A mini-introduction to topological K-theory [1]

Wen Jiang
Mathematical Institute, Oxford

[1] Notes written for Modave Summer School 2006. Please e-mail any comments to the author at `jiang@cantab.net`.

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Chapter 1

Introduction

K-theory has become a very important tool in many branches of mathematics and, more recently, in theoretical physics. Despite its seeming esotericism, topological K-theory may be introduced quite naturally via vector bundles over a topological space. It will turn out to have very nice properties, such as periodicity and similarities to a cohomology theory. We will content ourselves with statements of relevant results and not delve deeper into the mathematical details.

For proofs missing in this talk, the reader is advised to consult [1] and [2], while the paper [3] contains a good overview placing K-theory in the context of topology in general.

Some materials in the following two chapters are not strictly necessary for our purpose, but serve to introduce related concepts important in their own right.

Chapter 2

Complex Vector Bundles

2.1 Generalities

2.1.1 Definitions

For the purpose of this talk, all the spaces involved will be compact topological manifolds, all the maps continuous and all the vector spaces complex, unless otherwise stated. Almost all the results hold for smooth vector bundles, but it must be noted that holomorphic vector bundles are completely different.

Vector bundles are familiar objects to most at this school. They can be visualised as families of vector spaces parametrised by points in a manifold and homomorphisms between them must be vector space homomorphisms when restricted to the fibres. A vector bundle $p : E \to X$ with fibre V must be locally trivial in the sense that around any point $x \in X$ is an open neighbourhood U such that E restricted to U is isomorphic to the Cartesian product $U \times V$. Such a U is called a trivialising neighbourhood.

Given that X is compact, we can find a finite open cover U_i of X such that over each U_i there is an isomorphism $\phi_i : E|_{U_i} \to U_i \times V$. Then for any $x \in U_i \cap U_j$, the map $\phi_i \phi_j^{-1}$ restricted to $\{x\} \times V$ gives an isomorphism $g_{ij}(x) : V \to V$. The function $g_{ij} : U_i \cap U_j \to GL(V)$ is often called a transition function. We may work backwards and glue together E from the data $\{U_i\}$ and $\{g_{ij}\}$. In fact, thinking about which sets of such data define equivalent bundles will lead to Čech cohomology, of which we will speak no further.

Note that any vector bundle admits a hermitian metric. The transition functions will consequently take values in $U(V)$. Thus complex line bundles are just $U(1)$ or circle bundles.

2.1.2 Constructions

Let us look at various ways of obtaining new bundles from old. Natural operations on vector spaces, such as taking the dual, quotient, direct sum and tensor product, all carry over to vector bundles. Thus, the set of (isomorphism classes of) vector bundles over X almost forms a ring (like $\mathbb{Z}$), except that we cannot subtract bundles. Attempting to define additive inverses will lead one directly to topological K-theory.

Another method of producing bundles is the pull-back. Suppose we have a map $f : Y \to X$, we can define the 'pull-back bundle' f^*E as a subset of $E \times Y$ by

$$f^*E = \{(e, y) \in E \times Y \ : \ p(e) = f(y)\}.$$

In words, a point $y \in Y$ is associated with the vector space over the point $f(y)$. If $E|_U \cong U \times V$, then $W = f^{-1}(U)$ is open and $(f^*E)|_W \cong W \times V$. Hence, f^*E is locally trivial. The properties of the pull-back bundle is summarised by the following commutative diagram.

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

In terms of transition functions, those for f^*E are simply $g_{ij} \circ f$ over the intersections $W_i \cap W_j$.

Fact: If $f_0, f_1 : Y \to X$ are homotopic maps, then f_0^*E and f_1^*E are isomorphic. It follows that homotopy-equivalent spaces have the same isomorphism classes of vector bundles over them. In particular, any bundle over a contractible space is trivial.

Example (Bundles over S^n): The n-sphere S^n may be covered by two open disks U_1 and U_2, the intersection of which is homotopy-equivalent to S^{n-1}. Any bundle $V \to E \to S^n$ is trivial when restricted to U_1 and U_2. From the gluing point of view, E is determined by the homotopy class of the transition function $g : S^{n-1} \to GL(V)$. Thus, isomorphism classes of vector bundles over S^n correspond to $[S^{n-1},\ GL(V)]$, the set of homotopy classes of maps from $S^{n-1} \to GL(V)$.

