The BRST antifield formalism.
Part II: Applications.

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In these lectures, we present an introduction to two applications of the Lagrangian BRST formalism (also called field-antifield or Belavin-Vilkovisky formalism): the construction of consistent interactions and the quantization of systems with gauge freedom.

The first topic of these lectures is the application of the antifield formalism to the "consistent local interaction problem". The problem is the following: given a "free" theory of gauge fields, what are all the consistent couplings that one can introduce to make these fields interact locally? By "consistent", one means that the number of degrees of freedom is unchanged.

The antifield offers a very systematic way to answer this question, through a reformulation in terms of cohomology groups for the various differentials presented in the previous sections (s, d, γ and δ). We will briefly present this topic and illustrate it on an example.

On the other hand, the BRST formalism is mainly known because it offers a systematic way to quantize theories with a gauge freedom. For these theories, the most naive implementation of the path integral does not work. Indeed, the integration over gauge orbits makes the path integral infinite, as e.g. for electromagnetism:

$$\int \mathcal{D}A_\mu \, e^{\frac{i}{\hbar}S[A_\mu]} = \infty \, .$$

One solution is to remove all the redundant degrees of freedom to work with the physical fields only: one chooses a gauge before making the integration. However, this procedure is not liked because it spoils the gauge invariance, makes locality non manifest, is not covariant, etc. A way out was found by Faddeev and Popov. They turned the infinities into new terms in the action, which involve a new kind of fields called "ghosts". The Faddeev-Popov action, which depends on the gauge fields and on the ghost fields, has no gauge invariance (that is, no symmetry transformations depending on an arbitrary function), but it exhibits a new global symmetry, called BRST symmetry, which mixes gauge fields and ghosts.

However, the quantization problem was not completely solved because this procedure is limited to "simple" gauge theories. For example, it does not work for general reducible theories or when the gauge algebra is open. Furthermore, the mathematical scheme behind the BRST symmetry and the whole construction was at first poorly understood. Their meaning became only clear with the developpement of the BRST approach and the antifield formalism.

In these lectures, we will show how the antifield formalism is used to build actions without gauge invariance that can be used in the path integral quantization procedure. This construction is systematic and can be done for any gauge theory. It can be related to the Faddeev-Popov results when the latter exist.

0.1 Consistent interactions

As announced above, we want to answer the following question: given a theory (that is, a set of fields and an action to describe their motion), can one introduce consistent local interactions into this theory?

This problem can be economically reformulated as a deformation problem, namely that of *deforming consistently the master equation.* Consider the "free" action $S_0[\phi^i]$ with "free" gauge symmetries

$$\delta_\varepsilon \phi^i = \overset{(0)}{R}{}^i_\alpha \varepsilon^\alpha , \tag{0.1.1}$$

$$\overset{(0)}{R}{}^i_\alpha \frac{\delta S_0}{\delta \phi^i} = 0 \ . \tag{0.1.2}$$

One wishes to introduce consistent interactions, i.e. to modify S_0

$$S_0 \longrightarrow S = S_0 + gS_1 + g^2 S_2 + ... \tag{0.1.3}$$

in such a way that one can consistently deform the original gauge symmetries,

$$\overset{(0)}{R}{}^i_\alpha \longrightarrow R^i_\alpha = \overset{(0)}{R}{}^i_\alpha + g \overset{(1)}{R}{}^i_\alpha + g^2 \overset{(2)}{R}{}^i_\alpha + ... \tag{0.1.4}$$

The deformed gauge transformations $\delta_\varepsilon \phi^i = R^i_\alpha \varepsilon^\alpha$ are consistent if they are gauge symmetries of the full action (0.1.3),

$$(\overset{(0)}{R}{}^i_\alpha + g \overset{(1)}{R}{}^i_\alpha + g^2 \overset{(2)}{R}{}^i_\alpha + ...) \frac{\delta(S_0 + gS_1 + g^2 S_2 + ...)}{\delta \phi^i} = 0 \quad . \tag{0.1.5}$$

If the original gauge transformations are reducible, one should also demand that (0.1.4) remain reducible. Indeed, the deformed theory then possesses the same number of (possibly deformed) independent gauge symmetries, reducibility identities, *etc.*, as the system one started with, so that the number of physical degrees of freedom is unchanged. Equation (0.1.5) is the key equation to be satisfied.

Interactions obtained by making field redefinitions are considered as trivial. They are characterized by the fact that the first-order deformation is proportional to the left-hand side of the equations of motion. Indeed, take the transformation $\phi^i \longrightarrow \bar{\phi}^i = \phi^i + gF^i + ...$. One has

$$\begin{aligned} S_0[\phi^i] \longrightarrow S[\bar{\phi}^i] \equiv S_0[\phi^i[\bar{\phi}^i]] &= S_0[\bar{\phi}^i - gF^i + ...] \\ &= S_0[\bar{\phi}^i] - g \frac{\delta S_0}{\delta \phi^i} F^i + ... \ . \end{aligned} \tag{0.1.6}$$

Let us now reformulate the problem in the antifield framework. If a consistent interacting theory exists, one can construct the corresponding solution W of the master equation

$$(W, W) = 0 \ .$$

The latter equation guaranties the consistency of the interaction.

