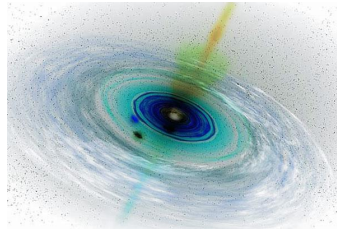


# An introduction to the mechanics of black holes

Lecture notes prepared for the Second Modave Summer  
School in Mathematical Physics



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**Abstract** These notes provide a self-contained introduction to the derivation of the zero, first and second laws of black hole mechanics. The prerequisite conservation laws in gauge and gravity theories are also briefly discussed. An explicit derivation of the first law in general relativity is performed in appendix.

**Pacs:** 04.20.-q, 04.70-s, 11.30.-j

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# Introduction

*Preliminary remark.* These notes (except the third chapter) are mainly based on previous reviews on thermodynamics of black holes [1, 2, 3, 4].

A black hole usually refers to a part of spacetime from which no future directed timelike or null line can escape to arbitrarily large distance into the outer asymptotic region. A white hole or white fountain is the time reversed concept which is believed not to be physically relevant, and will not be treated.

More precisely, if we denote by  $\mathfrak{J}^+$  the future asymptotic region of a spacetime  $(\mathcal{M}, g_{\mu\nu})$ , e.g. null infinity for asymptotically flat spacetimes and timelike infinity for asymptotically anti-de Sitter spacetimes, the black hole region  $\mathcal{B}$  is defined as

$$\mathcal{B} \equiv \mathcal{M} - I^-(\mathfrak{J}^+), \quad (1)$$

where  $I^-$  denotes the chronological past. The region  $I^-(\mathfrak{J}^+)$  is what is usually referred to as the *domain of outer communication*, it is the set of points for which it is possible to construct a future directed timelike line to arbitrary large distance in the outer region.

The *event horizon*  $\mathcal{H}$  of a black hole is then the boundary of  $\mathcal{B}$ . Let us denote  $J^-(U)$  the causal past of a set of points  $U \subset \mathcal{M}$  and  $\bar{J}^-(U)$  the topological closure of  $J^-$ . We have  $I^-(U) \subset J^-(U)$ . The (future) event horizon of  $\mathcal{M}$  can then equivalently be defined as

$$\mathcal{H} \equiv \bar{J}^-(\mathfrak{J}^+) - J^-(\mathfrak{J}^+), \quad (2)$$

i.e. the boundary of the closure of the causal past of  $\mathfrak{J}^+$ . See Fig. 1 for an example. The event horizon is a concept defined with respect to the entire causal structure of  $\mathcal{M}$ .

The event horizons are null hypersurfaces with peculiar properties. We shall develop their properties in section 1, what will allow us to sketch the proof of the area theorem [5]. The area theorem also called the second law of

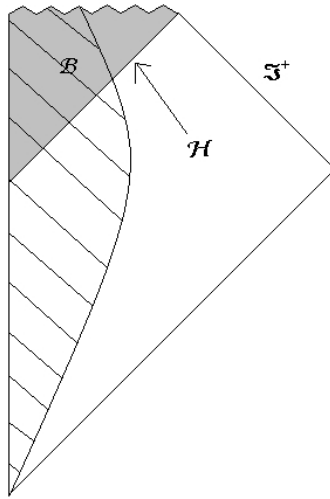


Figure 1: Penrose diagram of an asymptotically flat spacetime with spherically symmetric collapsing star. Each point is a  $n - 2$ -dimensional sphere. Light rays propagate along  $45^\circ$  diagonals. The star region is hatched and the black hole region is indicated in grey.

black hole mechanics because of its similarity with classical thermodynamics [6] is concerned with the dynamical evolution of sections of the event horizon at successive times.

In section 2, we introduce the notion of Killing horizon. This concept is adapted for black holes in equilibrium in stationary spacetimes. We will show how the zero law of mechanics, the consistency of a specific quantity defined on the horizon, directly comes out of the definitions.

The tools necessary to handle with the conservations laws in gravity theories are briefly introduced in section 3. In section 4, we derive the first law for two infinitesimally close equilibrium black holes [7].

Remark that the zero and first law of black hole mechanics may also be generalized to black holes in non-stationary spacetimes. This was done in the framework of “isolated horizons” very recently [8, 9]. However, in this introduction, we limit ourselves to the original notion of Killing horizon.

*Notation* In what follows,  $\partial_\mu f = f_{,\mu}$  is the partial derivative, while  $D_\mu f = f_{;\mu}$  denotes the covariant derivative.

# Chapter 1

## Event horizons

### 1.1 Null hypersurfaces

Let  $S(x^\mu)$  be a smooth function and consider the  $n - 1$  dimensional null hypersurface  $S(x) = 0$ , which we denote by  $\mathcal{H}$ . This surface will be the black hole horizon in the subsequent sections. It is a null hypersurface, i.e. such that its normal  $\xi^\mu \sim g^{\mu\nu} \partial_\nu S$  is null,

$$\xi^\mu \xi_\mu \stackrel{\mathcal{H}}{=} 0. \quad (1.1)$$

The vectors  $\eta^\mu$  tangent to  $\mathcal{H}$  obey  $\eta_\mu \xi^\mu|_{\mathcal{H}} = 0$  by definition. Since  $\mathcal{H}$  is null,  $\xi^\mu$  itself is a tangent vector, i.e.

$$\xi^\mu = \frac{dx^\mu(t)}{dt} \quad (1.2)$$

for some null curve  $x^\mu(t)$  inside  $\mathcal{H}$ . One can then prove that  $x^\mu(t)$  are null geodesics<sup>1</sup>

$$\xi^\nu \xi^\mu_{;\nu} \stackrel{\mathcal{H}}{=} \kappa \xi^\mu, \quad (1.4)$$

---

<sup>1</sup>Proof: Let  $\xi_\mu = \tilde{f} S_{,\mu}$ . We have

$$\begin{aligned} \xi^\nu \xi_{\mu;\nu} &= \xi^\nu \partial_\nu \tilde{f} S_{,\mu} + \tilde{f} \xi^\nu S_{,\mu;\nu} \\ &= \xi^\nu \partial_\nu \ln \tilde{f} \xi_\mu + \tilde{f} \xi^\nu S_{,\nu;\mu} \\ &= \xi^\nu \partial_\nu \ln \tilde{f} \xi_\mu + \tilde{f} \xi^\nu (\tilde{f}^{-1} \xi_\nu)_{;\mu} \\ &= \xi^\nu \partial_\nu \ln \tilde{f} \xi_\mu + \frac{1}{2} (\xi^2)_{;\mu} - \partial_\mu \ln \tilde{f} \xi^2. \end{aligned} \quad (1.3)$$

Now, as  $\xi$  is null on the horizon, any tangent vector  $\eta$  to  $\mathcal{H}$  satisfy  $(\xi^2)_{;\mu} \eta^\mu = 0$ . Therefore,  $(\xi^2)_{;\mu} \sim \xi_\mu$  and the right-hand side of (1.3) is proportional to  $\xi_\mu$  on the horizon.

where  $\kappa$  measure the extent to which the parameterization is not affine. If we denote by  $l$  the normal to  $\mathcal{H}$  which corresponds to an affine parameterization  $l^\nu l^\mu_{;\nu} = 0$  and  $\xi = f(x)l$  for some function  $f(x)$ , then  $\kappa = \xi^\mu \partial_\mu \ln |f|$ .

According to the Frobenius' theorem, a vector field  $v$  is hypersurface orthogonal if and only if it satisfies  $v_{[\mu} \partial_\nu v_{\rho]} = 0$ , see e.g. [10]. Therefore, the vector  $\xi$  satisfies the irrotationality condition

$$\xi_{[\mu} \partial_\nu \xi_{\rho]} \stackrel{\mathcal{H}}{=} 0. \quad (1.5)$$

A congruence is a family of curves such that precisely one curve of the family passes through each point. In particular, any smooth vector field define a congruence. Indeed, a vector field define at each point a direction which can be uniquely “integrated” along a curve starting from an arbitrary point.

