

β function and asymptotic freedom in QCD

Lecture notes
Modave Summer School in Mathematical Physics
June 2005

Glenn Barnich*

*Physique Théorique et Mathématique
Université Libre de Bruxelles
and
International Solvay Institutes
Campus Plaine C.P. 231, B-1050 Bruxelles
Belgique*

ABSTRACT. Prerequisites for the lectures were a first course on quantum field theory including the path integral representation for Green's functions. The aim of the course was a self-contained computation of the 1-loop β in QCD and the interpretation of this result in terms of the high energy behavior of quarks. These lecture notes highlight the main results and serve as a guide to standard literature where all the intermediate steps of the computations can be found.

* Maître de Recherches du Fonds National Belge de la Recherche Scientifique

Contents

1	Effective action at one loop	3
1.1	Legendre transform	3
1.2	Semi-classical expansion	3
1.3	Structure of divergences	4
1.4	Symmetries	5
1.5	Background fields	5
1.6	Guide to the literature	5
2	Asymptotic behavior	6
2.1	Dilatation invariance	6
2.2	Callan-Symanzik equation	6
2.3	High energy behavior and massless theory	8
2.4	Running coupling constant	8
3	Quantum gauge fields	10
	References	11

1 Effective action at one loop

1.1 Legendre transform

The generating functional for Green's function $Z(J)$ admits the following path integral representation

$$Z(J) = \int \mathcal{D}\phi \exp \frac{i}{\hbar} (S[\phi] + J \cdot \phi), \quad (1)$$

where $J \cdot \phi \equiv \int d^n x J_A(x) \phi^A(x)$ with $J_A(x)$ external sources and $\phi^A(x)$ the quantum fields. For simplicity, we consider in the first two sections the case of a single real scalar field ϕ with action

$$S = \int d^d x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) + i\epsilon \text{ terms} \right]. \quad (2)$$

If $S = S_0 + S_1$ with $S_0 = -\frac{1}{2} \int d^d x \int d^d y \mathcal{D}(x, y) \phi(x) \phi(y)$, and $S_1 = -\int d^d x V[\phi]$, the $i\epsilon$ terms implement the boundary conditions and guarantee that the propagator, the inverse of the quadratic part of the action, coincides up to normalization with the two point function of the free theory,

$$\frac{\hbar}{i} \mathcal{D}^{-1}(x, y) = \langle 0 | T \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle. \quad (3)$$

After introducing $W[J]$ defined by

$$\exp \frac{\hbar}{i} W[J] = \frac{Z[J]}{Z[0]}, \quad (4)$$

which can be shown to be the generating functional of connected Green's functions, the effective action is defined as the Legendre transform of $W[J]$ with respect to J :

$$\phi = \frac{\delta W}{\delta J} \iff J = J[\phi], \quad \Gamma[\phi] = (W[J] - J \cdot \phi)|_{J[\phi]}. \quad (5)$$

1.2 Semi-classical expansion

Expanding the path integral around the unique classical solution $\phi_0[J]$ of the equation

$$\frac{\delta S}{\delta \phi} = -J, \quad (6)$$

one finds in a first step

$$W[J] = S[\phi_0[J]] + J \cdot \phi_0[J] - \frac{\hbar}{2i} \text{tr} \ln \left(\delta^4(x, y) + \mathcal{D}^{-1}(x, y) V''[\phi_0[J](y)] \right) + O(\hbar^2). \quad (7)$$

After performing the Legendre transformation, one gets $\Gamma[\phi] = S[\phi] + \hbar \Gamma^{(1)}[\phi] + O(\hbar^2)$, where

$$\Gamma^{(1)}[\phi] = -\frac{1}{2i} \text{tr} \ln \left(\delta^4(x, y) + \mathcal{D}^{-1}(x, y) V''[\phi(y)] \right). \quad (8)$$

Expanding the logarithm as a power series and using the Fourier representation of the propagator gives, after a change of variables,

$$\begin{aligned} \Gamma^{(1)}[\phi] &= \frac{1}{2i} \sum_{n=1} \frac{(-)^n}{n} \int d^d z_1 \dots \int d^d z_n V''[\phi(z_1)] \dots V''[\phi(z_n)] \\ &\int \frac{d^d q_2}{(2\pi)^d} \dots \int \frac{d^d q_n}{(2\pi)^d} \exp -iz_1(q_2 + \dots + q_n) \exp iz_2 q_2 \dots \exp iz_n q_n \gamma^{(n)}(q_2, \dots, q_n), \quad (9) \\ &= \frac{1}{2i} \sum_{n=1} \frac{(-)^n}{n} \int d^d z_1 \dots \int d^d z_n V''[\phi(z_1)] \dots V''[\phi(z_n)] \\ &\int \frac{d^d q_1}{(2\pi)^d} \dots \int \frac{d^d q_n}{(2\pi)^d} \exp iq_1 z_1 \exp iz_2 q_2 \dots \exp iz_n q_n \tilde{\gamma}^{(n)}(q_1, \dots, q_n), \quad (10) \end{aligned}$$

where

$$\gamma^{(n)}(q_2, \dots, q_n) = \int \frac{d^d q_n}{(2\pi)^d} \frac{1}{q^2 + m^2 - i\epsilon} \frac{1}{(q + q_2)^2 + m^2 - i\epsilon} \cdots \frac{1}{(q + q_2 + \cdots + q_n)^2 + m^2 - i\epsilon} \quad (11)$$

while

$$\tilde{\gamma}^{(n)}(q_1, \dots, q_n) = (2\pi)^4 \delta^4(q_1 + \cdots + q_n) \gamma^{(n)}(q_2, \dots, q_n). \quad (12)$$

