single line between two vertices means that the corresponding simple roots have the same length and that the angle between them is $2\pi/3$. From this one can deduce that also the fundamental weights or of equal length (but are shorter than the simple roots), with an angle $\pi/3$ in between. Completing the representations of the $\mathfrak{sl}(2)$ subalgebras, we thus end up with the following picture:

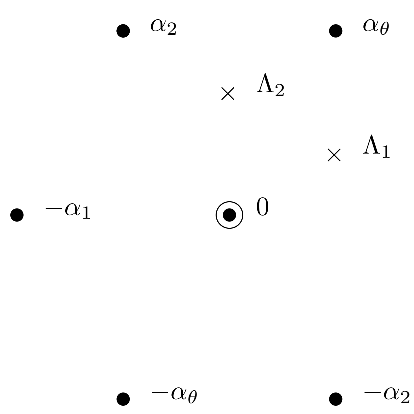


We have here marked the fundamental weights with crosses, the weights of the representation with dots, and the components of the vectors are given in the basis of fundamental weights. We have denoted the double multiplicity in $(0, 0)$ with a circle around the dot.

6.3 Implications for the root system

If we compare the definition of weights above with the definition of roots in Chapter 1 we find that the roots of a Kac-Moody algebra are nothing but the nonzero weights of its adjoint representation. A finite Kac-Moody algebra is finite-dimensional and simple, which means that its adjoint representation is finite-dimensional and irreducible. Thus it follows by Theorem 13 that the adjoint representation of a finite Kac-Moody algebra is a highest weight representation. The highest weight of the adjoint representation is accordingly called the **highest root** of the finite Kac-Moody algebra.

The example that we have studied above is in fact the adjoint representation of $\mathfrak{sl}(3)$. The number of weights, counted with multiplicity, is indeed 8, the dimension of $\mathfrak{sl}(3)$. The double weight (0, 0) corresponds to the two-dimensional Cartan subalgebra. We illustrate this with a new version of the figure above.



We would get the same result for the adjoint representation of any finite Kac-Moody algebra: all weights appear with multiplicity one, except for the zero weight, whose multiplicity is equal to the rank of the algebra. Thus we conclude that the roots of a finite Kac-Moody algebra always have multiplicity one, as we have already mentioned.

6.4 Integrable representations

Since affine and indefinite Kac-Moody algebras are infinite-dimensional there are no highest root, and consequently the adjoint representation is no highest weight representation. We would like to generalize the concept of highest weight representation so that we can describe also the adjoint representation of an affine or indefinite Kac-Moody algebra. To that end, we define an **integrable** representation as a module V spanned by (possibly infinitely many) eigenvectors of the Cartan elements where for any $v \in V$ and any $i = 1, 2, \ldots, r$, there is a positive integer N such that $(e_i)^N v = (f_i)^N v = 0$. Thus the representation of any $\mathfrak{sl}(2)$ subalgebra corresponding to a simple root is a direct sum of finite-dimensional representations, but in general not for an $\mathfrak{sl}(2)$ subalgebra corresponding to a nonsimple root. It follows from the Serre relations that the adjoint representation of a Kac-Moody algebra is integrable.

Theorem 14. *If λ is a weight of an integrable module and $\lambda + \alpha_i$ is not a weight, then*

$$2\frac{(\lambda|\alpha_i)}{(\lambda_i|\alpha_i)} \geq 0. \tag{6.11}$$

If $\lambda - \alpha_i$ is not a weight, then

$$2\frac{(\lambda|\alpha_i)}{(\lambda_i|\alpha_i)} \leq 0. \tag{6.12}$$

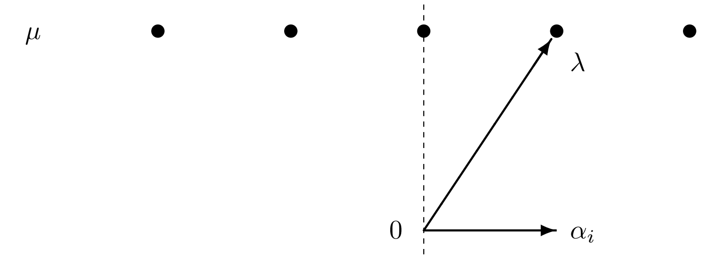
Proof. We would like to know for which integers t the vector $\mu = \lambda + t\alpha_i$ is a weight. We denote the corresponding eigenvector by v_μ. Then we have

$$h_i v_\mu = \mu(h_i) v_\mu = 2\frac{(\mu|\alpha_i)}{(\alpha_i|\alpha_i)} v_\mu. \tag{6.13}$$

But according to Theorem 11 the eigenvalue $\frac{(\mu,\,\alpha_i)}{(\alpha_i|\alpha_i)}$ must belong to the set $\{-p, -p+2, \ldots, 0, \ldots, p-2, p\}$ for some integer $p \geq 0$. Then the theorem follows. ☐

We illustrate the situation below for $p = 2$.