Since  $S(x)$  is also defined outside  $\mathcal{H}$ , the normal  $\xi$  defines a congruence but which is a null congruence only when restricted to  $\mathcal{H}$ . In order to study this congruence outside  $\mathcal{H}$ , it is useful to define a transverse null vector  $n^\mu$  cutting off the congruence with

$$n^\mu n_\mu = 0, \quad n_\mu \xi^\mu = -1. \quad (1.6)$$

The normalization  $-1$  is chosen so that if we consider  $\xi$  to be tangent to an outgoing radial null geodesic, then  $n$  is tangent to an ingoing one, see Fig. (1.1). The normalization conditions (1.6) (imposed everywhere,  $(n^2)_{;\nu} = 0 = (n \cdot \xi)_{;\nu}$ ) do not fix uniquely  $n$ . Let us choose one such  $n$  arbitrarily. The extent to which the family of hypersurfaces  $S(x) = \text{const}$  are not null is given by

$$\varsigma \equiv \frac{1}{2}(\xi^2)_{;\mu} n^\mu \neq 0. \quad (1.7)$$

The vectors  $\eta$  orthogonal to both  $\xi$  and  $n$ ,

$$\eta^\mu \xi_\mu = 0 = \eta^\mu n_\mu, \quad (1.8)$$

span a  $n-2$  dimensional spacelike subspace of  $\mathcal{H}$ . The metric can be written as

$$g_{\mu\nu} = -\xi_\mu n_\nu - \xi_\nu n_\mu + \gamma_{\mu\nu} \quad (1.9)$$

where  $\gamma_{\mu\nu} = \gamma_{(\mu\nu)}$  is a positive definite metric with  $\gamma_{\mu\nu} \xi^\mu = 0 = \gamma_{\mu\nu} n^\mu$ . The tensor  $\gamma^\mu{}_\nu = g^{\mu\alpha} \gamma_{\alpha\nu}$  provides a projector onto the  $n-2$  spacelike tangent space to  $\mathcal{H}$ .

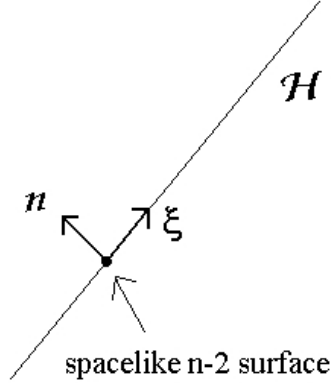


Figure 1.1: The null vector  $n$  is defined with respect to  $\xi$ .

For future convenience, we also consider the hypersurface orthogonal null congruence  $l^\mu$  with affine parameter  $\tau$  that is proportional to  $\xi^\mu$  on  $\mathcal{H}^2$ ,

$$l^\mu l_\mu = 0, \quad l^\nu l^\mu{}_{;\nu} = 0, \quad l^\mu \stackrel{\mathcal{H}}{\sim} \xi^\mu. \quad (1.10)$$

The vector field  $l$  extends  $\xi$  outside the horizon while keeping the null property.

## 1.2 The Raychaudhuri equation

In this section, we shall closely follow the reference [2]. We introduce part of the material needed to prove the area law.

Firstly, let us decompose the tensor  $D_\mu \xi_\nu$  into the tensorial products of  $\xi$ ,  $n$  and spacelike vectors  $\eta$  tangent to  $\mathcal{H}$ ,<sup>3</sup>

$$D_\mu \xi_\nu \stackrel{\mathcal{H}}{=} v_{\mu\nu} - \xi_\nu (\kappa n_\mu + \gamma_\mu^\alpha n^\beta D_\alpha \xi_\beta) - \xi_\mu n^\alpha D_\alpha \xi_\nu, \quad (1.12)$$

<sup>2</sup>We shall reserve the notation  $\xi^\mu$  for vectors coinciding with  $l^\mu$  on the horizon but which are not null outside the horizon.

<sup>3</sup>Proof: Let us first decompose  $D_\mu \xi_\nu$  as

$$D_\mu \xi_\nu = v_{\mu\nu} + n_\mu (C_1 n_\nu + C_2 \xi_\nu + C_3 \eta_\nu) + \tilde{\eta}_\mu \xi_\nu + \hat{\eta}_\mu n_\nu - \xi_\mu \alpha_\nu, \quad (1.11)$$

where  $v_{\mu\nu} = \gamma_\mu^\alpha \gamma_\nu^\beta v_{\alpha\beta}$  and  $\eta^\mu$ ,  $\tilde{\eta}^\mu$ ,  $\hat{\eta}^\mu$  are spacelike tangents to  $\mathcal{H}$ . Contracting with  $\xi^\mu$  and using (1.4), we find  $C_1 = 0 = C_3$ ,  $C_2 = -\kappa$ . Contracting with  $\gamma_\mu^\alpha n^\nu$ , we find  $\tilde{\eta}_\mu = -\gamma_\mu^\alpha n^\beta D_\alpha \xi_\beta$ . Contracting with  $\gamma_\mu^\alpha \xi^\nu$ , we find finally  $\hat{\eta}_\mu = -1/2 \gamma_\mu^\alpha D_\alpha (\xi^2) = 0$  thanks to (1.1).

where the orthogonal projection  $v_{\mu\nu} = \gamma^\alpha_\mu \gamma^\beta_\nu D_\alpha \xi_\beta$  can itself be decomposed in symmetric and antisymmetric parts

$$v_{\mu\nu} = \theta_{\mu\nu} + \omega_{\mu\nu}, \quad \theta_{[\mu\nu]} = 0, \quad \omega_{(\mu\nu)} = 0. \quad (1.13)$$

The Frobenius irrotationality condition (1.5) is equivalent to  $\omega_{\mu\nu}|_{\mathcal{H}} = 0^4$ . The tensor  $\theta_{\mu\nu}$  is interpreted as the expansion rate tensor of the congruence while its trace  $\theta = \theta^\mu_\mu$  is the divergence of the congruence. Any smooth  $n - 2$  dimensional area element evolves according to

$$\frac{d}{dt}(d\mathcal{A}) = \theta d\mathcal{A}. \quad (1.15)$$

The shear rate is the trace free part of the strain rate tensor,

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{n-2} \theta \gamma_{\mu\nu}. \quad (1.16)$$

Defining the scalar  $\sigma^2 = (n-2)\sigma_{\mu\nu}\sigma^{\mu\nu}$ , one has

$$\xi_{\mu;\nu}\xi^{\nu;\mu} \stackrel{\mathcal{H}}{=} \frac{1}{n-2}(\theta^2 + \sigma^2) + \kappa^2 + \varsigma^2, \quad (1.17)$$

where  $\varsigma$  was defined in (1.7). Note also that the divergence of the vector field has three contributions,

$$\xi^\mu_{;\mu} \stackrel{\mathcal{H}}{=} \theta + \kappa - \varsigma. \quad (1.18)$$

Now, the contraction of the Ricci identity

$$v^\alpha_{;\mu;\nu} - v^\alpha_{;\nu;\mu} = -R^\alpha_{\lambda\mu\nu} v^\lambda, \quad (1.19)$$

implies the following identity

$$(v^\nu_{;\nu})_{;\mu} v^\mu = (v^\nu v^\mu_{;\nu})_{;\mu} - v^{\nu;\mu} v_{\mu;\nu} - R_{\mu\nu} v^\mu v^\nu, \quad (1.20)$$

valid for any vector field  $v$ . The formulae (1.17)-(1.18) have their equivalent for  $l$  as

$$l_{\mu;\nu} l^{\nu;\mu} = \frac{1}{n-2}(\theta_{(0)}^2 + \sigma_{(0)}^2), \quad l^\mu_{;\mu} = \theta_{(0)}, \quad (1.21)$$

---

<sup>4</sup>Proof: We have

$$\xi_{[\mu}\partial_\nu\xi_{\rho]} = \xi_{[\mu}D_\nu\xi_{\rho]} = \xi_{[\mu}v_{\nu\rho]} = \xi_{[\mu}\omega_{\nu\rho]}. \quad (1.14)$$

where the right hand side are expressed in terms of the expansion rate  $\theta_{(0)} = \theta \frac{dt}{d\tau}$  and shear rate  $\sigma_{(0)} = \sigma \frac{dt}{d\tau}$  with respect to the affine parameter  $\tau$ . The identity (1.20) becomes

$$\frac{d\theta_{(0)}}{d\tau} = \dot{\theta}_{(0)} \stackrel{\mathcal{H}}{=} -\frac{1}{n-2}(\theta_{(0)}^2 + \sigma_{(0)}^2) - R_{\mu\nu}l^\mu l^\nu, \quad (1.22)$$

where the dot indicate a derivation along the generator. It is the final form of the Raychaudhuri equation for hypersurface orthogonal null geodesic congruences in  $n$  dimensions.

### 1.3 Properties of event horizons

As we have already mentioned, the main characteristic of event horizons is that they are null hypersurfaces. In the early seventies, Penrose and Hawking further investigated the generic properties of past boundaries, as event horizons. We shall only enumerate these properties below and refer the reader to the references [11, 3] for explicit proofs. These properties are crucial in order to prove the area theorem.