For $n = 1$, this set contains a single element, as $GL(V)$ is path-connected. Therefore, any complex vector bundle over S^1 is trivial – this is easy to see directly anyway.

When $n = 2$, $[S^1,\ GL(V)]$ is the same as the fundamental group $\pi_1(GL(V))$, which is $\mathbb{Z}$ for any V of dimension > 0. This integer invariant can be identified with the first Chern class of the bundle. Hence, two vector bundles over S^2 are isomorphic if and only if they have the same rank and first Chern class.

Example (Tautological bundles): A projective space is naturally equipped with a line bundle. Each point $[l]$ in the space $\mathbb{CP}^n$ by definition labels a 1-dimensional subspace l of $V = \mathbb{C}^{n+1}$. The bundle $H^* \to \mathbb{CP}^n$ simply assigns l as the fibre over $[l]$. Its dual is denoted by H and the direct sum (resp. tensor product) of m copies of H is written as mH (resp. H^m). Since $\mathbb{CP}^1 = S^2$, from the previous example, we deduce that any line bundle over S^2 is isomorphic to H^m for some $m \in \mathbb{Z}$.

A slight generalisation of a projective space is a Grassmanian $G_k(V)$, the set of subspaces of V of dimension k . So $\mathbb{CP}^n$ is simply $G_1(V)$. As above, the tautological bundle Q assigns the vector space $u \subset V$ to the point $[u] \in G_k(V)$. It is also called the classifying bundle over $G_k(V)$.

Now any map $f : Y \to G_k(V)$ gives a vector bundle f^*Q . We will see later that any rank k bundle over Y is of this form for some $V = \mathbb{C}^N$.

2.1.3 Complementary bundles

Certainly not every vector bundle $V \to E \to X$ is trivial. However, provided that the base space X is compact, as we assume, it can be shown that there always exists a bundle $V' \to E' \to X$ such that

$$E \oplus E' \cong X \times \mathbb{C}^N,$$

where $N = \text{dim}V + \text{dim}V'$. Such an E' is called complementary to E .

The existence of complementary bundles is due to the fact that any (compact) manifold admits a partition of unity. Given any (finite) open cover $\{U_i\}$ of X , a partition of unity subordinate to $\{U_i\}$ is a collection of continuous functions $f_i : X \to \mathbb{R}$ such that

- $f_i(x) \geq 0$, for any $x \in X$ and any i ;
- the support of each f_i lies in U_i ;
- $\sum_i f_i(x) = 1$, for any $x \in X$.

Partitions of unity are used to patch up local definitions into a global one. Suppose that the triviality of $E|_{U_i}$ allows us to produce a map h_i on U_i , then the object $\sum_i f_i h_i$ is well-defined over the whole manifold.

Example (Hermitian metrics): If we take h_i to be a hermitian metric on $E|_{U_i}$, we obtain a hermitian metric on the whole of E . As a consequence, any short exact sequence of vector bundles

$$0 \to E_1 \to E \to E_2 \to 0$$

splits, that is to say, $E \cong E_1 \oplus E_2$. To see this, note that $E \cong E_1 \oplus E_1^\perp$ given any metric. The exact sequence, however, implies that $E_2 \cong E/E_1$, which is in turn isomorphic to $E_1^\perp$.

2.2 Classifying space for vector bundles

The importance of complementary bundles will become apparent in this section, as we find a homotopy characterisation of $\mathrm{Vect}_n(X)$, the set of isomorphic classes of rank n bundles on X.

Let $E \to X$ be a rank n vector bundle as before. Take a complementary bundle $E' \to X$, so that $E \oplus E' \cong X \times \mathbb{C}^N$ for some N . Then we have an inclusion $i : E \to X \times \mathbb{C}^N$, a injective bundle morphism. Over each point $x \in X$, the map $i_x : E_x \to \mathbb{C}^N$ is an injective vector space homomorphism and the image $\mathrm{Im}\, i_x$ is of codimension n . We can then define a map

$$f : X \to G_n(\mathbb{C}^N); \quad x \mapsto \mathrm{Im}\, i_x.$$

Recall that the Grassmanian $G_n(\mathbb{C}^N)$ has a classifying bundle Q as described in Section 2.1.2. To see that the pull-back f^*Q is isomorphic to E , note that the fibre of f^*Q over $x \in X$ is the fibre of Q over $f(x)$, which is tautologically $\mathrm{Im} i_x$, naturally isomorphic to E_x . Hence, $f^*Q \cong E$.