To make contact with the free theory, the generator W should admit the generator W_0 of the free theory as its vanishing coupling limit,

$$W = W_0 + gW_1 + g^2 W_2 + \dots \,.$$

Inserting this expression into the master equation and splitting it according to the power in the coupling g yields

$$\begin{aligned}
(W_0, W_0) &= 0 && (0.1.7)\\
2(W_0, W_1) &= 0 && (0.1.8)\\
2(W_0, W_2) + (W_1, W_1) &= 0 && (0.1.9)\\
&\vdots
\end{aligned}$$

The first equation is satisfied by assumption, while the second implies that W_1 is a cocycle for the free BRST-differential $s \equiv (W_0, \cdot)$.

Suppose that W_1 is a coboundary, $W_1 = (W_0, T)$. This corresponds to a trivial deformation because S_0 is then modified as in (0.1.6)

$$\begin{aligned}
S_0 \longrightarrow S_0 + g\ [(W_0, T)]_{\Phi^*=0} &= S_0 + g\ \Big[\frac{\delta^R W_0}{\delta \Phi^A}\frac{\delta^L T}{\delta \Phi^*_A} - \frac{\delta^R W_0}{\delta \Phi^*_A}\frac{\delta^L T}{\delta \Phi^A}\Big]_{\Phi^*=0}\\
&= S_0 + g\ \frac{\delta^R S_0}{\delta \phi^i}\ \Big[\frac{\delta^L T}{\delta \phi^*_i}\Big]_{\Phi^*=0} \qquad (0.1.10)
\end{aligned}$$

Trivial deformations thus correspond to s-exact quantities, i.e. trivial elements of the cohomological space $H(s)$ of the undeformed theory in ghost number zero. Since the deformations must be s-cocycles, nontrivial deformations are thus determined by the equivalence classes of $H(s)$ in ghost number zero.

The next equation, Eq.(0.1.9), implies that W_1 should be such that (W_1, W_1) is trivial in $H(s)$ in ghost number one.

In practice, one prefers to work with the Lagrangian. So, let $W_k = \int \mathcal{L}_k$ where $\mathcal{L}_k$ is a local n-form, which thus depends on the field variables and only a finite number of their derivatives. In terms of the integrands $\mathcal{L}_k$, the equations (0.1.8-0.1.9) for W_k read [1]

$$\begin{aligned}
2s\mathcal{L}_1 &= dj_1 && (0.1.12)\\
s\mathcal{L}_2 + (\mathcal{L}_1, \mathcal{L}_1) &= dj_2 && (0.1.13)\\
&\vdots \quad .
\end{aligned}$$

[1] We also denote by (a, b) the antibracket for n-forms, i.e.,

$$(A, B) = \int (a, b) \qquad (0.1.11)$$

if $A = \int a$ and $B = \int b$. Because (A, B) is a local functional, there exists (a, b) such that Eq.(0.1.11) holds, but (a, b) is defined only up to d-exact terms.

The equation (0.1.12) expresses that $\mathcal{L}_1$ should be BRST-closed modulo d and again, it is easy to see that a BRST-exact term modulo d corresponds to trivial deformations. Nontrivial local deformations of the master equation are thus determined by $H^{n,0}(s|d)$, the cohomology of the BRST-differential s modulo the total derivative d, in maximal form-degree n and in ghost number 0.

0.1.1 Example: Yang-Mills

Let us illustrate the above procedure and sketch how one then goes on for electromagnetism. We will consider N vector fields and sketch the computations to find all their consistent local Poincaré-invariant interactions with at most two derivatives.

The set of fields and antifields is

- the vector fields A^a_μ, of vanishing parity and ghost number;
- the ghosts C^a, one for each gauge transformation, of odd parity and ghost number one;
- the antifields $A^{*\mu}_a$, of odd parity and ghost number -1;
- the antifields C^*_a, of even parity and ghost number -2.

Defining the field strength $F^a_{\mu\nu}$ by

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \ ,$$

the BRST action reads

$$W = \int d^dx \Big(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + A^*_{a\mu} \partial^\mu C^a \Big) \ ,$$

and the BRST differential is $s = \gamma + \delta$, where the action of γ and δ is zero on any (anti)field except in the following cases:

$$\begin{aligned} &\gamma A^{a\mu} = \partial^\mu C^b \ , \\ &\delta A^*_{a\mu} = \partial^\nu F_{a\nu\mu} \ , \qquad \delta C^*_a = \partial^\mu A^*_{a\mu} \ . \end{aligned}$$

One easily checks that $\gamma^2 = 0$, $\delta^2 = 0$ and $\gamma\delta + \delta\gamma = 0$.

To find all first-order deformations of the Lagrangian, we must compute $H(\delta|d)$, i.e. determine the general solution of the cocycle condition

$$s a^{n,0} + d b^{n-1,1} = 0, \tag{0.1.14}$$

where $a^{n,0}$ is a topform[2] of ghost number zero and $b^{n-1,1}$ a $(n-1)$-form of ghost number one, with the understanding that two solutions of Eq.(0.1.14) that differ by a trivial solution should be identified

$$a^{n,0} \sim a^{n,0} + s m^{n,-1} + d n^{n-1,0} \ .$$

[2]We denote the number of dimensions by n.