1. *Achronicity property.* No two points of the horizon can be connected by a timelike curve.
2. The null geodesic generators of  $\mathcal{H}$  may have past end-points in the sense that the continuation of the geodesic further into the past is no longer in  $\mathcal{H}$ .
3. The generators of  $\mathcal{H}$  have no future end-points, i.e. no generator may leave the horizon.

The second property hold for example for collapsing stars where the past continuation of all generators leave the horizon at the time the horizon was formed. As a consequence of properties 2 and 3, null geodesics may enter  $\mathcal{H}$  but not leave it.

### 1.4 The area theorem

The area theorem was initially demonstrated by Hawking [5]. We shall follow closely the reviews by Carter [2] and Townsend [3]. The theorem reads as follows.

**Theorem 1** *Area law.* If

- (i) Einstein's equations hold with a matter stress-tensor satisfying the null energy condition,  $T_{\mu\nu}k^\mu k^\nu \geq 0$ , for all null  $k^\mu$ ,
- (ii) The spacetime is “strongly asymptotically predictable”

then the surface area  $\mathcal{A}$  of the event horizon can never decrease with time.

The theorem was stated originally in 4 dimensions but it is actually valid in any dimension  $n \geq 3$ .

In order to understand the second requirement, let us remind some definitions. The future domain of dependence  $D^+(\Sigma)$  of an hypersurface  $\Sigma$  is the set of points  $p$  in the manifold for which every causal curve through  $p$  that has no past end-point intersects  $\Sigma$ . The significance of  $D^+(\Sigma)$  is that the behavior of solutions of hyperbolic PDE's *outside*  $D^+(\Sigma)$  is not determined by initial data on  $\Sigma$ . If no causal curves have past end-points, then the behavior of solutions inside  $D^+(\Sigma)$  is entirely determined in terms of data on  $\Sigma$ . The past domain of dependence  $D^-(\Sigma)$  is defined similarly.

A Cauchy surface is a spacelike hypersurface which every non-spacelike curve intersects exactly once. It has as domain of dependence  $D^+(\Sigma) \cup D^-(\Sigma)$  the manifold itself. If an open set  $\mathcal{N}$  admits a Cauchy surface then the Cauchy problem for any PDE with initial data on  $\mathcal{N}$  is well-defined. This is also equivalent to say that  $\mathcal{N}$  is globally hyperbolic.

The requirement (ii) means that it should exist a globally hyperbolic submanifold of spacetime containing both the exterior spacetime *and* the horizon. It is equivalent to say there exists a family of Cauchy hypersurfaces  $\Sigma(\tau)$ , such that  $\Sigma(\tau')$  is inside the domain of dependence of  $\Sigma(\tau)$  if  $\tau' > \tau$ .

Now, the boundary of the black hole is the past event horizon  $\mathcal{H}$ . It is a null hypersurface with generator  $l^\mu$  (that is proportional to  $\xi$  on  $\mathcal{H}$ ). We can choose to parameterize the Cauchy surfaces  $\Sigma(\tau)$  using the affine parameter  $\tau$  of the null geodesic generator  $l$ .

The *area of the horizon*  $\mathcal{A}(\tau)$  is then the area of the intersection of  $\Sigma(\tau)$  with  $\mathcal{H}$ . We have to prove that  $\mathcal{A}(\tau') > \mathcal{A}(\tau)$  if  $\tau' > \tau$ .

**Sketch of the proof:**

The Raychaudhuri equation for the null generator  $l$  reads as (1.22). Therefore, wherever the energy condition  $R_{\mu\nu}l^\mu l^\nu \geq 0$  hold, the null generator will evolve subject to the inequality

$$\frac{d\theta_{(0)}}{d\tau} \leq -\frac{1}{n-2}\theta_{(0)}^2, \tag{1.23}$$

except on possible singular points as caustics. It follows that if  $\theta_{(0)}$  becomes negative at any point  $p$  on the horizon (i.e. if there is a convergence) then the null generator can continue in the horizon for at most a finite affine distance before reaching a point  $p$  at which  $\theta_{(0)} \rightarrow -\infty$ , i.e. a point of infinite convergence representing a caustic beyond which the generators intersect.

Now, from the third property of event horizons above, the generators cannot leave the horizon. Therefore at least two generators cross at  $p$  inside  $\mathcal{H}$  and, following Hawking and Ellis (Prop 4.5.12 of [11]), they may be deformed to a timelike curve, see figure 1.2. This is however impossible because of the achronicity property of event horizons. Therefore, in order to avoid the contradiction, the point  $p$  cannot exist and  $\theta_{(0)}$  cannot be negative.

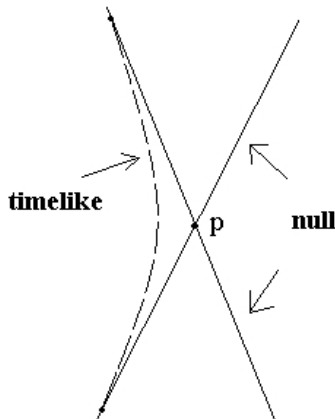


Figure 1.2: If two null generators of  $\mathcal{H}$  cross, they may be deformed to a timelike curve.

Since (at points where the horizon is not smooth) new null generators may begin but old ones cannot end, equation (1.15) implies that the total area  $\mathcal{A}(\tau)$  cannot decrease with increasing  $\tau$ ,

$$\frac{d}{d\tau}\mathcal{A} \geq \oint \theta_{(0)} d\mathcal{A} \geq 0. \quad (1.24)$$

This completes the proof.

In particular, if two black holes with area  $\mathcal{A}_1$  and  $\mathcal{A}_2$  merge then the area  $\mathcal{A}_3$  of the combined black hole have to satisfy

$$\mathcal{A}_3 > \mathcal{A}_1 + \mathcal{A}_2. \quad (1.25)$$

The area  $\mathcal{A}(\tau)$  do not change if  $\theta = 0$  on the entire horizon  $\mathcal{H}$ . The black hole is then stationary.

Note that this derivation implicitly assume regularity properties of the horizon (as piecewise  $C^2$ ) which may not be true for generic black holes. Recently these gaps in the derivation have been totally filled in [12, 4].

## Chapter 2

# Equilibrium states

### 2.1 Killing horizons

In any stationary and asymptotically flat spacetime with a black hole, the event horizon is a Killing horizon [11]. This theorem firstly proven by Hawking is called the rigidity theorem. It provides an essential link between event horizons and Killing horizons.<sup>1</sup>

A Killing horizon is a null hypersurface whose normal  $\xi$  is a Killing vector

$$\mathcal{L}_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \quad (2.1)$$

This additional property will allow us to explore many characteristics of black holes.

The parameter  $\kappa$  which we call now the surface gravity of  $\mathcal{H}$  is defined in (1.4). In asymptotically flat spacetimes, the normalization of  $\kappa$  is fixed by requiring  $\xi^2 \rightarrow -1$  at infinity (similarly, we impose  $\xi^2 \rightarrow -\frac{r^2}{l^2}$  in asymptotically anti-de Sitter spacetimes).

For Killing horizons, the expansion rate  $\theta_{\mu\nu} = \gamma_{(\mu}^{\alpha} \gamma_{\nu)}^{\beta} D_{\alpha} \xi_{\beta} = 0$ , so  $\theta = \sigma = 0$ . Moreover, from (1.18) and (2.1), we deduce  $\varsigma = \kappa$ . Equation (1.17) then provides an alternative definition for the surface gravity,

$$\kappa^2 = -\frac{1}{2} \xi_{\mu;\nu} \xi^{\mu;\nu} |_{\mathcal{H}}. \quad (2.2)$$

---

<sup>1</sup>The theorem further assumes the geometry is analytic around the horizon. Actually, there exist a counter-example to the rigidity theorem as stated in Hawking and Ellis [11] but under additional assumptions such as global hyperbolicity and simple connectedness of the spacetime, the result is totally valid [13].

Contracting (1.4) with the transverse null vector  $n$ , one has also

$$\kappa = \xi_{\mu;\nu} \xi^\mu n^\nu|_{\mathcal{H}} = \frac{1}{2} (\xi^2)_{;\mu} n^\mu|_{\mathcal{H}}. \quad (2.3)$$

The Raychaudhuri equation (1.22) also states in this case that

$$R_{\mu\nu} \xi^\mu \xi^\nu \stackrel{\mathcal{H}}{=} 0, \quad (2.4)$$

because  $l$  is proportional to  $\xi$  on the horizon.