It can be shown that different choices of E' lead to homotopic maps f , and thus isomorphic bundles. Denote by $[X,\ G_n(\mathbb{C}^N)]$ the set of homotopy classes of maps $f : X \to G_n(\mathbb{C}^N)$.

Theorem: The map $f \mapsto f^*Q$ induces an isomorphism

$$[X,\ G_n(\mathbb{C}^N)] \to \mathrm{Vect}_n(X),$$

where the natural number N only depends on the compact space X .

The number N increases when X becomes more complicated, that is, if a large number of trivialising neighbourhoods are required to cover X . The theorem above holds for any X if $G_n(\mathbb{C}^N)$ is replaced by a construction called the direct limit, $G_n(\mathbb{C}^\infty) = \varinjlim_N G_n(\mathbb{C}^N)$. It is a 'classifying space' for complex rank n bundles. We will not elaborate on this. For more details, see [6] or [9].

Chapter 3

Characteristic Classes

The aim of this chapter is to make a detour into the theory of characteristic classes, so that one could introduce those the Chern character and Todd class, which are often encountered in K-theory. We begin with Chern classes, not only because they are the most commonly encountered characteristic classes, but also because they form the building blocks of others, as we shall see.

The distinction between topological and smooth vector bundles will be blurred in this chapter. This poses no real ambiguity as any topological vector bundle we are considering could be given differential structures and any continuous maps between them can be approximated by smooth ones.

3.1 Chern classes

3.1.1 Topological approach

There are a number of definitions of Chern classes for complex vector bundles. They are usually introduced to physicists through the Chern-Weil homomorphism: given a hermitian metric and a connection on a complex bundle $E \to X$, the total Chern class $c(E) \in H^*(M, \mathbb{R})$ is represented by the closed form

$$\det(I + \frac{i}{2\pi}F),$$

where F is the curvature of the connection. It then has to be shown that this definition is independent of the metric and connection.

Having introduced the classifying bundle, we can give an alternative, purely topological, definition. Given a bundle $E \to X$, represent it as f^*Q for some map $f : X \to G_n(\mathbb{C}^\infty)$, as in the theorem in the last section. The cohomology ring of $G_n(\mathbb{C}^\infty)$ is a free polynomial ring on n generators,

$$\mathbb{R}[c_1, \ \ldots, \ c_n].$$

Then set the kth Chern class of E to be $c_k(E) = f^* c_k$. This definition can be verified to agree with the previous one, if the c_k's are normalised appropriately.

The definition is not as important as the properties. As a matter of fact, Chern classes are uniquely determined by the following of its properties.

- For any bundle $E \to X$ and continuous map $f : Y \to X$, $c(f^*E) = f^*(c(E))$, where f^*E is the pull-back bundle and $f^*(c(E))$ is the pull-back of the cohomology class on X.
- If $E \cong E_1 \oplus E_2$ on X, then $c(E) = c(E_1)\, c(E_2)$.
- The Chern class of a complex line bundle $L \to X$ is $c(L) = 1 + e(L_{\mathbb{R}})$, where $e(L_{\mathbb{R}})$ is the Euler class of L regarded as a rank 2 real vector bundle.

The last is the key, as it indicates that vector bundles can be built from line bundles as far as Chern classes are concerned. To explain this, we need to learn about the splitting principle, the subject of the next section. If the reader is not familiar with the Euler class, he could think of it as $iF/2\pi$, in terms of a $U(1)$ connection on L, or consult [4] for an elementary exposition.

Before carrying on, let's note a few results. If L, L' are line bundles, then

$$c_1(L \oplus L') = c_1(L)\, c_1(L') \quad \text{and} \quad c_1(L \otimes L') = c_1(L) + c_1(L').$$

Let L^* be the dual line bundle of L. Since $L \otimes L^* = X \times \mathbb{C}$ and c_1 of a trivial bundle is 0, we have $c_1(L^*) = -c_1(L)$.