The cocycles and coboundaries $a, b, m, n, \ldots$ are local forms of the field variables (including ghosts and antifields)

We will not go through the whole computation here. The main idea is to expand a, b and the equation (0.1.14) according to the antighost number. So $a = a_0 + a_1 + \ldots + a_k$ where a_i has antifield number i, a similar expansion holds for b, and they have to satisfy

$$\begin{aligned}
\delta a_1 + \gamma a_0 + d b_0 &= 0\,, \\
\delta a_2 + \gamma a_1 + d b_1 &= 0\,, \\
&\vdots \\
\delta a_k + \gamma a_{k-1} + d b_{k-1} &= 0\,, \qquad (0.1.15)\\
\gamma a_k &= 0\,. \qquad (0.1.16)
\end{aligned}$$

One calls such a system of equations a "descent". One starts to solve it from the bottom, using the knowledge of cohomological groups like $H(\gamma)$, $H(\delta|d)$ in antighost number k, $H(\gamma|d)$, etc.

With the assumption that the deformation of the Lagrangian should have at most two derivatives, one finds that a_k vanishes (or can be removed by trivial redefinitions) for any $k > 2$, and that the only nontrivial a_2 must be of the form

$$a_2 = f^a_{bc} C^*_a C^b C^c d^n x\,,$$

where f^a_{bc} is a constant. Note that f^a_{bc} is antisymmetric in its indices b and c because the ghosts are anticommutating. We will not prove this statement here because the proof, though very interesting, is quite technical.

One then tries to "lift" a_2, i.e. to find a_1 and a_0 that satisfy the equations of the descent. Inserting a_2 into Eq.(0.1.15), one finds

$$\begin{aligned}
\gamma a_1 &= -f^a_{bc} \partial^\mu A^*_{a\mu} C^b C^c d^n x - d b_1 \\
&= 2\, f^a_{bc} A^*_{a\mu} \partial^\mu (C^b) C^c d^n x + d\tilde{b}_1 \\
&= 2\, f^a_{bc} A^*_{a\mu} \gamma(A^{b\mu}) C^c d^n x + d\tilde{b}_1 \\
&= \gamma(-2\, f^a_{bc} A^*_{a\mu} A^{b\mu} C^c d^n x) + d\tilde{b}_1\,,
\end{aligned}$$

so one can take

$$a_1 = -2\, f^a_{bc} A^*_{a\mu} A^{b\mu} C^c d^n x\,.$$

Inserting this expression into the top equation of the descent, one finds

$$\begin{aligned}
\gamma a_0 &= 2\, f^a_{bc} \partial^\nu F_{a\nu\mu} A^{b\mu} C^c d^n x - d b_0 \\
&= -2\, f^a_{bc} F_{a\nu\mu} \partial^\nu A^{b\mu} C^c d^n x - 2\, f^a_{bc} F_{a\nu\mu} A^{b\mu} \partial^\nu C^c d^n x - d\tilde{b}_0 \\
&= -f^a_{bc} F_{a\nu\mu} F^{b\nu\mu} C^c d^n x + \gamma(-2\, f^a_{bc} F_{a\nu\mu} A^{b\mu} A^{c\nu} d^n x) - d\tilde{b}_0\,.
\end{aligned}$$

It can be shown that the first term on the right-hand side is not γ-exact modulo d, so this equation can be solved only if it vanishes. This implies that the structure function f^a_{bc} is antisymmetric in a and b, i.e. it must be completely antisymmetric.

We have now obtained a deformation of the free theory that is consistent at first-order in the coupling constant:

$$\mathcal{L}_1 = a_0 + a_1 + a_2 \; . \tag{0.1.17}$$

This deformation reproduces the first-order term of a Yang-Mills theory, except that the structure functions of Yang-Mills have to obey the Jacobi identity. The next step is to find the second-order deformation of the theory. We will see that it exists only if f^a_{bc} satisfies the Jacobi identity.

The equation to be solved is

$$s\mathcal{L}_2 + (\mathcal{L}_1, \mathcal{L}_1) = dj_2 \; .$$

A first-consistency check is that $(\mathcal{L}_1, \mathcal{L}_1)$ must be s-exact modulo d. Using some theorems, one shows that this implies that the term with highest antighost number in $(\mathcal{L}_1, \mathcal{L}_1)$ must be γ-exact. However, this term is

$$(a_2, a_2) \propto f^a_{bc} C^b C^c f^e_{af} C^*_e C^f d^n x$$

and this espression is nontrivial in $H(\gamma)$. It must thus vanish. Because the product $C^b C^c C^f$ is antisymmetric in its indices, it is enough that the part of $f^a_{bc} f^e_{fa}$ antisymmetric in the indices b, c and f vanishes, i.e. that the structure functions f^a_{bc} obey the Jacobi identity.

The end of the computations is straightforward. We know that the only $\mathcal{L}_1$ is given by Eq.(0.1.17) plus the Jacobi identity. We know that Yang-Mills theories are consistent and have exactly this first-order deformation. Putting the two facts together, we see that the only consistent interaction with at most two derivatives in the Lagrangian must be Yang-Mills.

We hope that this example, though incompletely treated, has given a taste of what the computations to find consistent interactions look like.