From the decomposition (1.12), the irrotationality condition (1.5) and the Killing property  $\xi_{[\mu;\nu]} = \xi_{\mu;\nu}$ , one can write

$$\xi_{\mu;\nu} \stackrel{\mathcal{H}}{=} \xi_\mu q_\nu - \xi_\nu q_\mu, \quad (2.5)$$

where the covector  $q_\mu$  can be fixed uniquely by the normalization  $q_\mu n^\mu = 0$ . Using (2.3), one can further decompose the last equation in terms of  $(n, \xi, \{\eta\})$  as

$$\xi_{\mu;\nu} \stackrel{\mathcal{H}}{=} -\kappa (\xi_\mu n_\nu - \xi_\nu n_\mu) + \xi_\mu \hat{\eta}_\nu - \hat{\eta}_\mu \xi_\nu, \quad (2.6)$$

where  $\hat{\eta}$  satisfy  $\hat{\eta} \cdot \xi = 0 = \hat{\eta} \cdot n$ . In particular, it shows that for any spacelike tangent vectors  $\eta, \tilde{\eta}$  to  $\mathcal{H}$ , one has  $\xi_{\mu;\nu} \eta^\mu \tilde{\eta}^\nu \stackrel{\mathcal{H}}{=} 0$ .

## 2.2 Zero law

We are now in position to prove that the surface gravity  $\kappa$  is constant on the horizon under generic conditions. More precisely,

**Theorem 2** *Zero law.* [7] If

- (i) The spacetime  $(M, g)$  admits a Killing vector  $\xi$  which is the generator of a Killing horizon  $\mathcal{H}$ ,
- (ii) Einstein's equations hold with matter satisfying the dominant energy condition, i.e.  $T_{\mu\nu} l^\nu$  is a non-spacelike vector for all  $l^\mu l_\mu \leq 0$ ,

then the surface gravity  $\kappa$  of the Killing horizon is constant over  $\mathcal{H}$ .

Using the aforementioned properties of null hypersurfaces and Killing horizons, together with

$$\xi_{\nu;\mu;\rho} = R_{\mu\nu\rho}{}^\tau \xi_\tau, \quad (2.7)$$

which is valid for Killing vectors, one can obtain (see [2] for a proof)

$$\dot{\kappa} = \kappa_{,\mu}\xi^\mu \stackrel{\mathcal{H}}{=} 0, \quad (2.8)$$

$$\kappa_{,\mu}\eta^\mu \stackrel{\mathcal{H}}{=} -R_{\mu\nu}\xi^\mu\eta^\nu, \quad (2.9)$$

for all spacelike tangent vectors  $\eta$ . Now, from the dominant energy condition,  $R_{\mu\nu}\xi^\mu$  is not spacelike. However, the Raychaudhuri equation implies (2.4). Therefore,  $R_{\mu\nu}\xi^\mu$  must be zero or proportional to  $\xi_\nu$  and  $R_{\mu\nu}\xi^\mu\eta^\nu = 0$ .

This theorem has an extension when gravity is coupled to electromagnetism. If the Killing vector field  $\xi$  is also a symmetry of the electromagnetic field up to a gauge transformation,  $\mathcal{L}_\xi A_\mu + \partial_\mu\epsilon = 0$ , one can also prove that the electric potential

$$\Phi = -(A_\mu\xi^\mu + \epsilon)|_{\mathcal{H}} \quad (2.10)$$

is constant on the horizon.

## Chapter 3

# Conservation laws

*“Anybody who looks for a magic formula for “local gravitational energy-momentum” is looking for the right answer to the wrong question. Unhappily, enormous time and effort were devoted in the past to trying to “answer this question” before investigators realized the futility of the enterprise”*

Misner, Thorne and Wheeler [14]

According to Misner, Thorne and Wheeler, the Principle of Equivalence forbids the existence of a localized energy-momentum stress-tensor for gravity. No experiment can be designed to measure a notion of local energy of the gravitational field because in a locally inertial frame the effect of gravity is locally suppressed. However, it is meaningful to ask what is energy content of *region* or of the *totality* of a spacetime. For the first issue, we refer the reader to the literature on recent quasi-local methods [15, 16], see also [17] for the link with pseudo-tensors. Here, we shall deal with the second issue, the best studied and oldest topic, namely, the definition of global conservation laws for gravity (and for general gauge theories).

It exists an overabundant literature over conservations laws. Some methods are prominent but none impose itself as the best one, each of them having overlapping advantages, drawbacks and scope of application. The following presentation will therefore reflect only a biased and narrow view on the topic.

### 3.1 The generalized Noether theorem

Let us begin the discussion by recalling the first Noether theorem.

**Theorem 3** *First Noether Theorem* Any equivalence class of continuous global symmetries of a lagrangian  $L d^n x$  is in one-to-one correspondence with an equivalence class of conserved currents  $J^\mu$ ,  $\partial_\mu J^\mu = 0$ .

Here, two global symmetries are equivalent if they differ by a gauge transformation and by a symmetry generated by a parameter vanishing on-shell. Two currents  $J^\mu$ ,  $J'^\mu$  are equivalent<sup>1</sup> if they differ by a trivial current,

$$J^\mu \sim J'^\mu + \partial_\nu k^{[\mu\nu]} + t^\mu \left( \frac{\delta L}{\delta \phi} \right), \quad t^\mu \approx 0, \quad (3.1)$$

where  $t^\mu$  depends on the equations of motion (i.e. vanishes on-shell). This theorem is essential in order to define the energy in classical mechanics or in field theories. For example, the total energy of the field associated with a time translation  $(\partial_t)^\mu$  on a spacelike Cauchy surface  $\Sigma$  is defined as  $E = \int_\Sigma J_\mu n^\mu$  where  $n^\mu$  is the unit normal to  $\Sigma$  and  $J_\mu = T_{\mu\nu} (\partial_t)^\nu$  where  $T_{\mu\nu}$  is the conserved stress-tensor of the field ( $\partial_\mu T^{\mu\nu} = 0$ ).

In diffeomorphic invariant theories, the infinitesimal coordinate transformations generated by a vector  $\xi$  are pure gauge transformations. The first Noether theorem implies that *all currents  $J_\xi$  associated to infinitesimal diffeomorphisms are trivial*<sup>2</sup>.

The main lesson of Theorem 3 is that for gauge theories, one should *not* look at conserved currents. In order to generalize the Noether theorem, it is convenient to introduce the notation for  $n - p$  forms,

$$(d^{n-p}x)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!(n-p)!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_{n-p+1}} \wedge \dots \wedge dx^{\mu_n}. \quad (3.2)$$

The Noether current can be reexpressed as a  $n - 1$  form as  $J = J^\mu (d^{n-1}x)_\mu$  which is closed, i.e.  $dJ = 0$ .

Now, we shall see that the conservation laws for gauge theories are *lower degree conservation laws*, involving conserved  $n-2$  forms,  $k = k^{[\mu\nu]} (d^{n-2}x)_{\mu\nu}$ , i.e. such that  $dk = 0$  or  $\partial_\nu k^{[\mu\nu]} = 0$ . Indeed, in the nineties, the following theorem was proved [18, 19], see also [20, 21] for related work and [22] for introductions to local cohomology,

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<sup>1</sup>In  $n = 1$  dimension, two currents differing by a constant  $J \in \mathbb{R}$  are also considered as equivalent.

<sup>2</sup>For example, in general relativity, one has  $\delta L = \delta g_{\mu\nu} \frac{\delta L}{\delta g_{\mu\nu}} + \partial_\mu \Theta^\mu(g, \delta g)$  for some  $\Theta(g, \delta g)$ . For a diffeomorphism, one has  $\delta g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu$ ,  $\mathcal{L}_\xi L = \partial_\mu (\xi^\mu L)$  and  $\delta g_{\mu\nu} \frac{\delta L}{\delta g_{\mu\nu}} = -2\sqrt{-g} D_\mu \xi^\mu = -2\partial_\mu (\sqrt{-g} G^{\mu\nu} \xi_\nu)$  where  $G_{\mu\nu}$  is the Einstein tensor. What is usually called the Noether current is  $J^\mu = \Theta^\mu(g, \mathcal{L}_\xi g) - \xi^\mu L$ . However, it is trivial because (as implied by Noether theorem) it exists a  $k^{\mu\nu} = k^{[\mu\nu]}$  such that  $J^\mu = -2\sqrt{-g} G^{\mu\nu} \xi_\nu + \partial_\nu k^{[\mu\nu]}$ .

**Theorem 4** *Generalized Noether Theorem* Any parameter of a gauge transformation vanishing on-shell such that the parameter itself is non zero on-shell is in one-to-one correspondence with  $n - 2$ -forms  $k$  that are conserved on-shell,  $dk \approx 0$  (up to trivial  $n - 2$ -forms<sup>3</sup> and up to the addition of the divergence of a  $n - 3$ -form).

Essentially, the theorem amounts first to identify the class of gauge transformations vanishing on-shell with non-trivial gauge parameters with the cohomology group  $H_2^n(\delta|d)$  and the class of non-trivial conserved  $n - 2$  forms with  $H_0^{n-2}(d|\delta)$ . The proof of the theorem then reduces to find an isomorphism between these cohomology classes.