Over $S^2 = \mathbb{CP}^1$, denote by 1 the trivial line bundle and by x the first Chern class $c_1(H)$ of the tautological line bundle H we introduced earlier. Then the total Chern classes of the two bundles $H^2 \oplus 1$ and $H \oplus H$ are

$$c(H^2 \oplus 1) = c(H^2)\, c(1) = c(H^2) = 1 + c_1(H^2) = 1 + 2x;$$

and

$$c(H \oplus H) = c(H)\, c(H) = (1 + x)^2 = 1 + 2x,$$

since x^2 is a degree 4 form, which vanishes on S^2. Now the rank and the first Chern class are the only topological invariants over S^2, as we saw in Section 2.1.2. Therefore,

$$H^2 \oplus 1 \cong H \oplus H.$$

One could also see this directly, by noting that the two bundles have transition functions $\begin{pmatrix} 1/z^2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1/z & 0 \\ 0 & 1/z \end{pmatrix}$ respectively. The following formula gives an explicit homotopy between the two. It is equal to the first matrix when $t = 0$ and to the second when $t = \pi/2$.

$$\begin{pmatrix} 1/z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1/z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

3.1.2 The splitting principle

Without going into details, we simply state that, given a vector bundle $E \to X$, there exists a compact space $F(E)$ and a map $\sigma : F(E) \to X$ such that

- $\sigma^* E \to F(E)$ splits into a sum of line bundles over $F(E)$,

$$\sigma^* E = L_1 \oplus \cdots \oplus L_n;$$

- The pull-back of cohomology $\sigma^* : H^*(X) \to H^*(F(E))$ is injective.

More generally, for two bundles E, E' on X, one can apply the above twice to split both E and E' on $F(E, E') := F(\sigma_E^* E')$.

To illustrate the use of this construction, suppose that we wish to prove a *polynomial* relation $P(c(E), c(E')) = 0$ between two vector bundles on X and assume that this relation holds when E and E' are sums of line bundles on any space. Pulling back by σ to $F(E, E')$ gives

$$\sigma^* P(c(E), c(E')) = P(\sigma^* c(E), \sigma^* c(E')) = P(c(\sigma^* E), c(\sigma^* E')) = 0,$$

since $\sigma^* E$ and $\sigma^* E'$ are sums of line bundles on $F(E, E')$. But the injectivity of σ^* implies that $P(c(E), c(E')) = 0$. (This argument of course goes unchanged if more than two bundles appear in P.) Thus, we have the following

Splitting Principle: To prove polynomial identities of Chern classes, it suffices to assume that the bundles involved are sums of line bundles.

This principle can in fact be used to *define* other characteristic classes in terms of Chern classes, as we will explain next.

3.2 More characteristic classes

The splitting principle allows us to write

$$\sigma^* c(E) = \prod_{i=1}^{n} (1 + x_i),$$

for certain $x_i \in H^2(F(E))$, the first Chern class of the line bundle L_i. Since the whole expression is symmetric in x_i, its degree k part, $s_k(x_1, \ldots, x_n)$, is also. It is called the kth elementary symmetric polynomial in x_i. For example,

$$s_1 = \sum_i x_i, \quad s_2 = \sum_{i<j} x_i x_j, \quad \ldots, \quad s_n = \prod_i x_i.$$

Fact: Any symmetric polynomial in x_i may be written as a polynomial in s_i.

Note that $s_k = \sigma^* c_k(E)$ by construction. Therefore, any symmetric polynomial $P(x_1, \ldots, x_n)$ is equal to a polynomial

$$P'(s_1, \ldots, s_n) = \sigma^* P'(c_1(E), \ldots, c_n(E)) \equiv \sigma^* \tau_P(E).$$

Since σ^* is injective, P uniquely determines a class $\tau_P(E) \in H^*(X)$ in this fashion. Such a 'characteristic' class evidently possesses the same functorial property as the Chern classes. One can in fact allow P to be a power series expression, since $H^m(X) = 0$ for $m > \dim X$ and so higher products of the x_i's eventually vanish.