0.2 Quantization

The purpose of this section is to bridge the gap between the antifield formalism presented in Section **??** and its use for the path integral quantization. On our way, we will meet Faddeev and Popov.

The outline of this section is as follows. In the path integral approach, one needs an action that has no gauge invariance. The BRST action could be a better candidate than the original action, but it still possesses some gauge invariance (Section 0.2.1). One can however fix this invariance in a clever way, by chosing a "gauge-fixing fermion", which removes the antifields. To build the latter fermion, it is necessary to introduce new fields, called *antighosts* (Section 0.2.2). In the end, one is left with an action that has no gauge invariance, but admits a global symmetry mixing all the fields (including ghosts and antighosts), called the BRST symmetry (Section 0.2.3). This action can be used in the path integral, modulo one last check: the correlation functions computed must be independent of the gauge-fixing procedure (Section 0.2.5). This requirement imposes conditions on the theory through the "quantum master equation".

We illustrate the gauge-fixing procedure on an example in Section 0.2.4.

0.2.1 Gauge invariance and gauge-fixing fermion

The BRST action built in the preceding section cannot be used as it is in the path integral because it still possesses gauge invariances. Indeed, differentiating the master equation

$$(W,W) = 2\frac{\delta^R W}{\delta\Phi^A}\frac{\delta^L W}{\delta\Phi^*_A} = 0$$

with respect to the fields or to the antifields, yields the following relations among the left-hand sides of the equations of motion:

$$0 = \frac{1}{2}\frac{\delta^R(W,W)}{\delta z^c} = \frac{\delta^R W}{\delta\Phi^A}\frac{\delta^L\delta^R W}{\delta\Phi^*_A\delta z^c} - \frac{\delta^R W}{\delta\Phi^*_A}\frac{\delta^L\delta^R W}{\delta\Phi^A\delta z^c}\,, \tag{0.2.18}$$

where z^c can be either Φ^C or Φ^*_C. Each of these Noether identities indicates the presence of a gauge symmetry. (Exercice: write them!)

One can show (see [2]) that the number of independent gauge transformations matches the number of fields (or equivalently the number of antifields), which we call N. In order to remove the gauge freedom, one thus has to fix the gauge by N gauge conditions. One usually uses these conditions to gauge-fix the antifields, that is, to express them in terms of the fields. This is implemented by choosing a function $\psi(\Phi^A)$ depending only on the fields and imposing the gauge conditions

$$\Phi^*_A = \frac{\delta\psi}{\delta\Phi^A}\,. \tag{0.2.19}$$

From this equation, one reads that ψ must be a fermion, and have the ghost number -1:

$$\epsilon(\psi) = 1\,, \quad gh(\psi) = -1\,.$$

Accordingly, the function ψ is called the "gauge-fixing fermion". Note that in Eq.(0.2.19) it is equivalent to take left- or right-derivatives (see rules in Appendix A).

We will now consider two issues. First, we will try to build a gauge-fixing fermion ψ and see that it is necessary to introduce new fields, the antighosts. We will illustrate this construction by an example. Secondly, we will check under which conditions the path integral is independent of the choice of ψ.

0.2.2 Antighost fields

We have said above that the gauge-fixing fermion should depend only on the fields and have ghost number -1. However, all the fields of the minimal set have positive ghost number (only the antifields have negative ghost number). It is thus necessary to add new fields (and antifields) in order to build an appropriate gauge-fixing fermion.

It is allowed to add new fields by pairs $\{\alpha, \beta\}$ that satisfy the following relations:

$$\begin{aligned} \epsilon(\alpha) &= \epsilon(\beta) + 1\,, \qquad & gh(\alpha) &= gh(\beta) + 1\,, \\ s\alpha &= 0\,, & s\beta &= \alpha\,. \end{aligned} \tag{0.2.20}$$

(one must also include their antifield counterparts $\{\alpha^*, \beta^*\}$). Such a couple will add no new contribution to the cohomology of s: all s-cocycles involving these fields are s-exact and thus trivial. One easily generates the BRST-variations (0.2.20) for α and β by adding a BRST-exact term to the BRST generator: $W^{minimal} \to W = W^{minimal} + W^{trivial}$, where $W^{trivial} = -\alpha\beta^*$. One can then read off the BRST variation of α^* and β^*, which is

$$s\alpha^* = (-)^{\epsilon(\beta)}\beta^* \quad and \quad s\beta^* = 0\,.$$

We will call these new fields "antighosts" because they correspond to the antighosts of Faddeev and Popov. In specific cases they have other nick-names, like B-fields, auxiliary fields, Nakanishi-Lautrup auxiliary field, Stuckelberg fields, "extra" ghosts.