As a first example, in electromagnetism (that may be coupled to gravity), the trivial gauge transformations  $\delta A_\mu = \partial_\mu c = 0$  are generated by constants  $c \in \mathbb{R}$ . The associated  $n - 2$ -form is simply

$$k_c[A, g] = c (d^{n-2}x)_{\mu\nu} \sqrt{-g} F^{\mu\nu} \quad (3.3)$$

which is indeed closed (outside sources that are not considered here) when the equations of motion hold. The closeness of  $k$  indicate that the electric charge

$$\mathcal{Q}_E = \oint_S k_{c=1}, \quad (3.4)$$

i.e. the integral of  $k$  over a closed surface  $S$  at constant time, does not depend on time and may be freely deformed in vacuum regions<sup>4</sup>.

In generally covariant theories, the trivial diffeomorphisms  $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$  are generated by the Killing vectors<sup>5</sup>

$$\mathcal{L}_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \quad (3.5)$$

However, here comes the problem: there is *no* solution to the Killing equation for *arbitrary* fields. Therefore, *none* vector  $\xi$  can be associated to a *generically* conserved  $n - 2$  form. The hope to associate a conserved quantity

$$\mathcal{Q}_\xi[g] = \oint_S K_\xi[g], \quad dK_\xi \approx 0, \quad K_\xi \approx 0, \quad (3.6)$$

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<sup>3</sup>Trivial  $n - 2$ -forms include superpotentials that vanish on-shell and topological superpotentials that are closed off-shell, see [22] for a discussion of topological conservation laws.

<sup>4</sup>Explicitly, let  $S$  be some surface  $r = \text{const}$ ,  $t = \text{const}$ . The  $r$  component of  $dk = 0$  is  $\partial_t k^{tr} + \partial_A k^{Ar} = 0$  where  $A = \theta, \phi$  are the angular coordinates. The time derivative of  $\mathcal{Q}$  is then given by  $\oint_S \partial_t k^{tr} (d^{n-2}x)_{tr} = - \oint_S \partial_A k^{Ar} dA = 0$  by Stokes theorem. Similarly, the  $t$  component of  $dk = 0$  is  $\partial_r k^{rt} + \partial_A k^{At} = 0$  and  $\partial_r \mathcal{Q} = 0$  too.

<sup>5</sup>Trivial diffeomorphisms must also satisfy  $\mathcal{L}_\xi \phi^i = 0$  if other fields  $\phi^i$  are present.

for a given vector  $\xi$  to all solutions  $g$  of general relativity is definitively annihilated.

What is the way out? In fact, there are many ways out with different methods and results. In these notes, we will weaken the requirements enormously by selecting special surfaces  $S$ , vectors  $\xi$  and classes of metrics  $g$  and by allowing the conserved quantities

$$\mathcal{Q}_\xi[g, \bar{g}] = \oint_S K_\xi[g, \bar{g}], \quad dK_\xi \approx 0, \quad K_\xi \approx 0, \quad (3.7)$$

to depend on some background solution  $\bar{g}$ . If we normalize the charge of the background (typically Minkowski or anti-de Sitter spacetime) to zero,  $\mathcal{Q}_\xi[g, \bar{g}]$  will provide a well-defined quantity associated to  $g$  and  $\xi$ . Let us now explain how the construction works for some particular cases in Einstein gravity.

### 3.2 The energy in general relativity

Let us only derive the conservation laws obtained originally by Arnowitt, Deser and Misner [23, 24] and Abbott and Deser [25]. For simplicity, only Einstein's gravity is discussed but other gauge theories admits similar structures.

Let us first linearize the Einstein-Hilbert lagrangian  $L^{EH}[g]$  with  $g = \bar{g} + h$  around a solution  $\bar{g}$ . It can be shown that the linearized lagrangian  $L^{free}[h]$  is gauge invariant under

$$\delta h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu}, \quad (3.8)$$

where  $\xi$  is an arbitrary vector. Now, if the background  $\bar{g}_{\mu\nu}$  admits exact Killing vectors, the generalized Noether theorem says that it exists  $n - 2$  forms  $k_\xi[h, \bar{g}]$  which are conserved when  $h$  satisfies the linearized equations of motion. In fact, the Killing vectors enumerate all non-trivial solutions to  $\mathcal{L}_\xi \bar{g}_{\mu\nu} \approx 0$  in that case and the  $n - 2$  forms  $k_\xi[h, \bar{g}]$  are thus the only non-trivial conserved forms [26].

For Einstein gravity, the  $n - 2$  form associated to a Killing vector  $\xi$  of  $\bar{g}$  is well-known to be [27, 28, 29, 30]

$$k_\xi[h, \bar{g}] = -\delta_h K_\xi[g] - i_\xi \Theta[h, \bar{g}] \quad (3.9)$$

where  $i_\xi = \xi^\mu \frac{\partial}{\partial x^\mu}$  is the inner product and the Komar term and the  $\Theta$  term

are given by

$$K_\xi[g] = \frac{\sqrt{-g}}{16\pi G} (D^\mu \xi^\nu - D^\nu \xi^\mu) (d^{n-2}x)_{\mu\nu}, \quad (3.10)$$

$$\Theta[h, g] = \frac{\sqrt{-g}}{16\pi G} (g^{\mu\alpha} D^\beta h_{\alpha\beta} - g^{\alpha\beta} D^\mu h_{\alpha\beta}) (d^{n-1}x)_\mu. \quad (3.11)$$

Here, the variation  $\delta_h$  acts on  $g$  and the result of the variation is evaluated on  $\bar{g}$ .

Now, the point is that this result may be lifted to the full interacting theory (at least) in two different ways :

1. For suitable classes of spacetimes  $(\mathcal{M}, g)$  with *boundary conditions* in an asymptotic region such that *the linearized theory applies* around some symmetric background  $\bar{g}$  in the asymptotic region.
2. For classes of solutions  $(\mathcal{M}, g)$  with a set of exact Killing vectors.

To illustrate the first case, let take as an example asymptotically flat spacetimes at spatial infinity  $r \rightarrow \infty$ . This class of spacetimes is constrained by the condition  $g_{\mu\nu} - \eta_{\mu\nu} = O(1/r)$ , where  $\eta$  is the Minkowski metric. We then consider the linearized field  $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$  around the background Minkowski metric. Under some appropriate additional boundary conditions, it can be shown that the linearized theory applies at infinity: since Minkowski spacetime admits Killing vectors  $\bar{\xi}$ ,  $k_{\bar{\xi}}[h, \bar{g}]^6$  are  $n - 2$ -form conserved in the asymptotic region, i.e. *asymptotically conserved*<sup>7</sup>. The translations, rotations and boosts of Minkowski spacetime are thus associated to energy-momentum and angular momentum. These are the familiar ADM expressions. The conserved quantities in anti-de Sitter spacetime can also be constructed that way.

In the second case, one applies the linearized theory around a family of solution  $g_{\mu\nu}$  which have  $\xi$  as an exact Killing vector. This allows one to compute the charge difference between  $g_{\mu\nu}$  and an infinitely close metric  $g_{\mu\nu} + \delta g_{\mu\nu}$ . As  $dk_\xi[\delta g, g] = 0$  in the whole spacetime, the charge difference  $\oint_S k_\xi[\delta g, g]$  does not depend one the choice of integration surface,

$$\oint_S k_\xi[\delta g, g] = \oint_{S'} k_\xi[\delta g, g], \quad (3.12)$$

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<sup>6</sup>Note for completeness that the theory of asymptotically conserved  $n - 2$  forms also allows for an extended notion of symmetry, asymptotic symmetries, where  $\mathcal{L}_\xi \bar{g}$  tends to zero only asymptotically.

<sup>7</sup>In fact, boundary conditions are chosen such that the charges are finite, conserved and form a representation of the Poincaré algebra.

where  $S, S'$  are any  $n - 2$  surfaces, usually chosen to be  $t = const, r = const$  in spherical coordinates. The total charge associated to  $\xi$  of a solution can then be defined by

$$Q_\xi[g, \bar{g}] = \oint_S \int_{\bar{g}}^g k_\xi[\delta g'; g'], \quad (3.13)$$

where  $\bar{g}_{\mu\nu}$  is a background solution with charge normalized to zero and  $g'$  is the integration variable. The outer integral is performed along a path of solutions. This definition is only meaningful if the charge does not depend on the path, which amount to what is called the *integrability condition*

$$\oint_S (\delta_1 k_\xi[\delta_2 g; g] - \delta_2 k_\xi[\delta_1 g; g]) = 0. \quad (3.14)$$

As a conclusion, we have sketched how one obtains the promised definition of charge (3.7) in the two aforementioned cases. In the first (asymptotic) case,  $K_\xi[g, \bar{g}] = k_\xi[h, \bar{g}]$  where  $h = g - \bar{g}$  is the linearized field at infinity. In the second (exact) case,  $K_\xi[g, \bar{g}] = \int_{\bar{g}}^g k_\xi[\delta g'; g']$ , where one integrates the linearized form  $k_\xi$  along a path of solutions.