1.3 Structure of divergences

If one goes to Euclidean momentum and uses spherical coordinates, one finds for large radius κ , i.e., when all momenta go to infinity together with a common factor, that $\gamma^{(n)} \sim \kappa^{d-1-2n}$, so that the integrals are convergent for $n > d/2$.

In 4 dimensions, only $\gamma^{(1)}$ and $\gamma^{(2)}(q_2)$ diverge. Furthermore, if one differentiates $\gamma^{(2)}$ with respect to q_2 , the result converges, so that only the q_2 independent part of $\gamma^{(2)}(q_2)$ diverges. One can thus write $\gamma^{(1)} = A + \gamma_{finite}^{(1)}$, $\gamma^{(2)}(q_2) = B + \gamma_{finite}^{(2)}(q_2)$, with A, B infinite constants. The one loop divergent part of the effective action thus becomes

$$\Gamma_\infty^{(1)}[\phi] = \int d^d x \left(-V''[\phi(x)] \frac{A}{2i} + (V''[\phi(x)])^2 \frac{B}{4i} \right). \quad (13)$$

It follows that the divergences at order 1 in \hbar are

- local functionals, i.e., integrals of polynomials in the fields and their derivatives, like the action itself;
- the effective action, and thus also all other Green's functions, can be made finite at first order by adding divergent counterterms to the action;
- if the canonical dimension of the two fields differentiating the potential is strictly bigger than 1 and if the potential is of canonical dimension less or equal to 4, then the canonical dimension of the first order divergences is less or equal to 4.

Furthermore, for $V(\phi) = g \frac{\phi^4}{4!}$, the theory can be made finite at one loop, by requiring $m_R = m - \hbar \frac{gA}{2i}$, $g_R = g + \hbar \frac{gB}{16i}$ to be finite.

At higher loops in order to get a finite theory, one needs to renormalize the wave function as well, $\phi_R = Z^{-1/2} \phi$ with $Z = 1 + \hbar(\text{divergent term})$. In fact, one can show that with appropriately chosen m_R, g_R, ϕ_R , the effective action can be made finite to all orders in \hbar . The finite theory is then defined through the renormalized effective action

$$\Gamma_R[\phi_R, m_R, g_R] = \lim_{\Lambda \rightarrow \infty} \Gamma_\Lambda[Z^{1/2}(\Lambda, m_R, g_R) \phi_R, m(\Lambda, m_R, g_R), g(\Lambda, m_R, g_R)]. \quad (14)$$

For $V(\phi) = g \frac{\phi^4}{4!}$ in 4 dimensions, an explicit computation of the divergent part using a cut-off Λ in the radial part of spherical coordinates for the Euclidean four-momentum gives

$$g_R = g - \hbar \frac{3g^2}{32\pi^2} \ln \frac{\Lambda}{m} + O(\hbar^2) \iff g = g_R + \hbar \frac{3g_R^2}{32\pi^2} \ln \frac{\Lambda}{m} + O(\hbar^2). \quad (15)$$

1.4 Symmetries

Let $\delta_Q \phi^A = Q^A[\phi]$ be a linear symmetry of the action,

$$0 = \delta_Q S = \int d^d x Q^A[\phi] \frac{\delta S}{\delta \phi^A(x)}. \quad (16)$$

If the path integral measure can be assumed to be invariant, one can prove, by a shift of integration variables, the Ward identities

$$J \cdot Q\left[\frac{\delta}{\delta J}\right] Z[J] = 0 = J \cdot Q\left[\frac{\delta}{\delta J}\right] W[J]. \quad (17)$$

For the effective action, this becomes

$$\delta_Q \Gamma = \int d^d x Q^A[\phi] \frac{\delta \Gamma}{\delta \phi^A(x)} = 0. \quad (18)$$

Under the above assumptions, one thus finds that the symmetries constrain the effective action, and also the one loop divergences, in the same way than they constrain the classical action.