This freedom to add trivial pairs is used in the gauge-fixing procedure to introduce fields with negative ghost number. So, e.g., in the irreducible case one adds one trivial pair $\{\bar{C}_{0\alpha_0}, \bar{\pi}_{0\alpha_0}\}$, which has the same index structure as the ghost $C_0^{\alpha_0}$. The ghost numbers and parities are

$$\epsilon[\bar{C}_{0\alpha_0}] = \epsilon[C_0^{\alpha_0}]\,, \quad gh[\bar{C}_{0\alpha_0}] = -1\,, \quad \epsilon[\bar{\pi}_{0\alpha_0}] = \epsilon[C_0^{\alpha_0}] + 1\,, \quad gh[\bar{\pi}_{0\alpha_0}] = 0\,.$$

For a first-stage reducible theory, one adds two more pairs $\{\bar{C}_{1\alpha_1}, \bar{\pi}_{1\alpha_1}\}$ and $\{C^1_{1\alpha_1}, \pi^1_{1\alpha_1}\}$. For an L-th stage reducible theory, for each integer s ranging from 0 to L, one adds $s+1$ trivial pairs having the same indices as the s-th stage ghost.

We will not enter into the particulars about why exactly these antighost fields should be added. Let us just state that, if the gauge-fixing fermion is properly chosen, they allow to remove all gauge and non-propagating degrees of freedom. More details about this can be found e.g. in [1].

As we just said, not every gauge-fixing fermion is allowed, one must make sure that it kills all the unwanted degrees of freedom. Nevertheless, in the end some freedom is left in the choice of the gauge-fixing fermion. One then chooses (or tries to choose !) the fermion that most simplifies the calculation one is to perform.

0.2.3 Gauge-fixed action and BRST symmetry

Having chosen the gauge-fixing fermion and used the gauge-fixing conditions, one is left with the gauge-fixed action

$$W_\psi = W(\Phi^A, \Phi^*_A = \frac{\delta\psi}{\delta\Phi^A}) = W(\Phi^A, \Phi^*_A)|_{\Sigma_\psi} \ ,$$

where we have denoted by Σ_ψ the surface on which $\Phi^*_A = \frac{\delta\psi}{\delta\Phi^A}$ holds.

Quite naturally, one defines the gauge-fixed BRST symmetry by

$$s_\psi X(\Phi^A) \equiv (W, X)|_{\Sigma_\psi} \ .$$

The gauge-fixed action is invariant under the gauge-fixed BRST symmetry. Indeed, using

$$\frac{\delta^L W_\psi}{\delta\Phi^A} = \frac{\delta^L W}{\delta\Phi^A}\Big|_{\Sigma_\psi} + \frac{\delta^L\delta^L\psi}{\delta\Phi^A\delta\Phi^B}\frac{\delta^L W_\psi}{\delta\Phi^*_B}\Big|_{\Sigma_\psi} \ ,$$

one has

$$\begin{aligned} s_\psi W_\psi &= (W, W_\psi)|_{\Sigma_\psi} = -\frac{\delta^R W}{\delta\Phi^*_A}\Big|_{\Sigma_\psi}\frac{\delta^L W_\psi}{\delta\Phi^A} \\ &= -\Big[\frac{\delta^R W}{\delta\Phi^*_A}\frac{\delta^L W}{\delta\Phi^A}\Big]\Big|_{\Sigma_\psi} - \Big[\frac{\delta^R W}{\delta\Phi^*_A}\frac{\delta^L\delta^L\psi}{\delta\Phi^A\delta\Phi^B}\frac{\delta^L W}{\delta\Phi^*_B}\Big]\Big|_{\Sigma_\psi} \\ &= 0 \ . \end{aligned}$$

The first term vanishes because of the master equation. The second term vanishes because of the statistics: using the formulae of the appendix A and reordering the factors, one gets $\frac{\delta^R W}{\delta\Phi^*_A}\frac{\delta^L\delta^L\psi}{\delta\Phi^A\delta\Phi^B}\frac{\delta^L W}{\delta\Phi^*_B} = -\frac{\delta^R W}{\delta\Phi^*_B}\frac{\delta^L\delta^L\psi}{\delta\Phi^B\delta\Phi^A}\frac{\delta^L W}{\delta\Phi^*_A}$, which must thus vanish.

The gauge-fixed BRST transformation is not always a differential. One has

$$\begin{aligned} s^2_\psi(X(\Phi^A)) &= (W, s_\psi X)|_{\Sigma_\psi} = -\frac{\delta^R W}{\delta\Phi^*_B}\Big|_{\Sigma_\psi}\frac{\delta^L s_\psi X}{\delta\Phi^B} \\ &= \frac{\delta^R W}{\delta\Phi^*_B}\Big|_{\Sigma_\psi}\Big[\frac{\delta^L\delta^R W}{\delta\Phi^B\delta\Phi^*_A}\frac{\delta^L X}{\delta\Phi^A} + \frac{\delta^L\delta^L\psi}{\delta\Phi^B\delta\Phi^C}\frac{\delta^L\delta^R W}{\delta\Phi^*_C\delta\Phi^*_A}\frac{\delta^L X}{\delta\Phi^A} \\ &\qquad + (-)^{\epsilon_B(\epsilon_A+1)}\frac{\delta^R W}{\delta\Phi^*_A}\frac{\delta^L\delta^L X}{\delta\Phi^B\delta\Phi^A}\Big]\Big|_{\Sigma_\psi} . \end{aligned} \tag{0.2.21}$$