$(\lambda,\, \alpha_i)$	-2	-1	0	1	2
t	-3	-2	-1	0	1



6.5 The Weyl group

One important consequence of the theorem in the previous subsection is the following corollary:

Theorem 15. *If λ is a root and α_i is a simple root, then*

$$\lambda - 2\frac{(\lambda|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i \tag{6.14}$$

is a root as well, with the same multiplicity as λ.

For any root α, define the reflection with respect to α by

$$\lambda \mapsto \lambda - 2\frac{(\lambda|\alpha)}{(\alpha|\alpha)}\alpha \tag{6.15}$$

This is a reflection in the plane perpendicular to the root α. The corollary thus says that the root system, and the root multiplicities, are preserved under the reflections with respect to the simple roots. Such reflections are called **fundamental Weyl reflections**. When $\alpha = \alpha_i$ we denote the fundamental Weyl reflection (6.14) by r_i.

Whenever some object is invariant under some operations, it is also invariant under the composition of the two such operations, and therefore it is interesting to study all operations that can obtained in this way, by successive compositions. In this case we are interested in the group generated by the fundamental Weyl reflections. This group is called the **Weyl group**, and it is a discrete subgroup of $GL(\mathfrak{h})$. Thus the root system, and the root multiplicities, are preserved under the Weyl group, which means that it is a subgroup to the permutation group of the roots. When the Kac-Moody algebra is finite, the Weyl group is also finite as a group, since there are only finitely many roots to permute.

Given any two elements of a group, it is of interest to know whether they commute or not. For the generators of the Weyl group we want to compare $r_i r_j$ and $r_j r_i$. Since the generators are reflections, each of them is its own inverse, $r_i r_i = 1$. Thus we have

$$r_i r_j = (r_j r_j) r_i r_j (r_i r_i) = r_j (r_j r_i)^2 r_i \tag{6.16}$$

and we see that $r_i r_j = r_j r_i$ if and only if $(r_j r_i)^2 = 1$.

When $(r_i r_j)^2 \neq 1$, so that $r_i r_j \neq r_j r_i$, one can ask if there is *any* integer $m \geq 2$ such that $(r_i r_j)^m = 1$. Let us therefore study $(r_i r_j)^m$ for a general integer $m \geq 2$. The vector space $\mathfrak{h}^\star$ is the direct sum of the plane spanned by α_i, α_j and its orthogonal complement. The orthogonal complement is invariant under both r_i and r_j, so it is enough to study the plane spanned by α_i, α_j. In this basis, the matrix realization of r_i and r_j are

$$r_i = \begin{pmatrix} -1 & -A_{ji} \\ 0 & 1 \end{pmatrix}, \qquad r_i = \begin{pmatrix} 1 & 0 \\ -A_{ij} & -1 \end{pmatrix}, \qquad r_i r_j = \begin{pmatrix} -1 + A_{ij}A_{ji} & A_{ji} \\ -A_{ij} & -1 \end{pmatrix}, \tag{6.17}$$

Set $M = r_i r_j$. Now we invoke the Cayley-Hamilton theorem, which says that a matrix satisfies its own characteristic equation. We have

$$\det(M - \lambda\mathbf{1}) = \lambda^2 + (2 - A_{ij}A_{ji})\lambda + \mathbf{1} \tag{6.18}$$

and so we get

$$M^2 = (A_{ij}A_{ji} - 2)M - \mathbf{1}. \tag{6.19}$$

Multiplying the equation (6.19) successively with M, and inserting it into the result, we obtain the equation

$$M^m = a_m M - a_{m-1}\mathbf{1} \tag{6.20}$$

where

$$a_m = (A_{ij}A_{ji} - 2)a_{m-1} - a_{m-2}, \qquad a_0 = 0, \qquad a_1 = 1. \tag{6.21}$$

For $A_{ij}A_{ji} - 2 \geq 2$ this sequence is increasing, and therefore there is no integer $m > 0$ such that $a_m = 0$ which is a necessary condition for $M^m = \mathbf{1}$. It is conventional to set $m = \infty$ in this case, $M^\infty = \mathbf{1}$. In the other cases, $0 \leq A_{ij}A_{ji} \leq 3$, one can easily check that m is given by the following table:

$A_{ij}A_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

(6.22)

Any group that is generated by a finite number n of generators r_i ($i = 1, 2, \ldots, n$) modulo the relations $(r_i r_j)^{m_{ij}} = 1$, for arbitrary integers $m_{ij} = m_{ji} \geq 2$, is called a **Coxeter group**. When the integers m_{ij} are restricted to the values 2, 3, 4, 6 and ∞ as above, the group is said to be a **crystallographic** Coxeter group. The conclusion of this last section is thus that the Weyl group of a Kac-Moody algebra is always a crystallographic Coxeter group. Conversely, any crystallographic Coxeter group is the Weyl group of a Kac-Moody algebra, but this is not uniquely given, as can be seen by (6.22). For further reading about Coxeter groups and the related *reflection groups*, we recommend the book [**?**] by Humphreys.