Finally note that all results presented here in covariant language have their analogue in Hamiltonian form [31, 29, 32].

## Chapter 4

# Quasi-equilibrium states

In 3+1 dimensions, stationary axisymmetric black holes are entirely characterized by their mass and their angular momentum. This fact is part of the *uniqueness theorems*, see [33] for a review. In  $n$  dimensions, the situation is more complicated. First, the black hole may rotate in different perpendicular planes. In 3+1 dimensions, the rotation group  $SO(3)$  has only one Casimir invariant, but in  $n$  dimensions, it has  $D \equiv \lfloor (n-1)/2 \rfloor$  Casimirs. Therefore, one expects that, in general, a black hole will have  $D$  conserved angular momenta. This is what happens in the higher dimensional Kerr and Reissner-Nordström black holes [34]. Remark that the generalization of rotating black holes to anti-de Sitter backgrounds was done only very recently [35]. So far so good; this is not a big deal with respect to uniqueness.

The worrying (but interesting) point is that higher dimensions allow for more exotic horizon topologies than the sphere. For example, *black ring* solutions were found [36] recently in 5 dimensions with horizon topology  $S^1 \times S^2$ . The initial idea of the uniqueness theorems that stationary axisymmetric black holes are entirely characterized by a few number of charges at infinity is thus not valid in higher dimensions. In what follows, we shall derive the first law of black hole mechanics without using uniqueness results.

From now, we restrict ourselves to stationary and axisymmetric black holes with Killing horizon, having  $\partial_t$  and  $\partial_{\varphi^a}$ ,  $1 \leq a \leq D$  as Killing vectors. We allow for arbitrary horizon topology, only assuming the horizon is connected. The Killing generator of the horizon is then a combination of the Killing vectors,

$$\xi = \frac{\partial}{\partial t} + \Omega^a \frac{\partial}{\partial \varphi^a}, \quad (4.1)$$

where  $\Omega^a$  are called the angular velocities at the horizon.

## 4.1 The first law for Einstein gravity

We are now set up to present the terms of the first law:

**Theorem 5** *First law.* Let  $(\mathcal{M}, g)$  and  $(\mathcal{M} + \delta\mathcal{M}, g + \delta g)$  be two slightly different stationary black hole solutions of Einstein's equations with Killing horizon. The difference of energy  $\mathcal{E}$ , angular momenta  $\mathcal{J}_a$  and area  $\mathcal{A}$  of the black hole are related by

$$\delta\mathcal{E} = \Omega^a \delta\mathcal{J}_a + \frac{\kappa}{8\pi} \delta\mathcal{A}, \quad (4.2)$$

where  $\Omega^a$  are the angular velocities at the horizon and  $\kappa$  is the surface gravity.

This *equilibrium state version* of the first law of black hole mechanics is essentially a balance sheet of energy between two stationary black holes<sup>1</sup>. We shall prove that it comes directly from the equality of the charge related to  $\xi$  at the horizon spacelike section  $H$  and at infinity, as (3.12),

$$\oint_{S^\infty} k_\xi[\delta g_{\mu\nu}; g_{\mu\nu}] = \oint_H k_\xi[\delta g_{\mu\nu}; g_{\mu\nu}]. \quad (4.3)$$

The energy and angular momenta of the black hole are defined as<sup>2</sup>

$$\delta\mathcal{E} = \oint_{S^\infty} k_{\partial_t}[\delta g_{\mu\nu}; g_{\mu\nu}], \quad \delta\mathcal{J}^a = - \oint_{S^\infty} k_{\partial_{\varphi^a}}[\delta g_{\mu\nu}; g_{\mu\nu}] \quad (4.4)$$

Therefore, the left-hand side of (4.3) is by definition given by

$$\oint_{S^\infty} k_\xi[\delta g_{\mu\nu}; g_{\mu\nu}] = \delta\mathcal{E} - \Omega\delta\mathcal{J} \quad (4.5)$$

Using (3.9), we may rewrite the right-hand side of (4.3) as

$$\oint_H k_\xi[\delta g_{\mu\nu}; g_{\mu\nu}] = -\delta \oint_H K_\xi[g] + \oint_H K_{\delta\xi}[g] - \oint_H \xi \cdot \Theta[\delta g; g], \quad (4.6)$$

where the variation of  $\xi$  that cancels between the two first terms on the right-hand side is put for later convenience.

<sup>1</sup>It also exists a physical process version, where an infinitesimal amount of matter is send through the horizon from infinity.

<sup>2</sup>The relative sign difference between the definitions of  $\mathcal{E}$  and  $\mathcal{J}^a$  trace its origin to the Lorentz signature of the metric [28].

On the horizon, the integration measure for  $n - 2$ -forms is given by

$$\sqrt{-g}(d^{n-2}x)_{\mu\nu} = \frac{1}{2}(\xi_\mu n_\nu - n_\mu \xi_\nu)d\mathcal{A}, \quad (4.7)$$

where  $d\mathcal{A}$  is the angular measure on  $H$ . Using the properties of Killing horizons, the Komar integral on the horizon becomes

$$\oint_H K_\xi[g] = -\frac{\kappa\mathcal{A}}{8\pi G}, \quad (4.8)$$

where  $\mathcal{A}$  is the area of the horizon. Now, it turns out that

$$\oint_H K_{\delta\xi}[g] - \oint_H \xi \cdot \Theta[\delta g; g] = -\delta\kappa \frac{\mathcal{A}}{8\pi G}. \quad (4.9)$$

The computation which is straightforward but lengthy is done explicitly in Appendix A<sup>3</sup> *without* assuming specific invariance properties under the variation as done in the original derivation [7] and subsequent derivations thereof [1, 28].

The right-hand side of (4.3) is finally given by

$$\oint_H k_\xi[\delta g_{\mu\nu}; g_{\mu\nu}] = \frac{\kappa}{8\pi G}\delta\mathcal{A}, \quad (4.10)$$

as it should and the first law is proven. We can see in this derivation that the first law is a *geometrical* law in the sense that it relates the geometry of Killing horizons to the geometric measure of energy and angular momenta.

Remark finally that the derivation was done in arbitrary dimensions, without hypotheses on the topology of the horizon and for arbitrary stationary variations. The first law also applies in particular for extremal black holes by taking  $\kappa = 0$ .

## 4.2 Extension to electromagnetism

It is straightforward to extend the present considerations to the coupled Einstein-Maxwell system. The original derivation was given in [7], see also [3, 37] for alternative derivations. We assume that no magnetic monopole is present, so that the potential  $A$  is regular everywhere.

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<sup>3</sup>I thanks G. Barnich for his suggestion of this computation.

According to the generalized Noether theorem, we first need to extend the notion of symmetry. The gauge transformations of the fields  $(g_{\mu\nu}, A_\mu)$  that are zero when the equations of motion are satisfied are given by

$$\mathcal{L}_\xi g_{\mu\nu} = 0, \quad (4.11)$$

$$\mathcal{L}_\xi A_\mu + \partial_\mu \epsilon = 0. \quad (4.12)$$

These equations are the generalized Killing equations. We consider only stationary and axisymmetric black holes with Killing horizon which have as a solution to these equations  $(\xi, \epsilon) = (\partial_t, 0)$ ,  $(\xi, \epsilon) = (\partial_{\varphi^a}, 0)$  for  $a = 1 \dots D$  and  $(\xi, \epsilon) = (0, \epsilon \in \mathbb{R})$ . The conserved quantities associated to these symmetry parameters are respectively the energy, the angular momenta and the electric charge. The existence of the electric charge independently from the other charges suggests that it will show up in the first law.