Notice that by ϵ_A, we mean the Grassmann parity of Φ^A, which is opposite to the parity of Φ^*_A . Using equation (0.2.18), the first term can be rewritten as $\frac{\delta^R W}{\delta\Phi^B}\frac{\delta^L\delta^R W}{\delta\Phi^*_B\delta\Phi^*_A}\frac{\delta^L X}{\delta\Phi^A}$. It can then be combined with the second term to yield

$$\Big(\frac{\delta^R W}{\delta\Phi^C} + \frac{\delta^R W}{\delta\Phi^*_B}\frac{\delta^R\delta^L\psi}{\delta\Phi^C\delta\Phi^B}\Big)\frac{\delta^L\delta^R W}{\delta\Phi^*_C\delta\Phi^*_A}\frac{\delta^L X}{\delta\Phi^A} = \frac{\delta^R W_\psi}{\delta\Phi^C}\frac{\delta^L\delta^R W}{\delta\Phi^*_C\delta\Phi^*_A}\frac{\delta^L X}{\delta\Phi^A} \ .$$

The last term of (0.2.21) identically vanishes because of symmetry properties. The final answer is thus

$$s_\psi^2(X(\Phi^A)) = \frac{\delta^R W_\psi}{\delta\Phi^C}\frac{\delta^L\delta^R W}{\delta\Phi^*_C\delta\Phi^*_A}\frac{\delta^L X}{\delta\Phi^A} ,$$

which means that generically the gauge-fixed action is only nilpotent on-shell, i.e. when the gauge-fixed equations of motion

$$\frac{\delta^R W_\psi}{\delta\Phi^C} = 0$$

are satisfied.

0.2.4 Example of a gauge-fixing fermion

Let us now illustrate this construction on an example. We will consider the Yang-Mills theory.

The set of fields is given by $\{A^a_\mu\}$, and their equations of motion are obtained from the action

$$S(A^a_\mu) = -\frac{1}{4}\int d^dx F^a_{\mu\nu}F^{\mu\nu}_a , \tag{0.2.22}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{bc}{}^a A^b_\mu A^c_\nu ,$$

and the structure functions $f_{bc}{}^a$ satisfy the Jacobi identity

$$f_{ab}{}^e f_{ec}{}^d + f_{bc}{}^e f_{ea}{}^d + f_{ca}{}^e f_{eb}{}^d = 0 .$$

The equations of motion read

$$D^\mu{}_a{}^b F_{b\mu\nu} = 0 ,$$

where

$$D^\mu{}_a{}^b \equiv \delta_a{}^b\partial^\mu + f_{ca}{}^b A^{c\mu} .$$

The equations of motion and the action are invariant under the gauge transformation

$$A^a_\mu \to A^a_\mu + D^a_{\mu b}\Lambda^b ,$$

where Λ^b is an arbitrary function and

$$D_\mu{}^a{}_b \equiv \delta_b{}^a\partial_\mu - f_{cb}{}^a A^c_\mu .$$

There are no reducibility relations.

The gauge invariance of the action implies the Noether identity

$$D^\nu{}_c{}^a D^\mu{}_a{}^b F_{b\mu\nu} = 0 .$$

Following the antifield construction, one introduces a ghost C^a for each gauge parameter Λ^a. For each field and ghost, one further adds an antifield. The set of fields (including ghosts) and antifields is thus $\{A^a_\mu, C^a\}$ and $\{A^{*\mu}_a, C^*_a\}$. Thier parity and ghost numbers are

$$\begin{array}{llll} \epsilon(A^a_\mu) = 0 \ , & \epsilon(C^a) = 1 \ , & \epsilon(A^{*\mu}_a) = 1 \ , & \epsilon(C^*_a) = 0 \ , \\ gh(A^a_\mu) = 0 \ , & gh(C^a) = 1 \ , & gh(A^{*\mu}_a) = -1 \ , & gh(C^*_a) = -2 \ . \end{array}$$

The BRST action reads

$$W = \int d^dx \Big(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + A^*_{a\mu} D^{\mu a}{}_b C^b + \frac{1}{2} C^*_c f_{ab}{}^c C^b C^a \Big) \ . \qquad (0.2.23)$$

One can check that W satisfies the master equation

$$(W, W) = 0 \ .$$

The action of the BRST differential on the various fields gives

$$\begin{aligned} sA^{a\mu} &= D^{\mu a}{}_b C^b \ , \\ sC^a &= -\frac{1}{2} f_{bc}{}^a C^c C^b \ , \\ sA^*_{a\mu} &= D^\nu{}_a{}^b F_{b\nu\mu} - A^*_{c\mu} f_{ab}{}^c C^b \ , \\ sC^*_a &= D^\mu{}_a{}^b A^*_{b\mu} - C^*_c f_{ab}{}^c C^b \ . \end{aligned}$$

Let us now fix the gauge invariance of the action (0.2.23). Since this theory is irreducible, one just introduces one pair of antighosts: $\{\bar{C}_a, \bar{\pi}_a\}$ (note that $\bar{\pi}_a$ is often denoted by B_a and called B-field), with

$$\epsilon(\bar{C}_a) = 1 \ , \quad gh(\bar{C}_a) = -1 \ , \quad \epsilon(\bar{\pi}_a) = 0 \ , \quad gh(\bar{\pi}_a) = 0 \ ,$$

as well as the corresponding antifields, $\{\bar{C}^{*a}, \bar{\pi}^{*a}\}$, with

$$\epsilon(\bar{C}^{*a}) = 0 \ , \quad gh(\bar{C}^{*a}) = 0 \ , \quad \epsilon(\bar{\pi}^{*a}) = 1 \ , \quad gh(\bar{\pi}^{*a}) = -1 \ .$$