We may now introduce the Chern character. Let P be the power series

$$f_{\mathrm{ch}} = \sum_{i=1}^{n} e^{x_i} = n + \sum x_i + \frac{1}{2} \sum x_i^2 + \cdots .$$

The class it determines is the total Chern character $\mathrm{ch}(E)$. In terms of the Chern classes,

$$\mathrm{ch}(E) = \mathrm{rank}\, E + c_1(E) + \left(\frac{1}{2} c_1(E)^2 - c_2(E) \right) + \cdots .$$

The importance of the Chern character is that, for any bundles $E, F \to X$,

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F) \quad \text{and} \quad \mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F).$$

To prove this, we use the splitting principle and assume that E and F are sums of line bundles, L_i and L'_j, with c_1 being x_i and y_j respectively. Then

$$c(E \oplus F) = c(\bigoplus_i L_i \oplus \bigoplus_j L'_j) = \prod_i (1 + x_i) \prod_j (1 + y_j),$$

so

$$\mathrm{ch}(E \oplus F) = \sum_i e^{x_i} + \sum_j e^{y_j} = \mathrm{ch}(E) + \mathrm{ch}(F).$$

For the second equality, we go through the same procedure:

$$c(E \otimes F) = c(\sum_{i,j} L_i \otimes L'_j) = \prod_{i,j} (1 + x_i + y_j)$$

implies that

$$\mathrm{ch}(E \otimes F) = \sum_{i,j} e^{x_i + y_j} = \sum_i e^{x_i} \sum_j e^{y_j} = \mathrm{ch}(E)\,\mathrm{ch}(F).$$

In the Chern-Weil theory, the Chern character may be represented by the following closed form

$$\mathrm{Tr}\,\exp\left(\frac{F}{4\pi i}\right).$$

More characteristic classes may be obtained by considering other choices of P. Of course, the only interesting ones are those which arise naturally from real problems. For example, in the Atiyah-Singer Index theorem, one encounters the formal object

$$\mathrm{ch}\,(\bigwedge^{\mathrm{even}} E - \bigwedge^{\mathrm{odd}} E) = \prod_i (1 - e^{x_i}),$$

where the $\bigwedge$ is the exterior product and we will worry about the '$-$' sign in the next section. Dividing the top Chern class $c_n(E) = \prod x_i$ by this leads one to define the Todd class through the symmetric power series

$$f_{\mathrm{Td}} = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} = \prod_{i=1}^{n} \left(1 + \frac{1}{2}x_i + \frac{1}{12}x_i^2 + \cdots\right).$$

In terms of Chern classes, $\mathrm{Td}_1 = c_1/2$, $\mathrm{Td}_2 = (c_1^2 + c_2)/12$, etc.

There is a similar story for real vector bundles. One uses the Pontryagin classes, which are Chern classes of the complexification, to build the $\hat{A}$-classes through the power series $\prod \frac{x_i/2}{\sinh(x_i/2)}$, and also the $\hat{L}$-classes where '$\sinh$' is replaced by '$\tanh$'.

3.3 Other cohomological invariants?

In the previous section, we constructed some characteristic classes as polynomials in the Chern classes. Are there any other characteristic classes not expressible in this way?

What do we mean then by a characteristic class? One may define it as a natural transformation τ between $\mathrm{Vect}_n(\cdot)$ and $H^*(\,\cdot\,;\mathbb{R})$, both contravariant functors from the category of compact manifolds to the category of sets. By this 'functorial statement', we simply mean that, for any bundle $E \to X$ and map $f : Y \to X$, $\tau(E)$ is a cohomology class in $H^*(X)$ and that

$$\tau(f^*E) = f^*\tau(E).$$

This is of course in complete analogy with the first property that Chern classes satisfy, as stated in the last section. From this point of view, the following theorem shows that Chern classes are the only cohomological invariants of complex vector bundles in this sense.

Theorem: Every natural transformation τ as above can be expressed as a polynomial in the Chern classes.

Proof. Given a bundle $E \to X$, let $f : X \to G_n(\mathbb{C}^\infty)$ be a classifying map, so that $E \cong f^*Q$. Now $\tau(Q) \in H^*(G_n(\mathbb{C}^\infty))$, which is a polynomial ring generated by c_k, as we said. (It does not matter here that $G_n(\mathbb{C}^\infty)$ is infinite-dimensional.) Hence, $\tau(Q) = P(c_1, \dots, c_n)$ for some polynomial. The 'functoriality' of τ then implies that

$$\tau(E) = \tau(f^*Q) = f^*\tau(Q) = P(f^*c_1, \dots, f^*c_n) = P(c_1(E), \dots, c_n(E)),$$

as we claimed. □

Chapter 4

K-theory

As appropriate to this school, we will concentrate on the basic ideas, which are fairly natural, even though proofs of results are generally technical and will be omitted here. Those who wish may find them in the copious literature, for example, [1], [5] and [7].