The conserved superpotential associated to a symmetry parameter  $(\xi, \epsilon)$  can be shown to be

$$k_{\xi, \epsilon}^{tot}[\delta g, \delta A; g, A] = k_\xi^{grav} - \delta K_{\xi, \epsilon}^{em} + K_{\delta\xi, \delta\epsilon}^{em} - \xi \cdot \Theta^{em} \quad (4.13)$$

where  $k_\xi^{grav}[\delta g, g]$  is the gravitational contribution (3.9)<sup>4</sup>,

$$K_{\xi, \epsilon}^{em}[g, A] = \frac{\sqrt{-g}}{16\pi G} [F^{\mu\nu}(\xi^\alpha A_\alpha + \epsilon)](d^{n-2}x)_{\mu\nu} \quad (4.14)$$

and

$$\Theta^{em}[g, A; \delta A] = \frac{\sqrt{-g}}{16\pi G} F^{\alpha\mu} \delta A_\alpha (d^{n-1}x)_\mu. \quad (4.15)$$

Let us now look at the fundamental equality (4.3) of the first law where we choose  $\xi$  as the Killing generator and  $\epsilon = 0$ . The superpotential  $k = k^{tot}$  should contain the electromagnetic contributions as well.

On the one hand, the energy and angular momenta are still defined by (4.4) where  $k$  is given by (4.13). For usual potentials, the electromagnetic contributions vanish at infinity and only the gravitational contributions are important<sup>5</sup>. Equation (4.5) still hold. On the other hand, at the horizon, the electromagnetic field is not negligible and we have

$$\oint_H k_{\xi, 0}^{tot} = \frac{\kappa}{8\pi} \delta \mathcal{A} - \delta \oint_H K_{\xi, \epsilon}^{em} + \oint_H K_{\delta\xi, 0}^{em} - \oint_H \xi \cdot \Theta^{em}. \quad (4.16)$$

<sup>4</sup>We use “geometrized units” for the electromagnetic field; the lagrangian is  $L = \frac{\sqrt{-g}}{16\pi G} (R - \frac{1}{4} F^2)$ .

<sup>5</sup>Note however that if the gauge potential tends to a constant at infinity, the quantities (4.4) will contain a contribution from the electric charge. We choose for convenience a potential vanishing at infinity. The contributions from the electromagnetic fields will then comes only from the surface integral over the horizon.

Now, remember that the zero law extended to electromagnetism said that  $\Phi = -(\xi^\alpha A_\alpha)|_H$  is constant on the horizon<sup>6</sup>. Therefore, we have directly

$$\oint_H K_{\xi,0}^{em} = -\Phi Q. \quad (4.17)$$

In order to work out the two remaining terms on the right hand side of (4.16), let us rewrite equation (4.12) as

$$\mathcal{L}_\xi A_\mu = \xi^\nu F_{\nu\mu} + (\xi^\alpha A_\alpha)_{,\mu} = 0. \quad (4.18)$$

Since  $\Phi$  is constant on the horizon, this equation says that the electric field  $E^\mu \equiv F^{\mu\nu} \xi_\nu$  with respect to  $\xi$  satisfy  $\eta_\mu E^\mu \stackrel{\mathcal{H}}{=} 0$  for all tangent vectors  $\eta^\mu$ , therefore  $E^\mu$  is proportional to  $\xi^\mu$  or more precisely,

$$F^{\mu\nu} \xi_\nu \stackrel{\mathcal{H}}{=} (F^{\nu\rho} \xi_\nu n_\rho) \xi^\mu. \quad (4.19)$$

Using the last relation with (4.7), it is easy to show that

$$\oint_H K_{\delta\xi,0}^{em} - \oint_H \xi \cdot \Theta^{em} = -\delta\Phi Q - \oint_H \frac{d\mathcal{A}}{4\pi G} n_\mu F^{\mu\nu} \delta A_\nu \xi^2. \quad (4.20)$$

The last term vanishes because of (1.1). Finally, we obtain the first law valid when electromagnetic fields are present,

$$\delta\mathcal{E} - \Omega_a \delta\mathcal{J}^a = \frac{\kappa}{8\pi G} \delta\mathcal{A} + \Phi \delta Q. \quad (4.21)$$

Note finally that the first applies for any theory of gravity, from arbitrary diffeomorphic invariant Lagrangians [28], to black holes in non-conventional background geometries [38, 39] or to black objects in string theory or supergravity [40, 41].

In addition, Hawking [42] discovered that  $T \equiv \frac{\kappa\hbar}{2\pi}$  is the temperature of the quantum radiation emitted by the black hole. Moreover, as tells us the second law,  $S \equiv \frac{\mathcal{A}}{4\hbar G}$  is a quantity that can classically only increase with time. It suggests [43] to associate an entropy  $S$  to the black hole. This striking occurrence of thermodynamics emerging out of the structure of gravity theories is a fundamental issue to be further explained in still elusive quantum gravity theories.

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<sup>6</sup>We previously assumed that  $\mathcal{L}_\xi A_\mu = 0$ . So  $\epsilon = 0$  here.

### 4.3 Extension to any theory of gravity

Let us finally briefly review the proposal of Iyer and Wald [28, 44] for the extension of the first law to an arbitrary diffeomorphic invariant Lagrangian. Let

$$L[\phi] = L[g_{\mu\nu}, R_{\mu\nu\rho\sigma;(\alpha)}], \quad (4.22)$$

be the lagrangian where  $\phi$  denote all the fields of the theory and  $(\alpha)$  denotes an arbitrary number of symmetrized derivatives.

According to the authors, the charge difference between the solutions  $\phi$  and  $\phi + \delta\phi$  corresponding to the vector  $\xi$  can be written as (4.6) [28, 44] where the Komar term  $\oint K_\xi$  is given by

$$\oint (d^{n-2}x)_{\mu\nu} \left( \sqrt{-g} \xi^\mu W^\nu[\phi] + Y^{\mu\nu}[\mathcal{L}_\xi \phi, \phi] - \frac{\delta L}{\delta R_{\mu\nu\alpha\beta}} \xi_{\alpha;\beta} - (\mu \leftrightarrow \nu) \right) \quad (4.23)$$

and where  $\Theta[\delta\phi; \phi]$ ,  $W^\mu[\phi]$  and  $Y^{\mu\nu}[\mathcal{L}_\xi \phi, \phi]$  have a generic form which we shall not need here. The energy and angular momentum are defined as previously by (4.4) where the  $n-2$  form  $k_\xi$  is defined by (4.6).

Therefore, equations (4.3)-(4.5)-(4.6) are still valid with appropriate  $K_\xi[\phi]$  and  $\Theta[\delta\phi; \phi]$ . Now, we assume that the surface gravity  $\kappa$  is not vanishing and that the horizon generators are geodesically complete to the past. Then, it exists [45] a special  $n-2$  spacelike surface on the horizon, the bifurcation surface, where the Killing vector  $\xi$  vanish. For Killing vectors  $\xi$ , i.e. such that  $\mathcal{L}_\xi \phi^i = 0$  and for solutions of the equations of motion,  $k_\xi$  is a closed form, (3.12) hold and we may evaluate the right-hand side of (4.6) on the bifurcation surface. There, assuming regularity conditions, only the third term in the Komar expression (4.23) contributes [28, 46] and the right-hand side of (4.6) becomes

$$\frac{\kappa}{2\pi} \delta \mathcal{S}, \quad (4.24)$$

where the ‘‘higher order in the curvature’’ entropy is defined by

$$\mathcal{S} = -8\pi \oint_H (d^{n-2}x)_{\mu\nu} \frac{\delta L}{\delta R_{\mu\nu\alpha\beta}} \xi_\alpha n_\beta. \quad (4.25)$$

The first law (4.2) therefore holds with appropriate definitions of energy, angular momentum and entropy.

For the Einstein-Hilbert lagrangian, we have

$$\frac{\delta L^{EH}}{\delta R_{\mu\nu\alpha\beta}} = \frac{\sqrt{-g}}{32\pi G} (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}), \quad (4.26)$$

and the entropy (4.25) reduces to the familiar expression  $\mathcal{A}/4G$ .

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## Appendix A

# Completion of the proof of the first law

Let us prove the relation (4.9). We consider any stationary variation of the fields  $\delta g_{\mu\nu}$ ,  $\delta\xi^\mu$ , i.e. such that

$$\mathcal{L}_\xi \delta g_{\mu\nu} + \mathcal{L}_{\delta\xi} g_{\mu\nu} = 0. \quad (\text{A.1})$$

The variation is chosen to commute with the total derivative, i.e. the coordinates are left unchanged  $\delta x^\mu = 0$ .

Using the decomposition (4.7), the left-hand side of equation (4.9) can be written explicitly as

$$\oint_H K_{\delta\xi}[g] - \oint_H \xi \cdot \Theta[\delta g; g] = \oint_H \frac{d\mathcal{A}}{16\pi G} \left( -\delta\xi^{\mu;\nu} (\xi_\mu n_\nu - \xi_\nu n_\mu) + \xi^\mu (\delta g_{\mu\nu}{}^{;\nu} - g^{\alpha\beta} \delta g_{\alpha\beta;\mu}) \right). \quad (\text{A.2})$$

We have to relate this expression to the variation of the surface gravity  $\kappa$ . This is merely an exercise of differential geometry.