The BRST transformations are now generated by the BRST action

$$W = \int d^dx \, [-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + A^*_{a\mu} D^{\mu a}{}_b C^b + \frac{1}{2} C^*_c f_{ab}{}^c C^b C^a - \bar{\pi}_a \bar{C}^{*a}] \ ,$$

which is (0.2.23) where we have added the trivial term $\bar{\pi}_a \bar{C}^{*a}$. The BRST transformations of the antighosts and their antifields is

$$s\,\bar{C}_a = \bar{\pi}_a \ , \quad s\,\bar{\pi}_a = 0 \ , \quad s\,\bar{C}^{*a} = 0 \ , \quad s\,\bar{\pi}^{*a} = \bar{C}^{*a} \ .$$

The gauge-fixing fermion can for example be chosen to be

$$\psi = \int d^dx \, \bar{C}_a (\bar{\pi}^a / 2\xi + \partial^\mu A^a_\mu) \ ,$$

where ξ is a parameter. The gauge-fixing conditions (0.2.19) then read

$$A^*_{a\mu} = -\partial_\mu \bar{C}_a \ , \quad C^*_a = 0 \ , \quad \bar{C}^{*a} = \bar{\pi}^a/2\xi + \partial^\mu A^a_\mu \ , \quad \bar{\pi}^*_a = \bar{C}_a/2\xi \ .$$

The gauge-fixed action is thus

$$W_\psi = \int d^dx \, [-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \partial_\mu \bar{C}_a D^{\mu a}{}_b C^b - (\bar{\pi}^a/2\xi + \partial^\mu A^a_\mu)\bar{\pi}_a] \ .$$

It is invariant under the gauge-fixed BRST transformation

$$s_\psi A^{a\mu} = D^{\mu a}{}_b C^b \ , \quad s_\psi C^a = \frac{1}{2} f_{bc}{}^a C^c C^b \ , \quad s_\psi \bar{C}_a = \bar{\pi}_a \ , \quad s_\psi \bar{\pi}_a = 0 \ .$$

One can check that this transformation squares to zero (as a consequence of the fact that the algebra is closed).

Note that the field $\bar{\pi}_a$ is an auxiliary field: one can solve its equations of motion to eliminate it. Indeed, the latter read

$$-\bar{\pi}^a/\xi - \partial^\mu A^a_\mu = 0 \, ,$$

or equivalently

$$\bar{\pi}^a = -\xi \partial^\mu A^a_\mu \, .$$

Inserting this expression for $\bar{\pi}^a$ into the action yields

$$W_\psi = \int d^dx \, [-\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \partial_\mu \bar{C}_a D^{\mu a}{}_b C^b + \tfrac{1}{2} \, \xi \partial^\mu A^a_\mu \partial^\nu A_{a\nu}] \ .$$

The latter action is known as the Faddeev-Popov action for Yang-Mills theories. It is invariant under the BRST symmetry

$$s_\psi A^{a\mu} = D^{\mu a}{}_b C^b \ , \quad s_\psi C^a = \frac{1}{2} f_{bc}{}^a C^c C^b \ , \quad s_\psi \bar{C}^a = -\xi \partial^\mu A^a_\mu \ ,$$

which is obtained from the previous one by using the equations of motion for $\bar{\pi}_a$ to eliminate the latter field. This symmetry is nilpotent only on-shell:

$$s^2_\psi \bar{C}^a = -\xi \partial_\mu D^{\mu a}{}_b C^b \propto \frac{\delta S_\psi}{\delta \bar{C}^a} \, .$$

verifier signes

0.2.5 Is the path integral independent of the gauge fixing?

We now want to use the gauge-fixed action to quantize our gauge theory in the path integral approach. There are no more problems of infinities due to infinite gauge orbits since there is no gauge invariance left. However, one must check that the computations do not depend on the gauge-fixing one chose. That is the point of this section.

The quantum quantities computed in the path integral approach are

$$I_\psi(X) = \int \mathcal{D}\Phi\mathcal{D}\Phi_*\, \delta(\Phi_*^A - \frac{\delta\psi}{\delta\Phi_A})\, e^{\frac{i}{\hbar}W^q[\Phi,\Phi_*]}\, X[\Phi,\Phi_*]\ ,$$

where X is the correlation function of interest. Note that W^q is not the BRST action W obtained in the previous sections, but its quantum generalization[3]. It coincides with W in the limit $\hbar \to 0$:

$$W^q = W + \sum_{n=1}^{\infty} \hbar^n W_n\ .$$

Let us define $\mathcal{I}_\psi$ as the integrand:

$$\mathcal{I}[\Phi,\Phi_*] \equiv e^{\frac{i}{\hbar}W^q[\Phi,\phi_*]}\, X[\Phi,\Phi_*]\ .$$