4.1 Basic objects

4.1.1 The group $K(X)$

We saw in Section 2.1.2 that vector bundles on a space X can be added and multiplied. To make the set of vector bundles $\mathrm{Vect}(X)$ into a ring, it would require one to define a 'subtraction' or, equivalently, additive inverses. This can be done formally. Let's recall the construction of negative numbers from natural numbers. Define a relation $\sim$ on $\mathbb{N}\times\mathbb{N}$ by

$$(a,b)\sim(c,d) \quad \text{if} \quad a+d=c+b.$$

Here (a,b) is to be thought of as '$a-b$'. This relation is evidently reflexive and symmetric. It is also transitive, since $(a,b)\sim(c,d)$ and $(c,d)\sim(e,f)$ implies that

$$a+f+c+d=b+e+c+d$$

and so cancelling $(c+d)$ gives $(a,b)\sim(e,f)$. Equivalence classes under $\sim$ then form the ring of integers $\mathbb{Z}$.

One can attempt the same procedure with $\mathrm{Vect}(X)\times\mathrm{Vect}(X)$, but one immediately runs into difficulty, as the 'cancellation' above does not hold for vector bundles. (A real bundle example is the tangent bundle of $S^2\subset\mathbb{R}^3$. Its direct sum with the normal bundle $S^2\times\mathbb{R}$ is $S^2\times\mathbb{R}^3$.) Therefore, one is led to define $(E,F)\sim(E',F')$ if there exists G such that

$$E\oplus F'\oplus G\cong E'\oplus F\oplus G.$$

Transitivity holds for this relation. The equivalence classes of these pairs $[(E,F)]$ form the additive abelian group $K(X)$. Note that indeed $[E\oplus F] = [E]+[F]$

Each vector bundle $E\to X$ determines an element $[E]\equiv[(E,0)]$ of $K(X)$ and since $(E,0)+(0,E)\sim(0,0)$, we have $-[E]=[(0,E)]$. Thus, every element of $K(X)$ is of the form $[E]-[F]$. Moreover, there is a complementary bundle F' such that $F\oplus F'\cong n$, the trivial rank n bundle. Consequently

$$[E]-[F] = [E\oplus F'] - [F\oplus F'] = [E\oplus F'] - [n],$$

and so all the elements of $K(X)$ are in fact of the form $[E]-[n]$. It follows that $[E]=[F]$ in $K(X)$ if and only if $E\oplus n = F\oplus n$ for some n. Such bundles are called stably isomorphic.

Any map $f:Y\to X$ enables one to pull back bundles from X to Y and so induces a map $f^*: K(X)\to K(Y)$, which only depends on the homotopy class of f. One can view $K(\,\cdot\,)$ as a contravariant functor from the category of compact topological spaces to the category of abelian groups. Furthermore, the tensor product between vector bundles on X induces a multiplication in $K(X)$, which makes $K(X)$ into a ring.

One can remove the appearance of an arbitrary natural number from elements $[E]-[n]$ of $K(X)$ as follows. Take any point $x_0\in X$, the inclusion $i:\{x_0\}\to X$ induces a map $i^*: K(X)\to K(x_0)\cong\mathbb{Z}$. Define $\widetilde{K}(X)$ to be the kernel of i^*. Hence the elements of $\widetilde{K}(X)$, if X is connected, are of the form $[E]-[\mathrm{rk}E]$ and $[E]=[F]$ in $\widetilde{K}(X)$ if and only if $E\oplus m\cong F\oplus n$ for some m and n. Such bundles are called stably equivalent.

The collapsing map $c: X\to\{x_0\}$ induces an splitting and

$$K(X)\cong\widetilde{K}(X)\oplus K(x_0)\cong\widetilde{K}(X)\oplus\mathbb{Z}.$$

Example ($K(S^k)$): Because of its properties mentioned above, $K(X)$ is a homotopy invariant, to calculate which is generally very difficult. However, Bott periodicity, to be introduced later, makes it possible for certain spaces. From our gluing point of view in Section 2.1.2, we see that $\mathrm{Vect}(S^1)=\mathbb{Z}$ and $\mathrm{Vect}(S^2)=\mathbb{Z}\times\mathbb{Z}$ as sets. If S^0 means the '0-sphere' $\{1,-1\}$, then $\mathrm{Vect}(S^0)=\mathbb{Z}\times\mathbb{Z}$. Evidently $K(X)=\mathrm{Vect}(X)$ for $X=S^0$ or S^1, since all the bundles are trivial over these spaces. We assert that this is also true when $X=S^2$. We see, therefore, that

$$K(S^0)=\mathbb{Z}\times\mathbb{Z},\quad K(S^1)=\mathbb{Z},\quad\text{and}\quad K(S^2)=\mathbb{Z}\times\mathbb{Z}.$$