Since the horizon  $S(x) = 0$  stays at the same location in  $x^\mu$ , the covariant vector normal to the horizon  $\xi_\mu = f \partial_\mu S$ , where  $f$  is a  $\kappa$ -dependent normalization function, satisfies

$$\delta\xi_\mu \stackrel{\mathcal{H}}{=} \delta f \xi_\mu, \quad (\text{A.3})$$

where  $\delta\xi_\mu \equiv \delta(g_{\mu\nu}\xi^\nu)$ . From the variation of (1.1) and of the second normalization condition (1.6), one obtains

$$\delta\xi^\mu \xi_\mu \stackrel{\mathcal{H}}{=} 0, \quad \delta n^\mu \xi_\mu \stackrel{\mathcal{H}}{=} \delta f, \quad (\text{A.4})$$

which shows that  $\delta\xi^\mu$  has no component along  $n^\mu$  and  $\delta n^\mu$  has a component along  $n^\mu$  which equals  $-\delta f^1$ .

Let us develop the variation of  $\kappa$  starting from the definition (2.3). One has

$$\delta\kappa = \frac{1}{2}(\xi^\mu\xi_\mu)_{;\nu}\delta n^\nu + \frac{1}{2}(\delta\xi_\mu\xi^\mu + \xi_\mu\delta\xi^\mu)_{;\nu}n^\nu, \quad (\text{A.6})$$

$$\begin{aligned} &= \frac{1}{2}\delta\xi_{\mu;\nu}(\xi^\mu n^\nu + \xi^\nu n^\mu) + \xi^\mu_{;\nu}(\delta\xi_\mu n^\nu + \xi_\mu\delta n^\nu) \\ &\quad + \frac{1}{2}n^\nu(\xi_\mu\delta\xi^\mu)_{;\nu} - \frac{1}{2}n^\nu\mathcal{L}_\xi\delta\xi_\nu, \end{aligned} \quad (\text{A.7})$$

where all expressions are implicitly pulled-back on the horizon. The first term in (A.7) is recognized as  $-\frac{1}{2}\delta\xi_{\mu;\nu}g^{\mu\nu}$  after using (1.9), (A.3) and (2.6). According to (A.3)-(A.4), the second term can be written as

$$\xi^\mu_{;\nu}(\delta\xi_\mu n^\nu + \xi_\mu\delta n^\nu) = \xi_{\mu;\nu}\xi^\mu\delta\eta^\nu, \quad (\text{A.8})$$

for some  $\delta\eta^\nu$  tangent to  $\mathcal{H}$ . This term vanishes thanks to (2.6). The third term can be written as

$$\frac{1}{2}n^\nu(\xi_\mu\delta\xi^\mu)_{;\nu} = -\frac{1}{2}n^\nu\mathcal{L}_{\delta\xi}\xi_\nu + n_\nu\xi_\mu\delta\xi^{\mu;\nu}. \quad (\text{A.9})$$

Now, the Lie derivative of  $\delta\xi_\mu$  along  $\xi$  can be expressed as

$$\mathcal{L}_\xi\delta\xi_\mu = -\mathcal{L}_{\delta\xi}\xi_\mu, \quad (\text{A.10})$$

by using the Killing equation (2.1) and its variation (A.1). The fourth term can then be written as

$$-\frac{1}{2}n^\nu\mathcal{L}_\xi\delta\xi_\nu = \frac{1}{2}n^\nu\mathcal{L}_{\delta\xi}\xi_\nu. \quad (\text{A.11})$$

Adding all the terms, the variation of the surface gravity becomes

$$\begin{aligned} \delta\kappa &= -\frac{1}{2}(\delta\xi_\mu)^{;\mu} + \delta\xi^{\mu;\nu}\xi_\mu n_\nu, \\ &= -\frac{1}{2}\delta g_{\mu\nu}{}^{;\mu}\xi^\nu - \frac{1}{2}\delta\xi^\mu{}_{;\mu} + \frac{1}{2}\delta\xi^{\mu;\nu}(\xi_\mu n_\nu - \xi_\nu n_\mu) + \frac{1}{2}\delta\xi^{\mu;\nu}(\xi_\mu n_\nu + \xi_\nu n_\mu), \\ &= -\frac{1}{2}\delta g_{\mu\nu}{}^{;\mu}\xi^\nu - \delta\xi^\mu{}_{;\mu} + \frac{1}{2}\delta\xi^{\mu;\nu}(\xi_\mu n_\nu - \xi_\nu n_\mu) + \frac{1}{2}\delta\xi^{\mu;\nu}\gamma_{\mu\nu}. \end{aligned} \quad (\text{A.12})$$

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<sup>1</sup>Note also the following property that is useful in order to prove the first law in the way of [28]. Using (2.5) and (A.3), we have

$$\delta\xi_{\mu,\nu} - \delta\xi_{\nu,\mu} \stackrel{\mathcal{H}}{=} \xi_\mu(\delta f q_\nu + \delta q_\nu) - \xi_\nu(\delta f q_\mu + \delta q_\mu). \quad (\text{A.5})$$

It implies in particular that the expression  $\delta\xi_{[\mu;\nu]}$  has no tangential-tangential component,  $\delta\xi_{[\mu;\nu]}\eta^\mu\tilde{\eta}^\nu \stackrel{\mathcal{H}}{=} 0$ ,  $\forall\eta, \tilde{\eta}$  orthogonal to  $\mathcal{H}$ .

The last line is a consequence of (1.9). Contracting (A.1) with  $g^{\mu\nu}$  we also have

$$\delta\xi^\mu{}_{;\mu} = -\frac{1}{2}\xi^\mu g^{\alpha\beta}\delta g_{\alpha\beta;\mu}. \quad (\text{A.13})$$

Finally, the last term in (A.12) reduces to  $\frac{1}{2}\delta t^\alpha{}_{|\alpha}$  where  $|_\alpha$  denotes the covariant derivative with respect to the  $n-2$  metric  $\gamma_{\mu\nu}$  and  $\delta t^\mu = \gamma^\mu{}_\nu\delta\xi^\nu$  is the pull-back of  $\delta\xi^\mu$  on  $\mathcal{H}$ . Indeed, one has

$$\frac{1}{2}\delta\xi^{\mu;\nu}\gamma_{\mu\nu} = \frac{1}{2}\delta t^{\mu;\nu}\gamma_{\mu\nu}, \quad (\text{A.14})$$

$$= \frac{1}{2}(\delta t^\mu{}_{;\nu}\gamma_\mu{}^\nu + \Gamma_{\mu;\nu\alpha}\gamma^{\mu\nu}\delta t^\alpha), \quad (\text{A.15})$$

$$= \frac{1}{2}\delta t^\mu{}_{|\mu}, \quad (\text{A.16})$$

where  $|\mu$  denotes the covariant derivative with respect to the  $n-2$  metric  $\gamma_{\mu\nu}$ . The first line uses (2.1)-(A.4) and  $\gamma_{\mu\nu}\xi^\nu = 0 = \gamma_{\mu\nu}n^\nu$ . The last line uses the decomposition (1.9) and  $\delta t^\alpha n_\alpha = 0 = \delta t^\alpha\xi_\alpha$ . We have finally the result

$$\delta\kappa = -\frac{1}{2}\delta g_{\mu\nu}{}^{;\mu}\xi^\nu + \frac{1}{2}\xi^\mu g^{\alpha\beta}\delta g_{\alpha\beta;\mu} + \frac{1}{2}\delta\xi^{\mu;\nu}(\xi_\mu n_\nu - \xi_\nu n_\mu) + \frac{1}{2}(\gamma^\mu{}_\nu\delta\xi^\nu)_{|\mu}. \quad (\text{A.17})$$

Expression (A.2) is therefore equals to

$$\oint_H K_{\delta\xi}[g] - \oint_H \xi \cdot \Theta[\delta g; g] = -\oint_H \frac{dA}{8\pi G}\delta\kappa, \quad (\text{A.18})$$

and the result (4.9) follows because  $\delta\kappa$  is constant on the horizon.

Remark that in classical derivations [7, 1], it is assumed that the Killing vectors  $\partial_t$  and  $\partial_\varphi$  have the same components before and after the variation,

$$\delta(\partial_t)^\mu = \delta(\partial_\varphi)^\mu = 0.$$

One then has  $\delta\xi^\mu = \delta\Omega(\partial_\varphi)^\mu$  and the variation of  $\kappa$  reduces to the well-known expression

$$\delta\kappa = -\frac{1}{2}\delta g_{\mu\nu}{}^{;\mu}\xi^\nu + \delta\Omega(\partial_\varphi)^{\mu;\nu}\xi_\mu n_\nu.$$

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