The variation of $I_\psi(X)$ under an infinitesimal change of ψ , is

$$\begin{aligned} I_{\psi+\delta\psi}(X) - I_\psi(X) &= \int \mathcal{D}\Phi\, \Big(\mathcal{I}[\Phi, \frac{\delta(\psi+\delta\psi)}{\delta\Phi}] - \mathcal{I}[\Phi, \frac{\delta\psi}{\delta\Phi}]\Big) \\ &= \int \mathcal{D}\Phi\, \Big(\frac{\delta^L(\delta\psi)}{\delta\Phi^A}\frac{\delta^L\mathcal{I}}{\delta\Phi^*_A} + \mathcal{O}((\delta\psi)^2)\Big) \\ &= \int \mathcal{D}\Phi\, \Big(\delta\psi\, \Delta\mathcal{I} + \mathcal{O}((\delta\psi)^2)\Big)\ , \end{aligned}$$

where

$$\Delta \equiv (-)^{\epsilon_A}\frac{\delta^L}{\delta\Phi^A}\frac{\delta^L}{\delta\Phi^*_A}\ . \tag{0.2.24}$$

(Some properties of this operator are given in the appendix B .) So the integral $I_\psi(X)$ is infinitesimally independent of ψ iff

$$\Delta\mathcal{I}(X) = 0\ .$$

This equation is not only a condition on the operators X of the correlation functions that one wants to compute, but also a condition on the quantum action W^q . Indeed, let us consider the simplest case, $X = 1$. The above equation reads

$$0 = \Delta e^{\frac{i}{\hbar}W^q} = e^{\frac{i}{\hbar}W^q}\Big(\frac{i}{\hbar}\Delta W^q - \frac{1}{2\hbar^2}(W^q,W^q)\Big)\ .$$

It thus implies

$$\frac{1}{2}(W^q,W^q) = i\hbar\Delta W^q\ ,$$

which is called the "quantum master equation". This equation is a consistency equation for the quantum extension of the BRST action, which

[3]The presence of the additionnal terms can be interpreted as a requirement to make the measure BRST-invariant.

ensures that the path integral measure is independent of the choice of the gauge-fixing fermion.

Decomposing it in powers of $\hbar$, the quantum master equation reads

$$(W,W)=0\ ,\quad (W_1,W)=i\Delta W\ ,$$
$$(W_p,W)=i\Delta W_{p-1}-\frac{1}{2}\sum_{n=1}^{p-1}(W_n,W_{p-n})\ ,\ p\geq 2\,.$$

The first of these equations is satisfied automatically, it is the classical master equation (**??**).

The second equation can be seen as an equation to determine W_1. It has a solution iff ΔW is s-exact, $\Delta W=sX$ for some X. In other words, ΔW must be in the trivial cohomology class of $H^1(s)$. If the latter group is trivial, then one can always find W_1. If it is not trivial, i.e. $H^1(s)\neq 0$, then there can be obstructions to the existence of W_1, and one must compute ΔW explicitely to check in which equivalence class it is. If ΔW is in a nontrivial class of $H^1(s)$, then W_1 (and thus the quantum completion of W) does not exist. One says that the theory has an anomaly.

A Left and right derivatives

The following formulae involving left and right derivatives are very useful in the calculations.

The link between left and right derivatives of a function X is

$$\frac{\delta^R X}{\delta\phi}=(-)^{\epsilon_\phi(\epsilon_X+1)}\frac{\delta^L X}{\delta\phi}\ .$$

The commutation relations between derivatives are given by

$$\frac{\delta^R\delta^R X}{\delta\phi^A\phi^B}=(-)^{\epsilon_A\epsilon_B}\frac{\delta^R\delta^R X}{\delta\phi^B\phi^A}\tag{A25}$$
$$\frac{\delta^L\delta^R X}{\delta\phi^A\phi^B}=\frac{\delta^R\delta^L X}{\delta\phi^B\phi^A}\tag{A26}$$
$$\frac{\delta^L\delta^L X}{\delta\phi^A\phi^B}=(-)^{\epsilon_A\epsilon_B}\frac{\delta^L\delta^L X}{\delta\phi^B\phi^A}\tag{A27}$$

The derivation chain rule reads

$$\frac{\delta^R X(Y(\phi))}{\delta\phi}=\frac{\delta^R X}{\delta Y}\frac{\delta^R Y}{\delta\phi}\ ,\quad \frac{\delta^L X(Y(\phi))}{\delta\phi}=\frac{\delta^L Y}{\delta\phi}\frac{\delta^L X}{\delta Y}\tag{A28}$$

B Some properties of the operator Δ

The operator Δ defined by Eq.(0.2.24) is of odd grassmannian parity and its ghost number is one,

$$\epsilon(\Delta) = 1 \ , \quad gh(\Delta) = 1 \ .$$

It is nilpotent

$$\Delta^2 = 0 \ .$$

Its action on an antibracket and a pointwise product yields

$$\begin{aligned} \Delta(\alpha, \beta) &= (\Delta\alpha, \beta) - (-)^{\epsilon_\alpha}(\alpha, \Delta\beta) \ , \\ \Delta(\alpha\beta) &= (\Delta\alpha)\beta + (-)^{\epsilon_\alpha}\alpha(\Delta\beta) + (-)^{\epsilon_\alpha}(\alpha, \beta) \ . \end{aligned} \tag{B29}$$

The proofs of these equations are straightforward by using the definitions and the rules of Appendix A .

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