The equality of $K(S^0)$ and $K(S^2)$ is no coincidence and is an example of Bott periodicity, which implies that, for any $k\in\mathbb{N}$,

$$\begin{aligned} K(S^{2k}) &= K(S^0),\\ K(S^{2k+1}) &= K(S^1). \end{aligned}$$

The ring, i.e. multiplicative, structure will be described later.

4.1.2 Suspension and $K^*(X)$

The group $K(X)$ is really $K^0(X)$ in a infinite sequence of K-groups. To define these, one has to introduce several topological constructions. For a space X with a distinguished base point x_0, let the suspension of X be

$$SX = ((X \times [0,1])/(X \times \{0\}))\,/(X \times \{1\}),$$

i.e. a cylinder with each end pinched to a point. The reduced suspension is

$$\Sigma X = SX/(\{x_0\} \times [0,1]),$$

i.e. the line segment through x_0 is further shrunk to a point. For example, for a sphere, $SS^n \cong \Sigma S^n \cong S^{n+1}$. Note that SX and ΣX are homotopy equivalent and so have identical K-groups.

When $Y \subset X$ is non-empty and compact, X/Y is the quotient space with Y/Y as the base point. For $Y = \emptyset$, let $X/\emptyset$ be the space $X_+ = X \sqcup \{*\}$, the disjoint union of X with a point.

Definition: For $n \geq 0$, let

$$\begin{aligned} \widetilde{K}^{-n}(X) &= \widetilde{K}(\Sigma^n X) \\ K^{-n}(X,Y) &= \widetilde{K}^{-n}(X/Y) \\ K^{-n}(X) &= K^{-n}(X,\emptyset) = \widetilde{K}(\Sigma^n(X_+)). \end{aligned}$$

The minus sign in the exponents was chosen so that the index increases in the following exact sequence (see [1] for proof)

$$\cdots \to K^{-2}(Y) \xrightarrow{\delta} K^{-1}(X,Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*}$$
$$K^{-1}(Y) \xrightarrow{\delta} K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y),$$

where $Y \subset X$ is compact and $i : Y \to X$ and $j : (X,\emptyset) \to (X,Y)$ are the inclusions. Please consult [1] for the description of the map δ.

Some may have noticed that this is similar in form to the long exact sequence in relative cohomology. In fact, $K*$ satisfies all the axioms of Eilenberg and Steenrod, except the dimension axiom, and is therefore an ‘extraordinary cohomology theory’. The Chern character, constructed in Section 3.2, gives rise to a ring homomorphism

$$\mathrm{ch} : K^0(X) \to H^{2*}(X;\mathbb{Q}),$$

which turns out to induce an isomorphism $K^0(X) \otimes \mathbb{Q} \to H^{2*}(X;\mathbb{Q})$ for ‘nice enough’ spaces.

4.2 Bott Periodicity

Bott originally discovered the periodicity named after him when he applied Morse theory to loop spaces of (compact) classical groups. This translates into a natural isomorphism of the K-groups: $K^{-n}(X) \xrightarrow{\sim} K^{-n-2}(X)$, for all $n \geq 0$. One could read Bott's original approach in Milnor's beautiful book [8], or a direct proof for K-theory in [1].

Unfortunately, there is no time to explain the precise nature of the isomorphism. We will take account of the periodicity and set, for $n > 0$, $K^n = K^{-n}$. Note that the long exact sequence in the previous section reduces to a six-term exact sequence.

$$\begin{array}{ccccc} K^0(X,Y) & \xrightarrow{j^*} & K^0(X) & \xrightarrow{i^*} & K^0(Y) \\ \delta \big\uparrow & & & & \big\downarrow \delta \\ K^1(Y) & \xleftarrow{i^*} & K^1(X) & \xleftarrow{j^*} & K^1(X,Y) \end{array}$$

Example ($K^*(S^n)$): In the case of a point,

$$K^{-n}(\mathrm{pt}) = \widetilde{K}(\Sigma^n(\mathrm{pt}_+)) = \widetilde{K}(S^n).$$

Therefore, the periodicity $K^{-n}(\mathrm{pt}) = K^{-n-2}(\mathrm{pt})$ implies that the K-groups of spheres are periodic, as we claimed before. The ring structure can be described as follows. We will look at the simpler reduced version, $\widetilde{K}^*(S^n) = \widetilde{K}^0(S^n) \oplus \widetilde{K}^1(S^n)$. There are only two distinct cases: $n = 1$ or 2.

The 0-th group $\widetilde{K}^0(S^2) \cong \mathbb{Z}$ is generated additively by $[H] - [1]$. We saw in Section 3.1.1 that $H^2 \oplus 1 \cong 2H$, which implies that

$$([H] - [1])^2 = 0$$

in K-theory. On the other hand, $\widetilde{K}^1(S^2) = \widetilde{K}(\Sigma S^2) = \widetilde{K}(S^3) \cong \widetilde{K}(S^1) = 0$. Thus, as rings,

$$\widetilde{K}^*(S^2) = \mathbb{Z}t \oplus 0, \quad \text{where } t^2 = 0.$$

The ring $\widetilde{K}^*(S^1)$ is the same except that the even-odd grading is reversed.

Aside. One could give some motivation to the use of suspension in defining K^{-n}. It could be seen as directly taking advantage of Bott periodicity for the classical groups. Let $U = \varinjlim U(n)$ and $BU = \varinjlim BU(n)$. Bott periodicity for $U(n)$ (deformation retract of $GL_n(\mathbb{C})$) says that the loop space $\Omega(U)$ is homotopy-equivalent to $BU \times \mathbb{Z}$. For X compact and connected, it is not difficult to see that $[X, BU]$, the set of (free) homotopy classes of maps $X \to BU$, corresponds to $\widetilde{K}(X)$, the set of classes of stably equivalent bundles on X. We need to consider based maps, so noting that $[X_+, BU]_{\text{based}} = [X, BU]_{\text{free}}$ and $K(X) = \widetilde{K}(X) \oplus \mathbb{Z}$, we have

$$\begin{aligned}
K(X) &= [X_+, BU \times \mathbb{Z}] && \text{since } X \text{ is connected,} \\
&= [X_+, \Omega(U)] && \text{by Bott periodicity,} \\
&= [\Sigma(X_+), U] && \text{as } \Sigma \text{ and } \Omega \text{ are adjoint,} \\
&= \widetilde{K}(\Sigma^2(X_+)) && \text{by the gluing construction,} \\
&= K^{-2}(X) .
\end{aligned}$$

Thus we have periodicity in K^*, too. (Here all maps are based.)

4.3 Appearance in the Index theorem

The first major application of K-theory was in Atiyah and Singer's proof of their Index theorem. We cannot go into any detail except to describe briefly the role of K-theory in this context.

Let X be a compact smooth manifold of dimension n and E, F be smooth vector bundles over X. An elliptic differential operator $P : \Gamma(E) \to \Gamma(F)$ determines a principal symbol $\sigma(P)$, a topological object that can be regarded as an element of $K_c^0(T^*M)$. Here the cotangent bundle T^*M is of course non-compact and $K_c^0(T^*M)$ is K-group with compact support. One then defines an 'analytical index', which is basically the index of the differential operator P:

$$\operatorname{Ind} P = \dim \ker P - \dim (\operatorname{Im} P)^{\perp}.$$

This quantity is defined through the analytical properties of P, but may in fact be expressed using purely topological data of the bundles. One could see this by defining a 'topological index' Ind_T and then showing that the two indices are equal.

The number Ind_T can be expressed explicitly as the integral over X of a cohomological class and the Atiyah-Singer Index theorem is

$$\operatorname{Ind} P = (-1)^{\frac{1}{2}n(n+1)} \int_X \pi_! \operatorname{ch}(\sigma(P)) \wedge \operatorname{Td}(TM \otimes \mathbb{C}).$$

For more details, please consult [6] or any of the many books written on this beautiful result.

Acknowledgement: The author would like to thank the organisers of Modave Summer School 2006 for inviting him to speak on this subject and to enjoy the wonderful village of Modave.

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