1.5 Background fields

Consider additional external sources, the background fields $\tilde{\phi}^A(x)$ and define

$$\tilde{Z}[J, \tilde{\phi}] = \int \mathcal{D}\phi \exp \frac{i}{\hbar} (S[\phi + \tilde{\phi}] + J \cdot \phi), \quad (19)$$

$$\tilde{W}[J, \tilde{\phi}] = \frac{\hbar}{i} \ln \frac{\tilde{Z}[J, \tilde{\phi}]}{\tilde{Z}[0, 0]}. \quad (20)$$

The Legendre transform on J is now performed in the presence of the background fields, which are passive,

$$\phi_{J, \tilde{\phi}}^A(x) = \frac{\delta \tilde{W}[J, \tilde{\phi}]}{\delta J_A(x)} \Leftrightarrow J_A(x) = J_A[\phi, \tilde{\phi}](x), \quad (21)$$

$$\tilde{\Gamma}[\phi, \tilde{\phi}] = W[J[\phi, \tilde{\phi}], \tilde{\phi}] - J[\phi, \tilde{\phi}] \cdot \phi. \quad (22)$$

By shifting integration variables in (53) from ϕ to $\phi - \tilde{\phi}$, one can prove that

$$\tilde{\Gamma}[\phi, \tilde{\phi}] = \Gamma[\phi + \tilde{\phi}], \quad (23)$$

and in particular, that $\Gamma[\tilde{\phi}] = \Gamma[0, \tilde{\phi}]$, i.e., the effective action can be obtained by computing only vacuum graphs, but in the presence of background fields.

1.6 Guide to the literature

The computation of the one loop contribution to the effective action can be found in chapter 9-2-2 of [1]. The structure of divergences at one loop order are discussed in chapter 9.1 of [2]. Background fields are briefly discussed in chapter 16 of [3] and with more details in [4]. Copies of detailed lecture notes (in french) for this part, with prerequisites and intermediate computations, can be found in [5], chapter 2 and in [6]. In the next section, we follow chapter 13 of [1].

2 Asymptotic behavior

2.1 Dilatation invariance

Consider a change of scale,

$$x' = \lambda x, \quad (24)$$

$$\phi^{A'}(x') = \lambda^{-\Delta^A} \phi^A(x). \quad (25)$$

The action

$$S = \int d^4x \mathcal{L}, \quad \mathcal{L} = -\left[\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} \phi^4\right]. \quad (26)$$

is invariant under such a transformation, provided $\Delta_\phi = 1$ and $m = 0$. The associated infinitesimal transformation for decreasing scale is

$$\delta\phi = \left(1 + x \cdot \frac{\partial}{\partial x}\right)\phi. \quad (27)$$

In the presence of a mass term, one finds

$$\delta S + m \frac{\partial}{\partial m} S = 0. \quad (28)$$

Because the transformation of the fields is linear, one naïvely expects, by repeating the reasoning that leads to the Ward identities, the effective action to satisfy the same relation,

$$\delta\Gamma + m \frac{\partial}{\partial m} \Gamma = 0. \quad (29)$$

We will now see how renormalization, i.e., the process of absorbing divergences, leads to quantum corrections.

2.2 Callan-Symanzik equation

In order to respect conventional dimensional analysis, one can adjust the finite part of the subtraction so that

$$Z(\Lambda, m_R, g_R) = Z\left(\frac{\Lambda}{m_R}, g_R\right), \quad g(\Lambda, m_R, g_R) = g\left(\frac{\Lambda}{m_R}, g_R\right), \quad \frac{m}{m_R}(\Lambda, m_R, g_R) = \frac{m}{m_R} \left(\frac{\Lambda}{m_R}, g_R\right), \quad (30)$$

and similarly for the renormalized in terms of the bare quantities. Defining $\Gamma_{\Delta, \Lambda}[\phi, m, g] = \frac{1}{2} m \frac{\partial \Gamma_\Lambda}{\partial m}[\phi, m, g]$ and varying m with Λ, g, ϕ fixed in (14), one finds

$$\lim_{\Lambda \rightarrow \infty} 2 \frac{dm}{m} \Gamma_{\Delta, \Lambda}[\phi, m, g] = \left[dm_R \frac{\partial}{\partial m_R} + dg_R \frac{\partial}{\partial g_R} + \int d^4x \delta\phi_R(x) \frac{\delta}{\delta\phi_R(x)} \right] \Gamma_R[\phi_R, m_R, g_R]. \quad (31)$$

As $\Gamma_R[\phi_R, m_R, g_R]$ is of power counting dimension zero, (there is no longer any Λ which changes the dimensional analysis), one also has

$$\left[dm_R \frac{\partial}{\partial m_R} + \int d^4x \left(1 + x \cdot \frac{\partial}{\partial x}\right) \phi_R(x) \frac{\delta}{\delta\phi_R(x)} \right] \Gamma_R[\phi_R, m_R, g_R] = 0. \quad (32)$$

Using $\delta\phi(x) = 0 = \frac{1}{2}Z^{-1/2}dZ\phi_R(x) + Z^{1/2}\delta\phi_R(x)$ which gives $\delta\phi_R(x) = -\frac{1}{2}d(\ln Z)\phi_R(x)$, multiplying (31) by $\frac{m_R}{dm_R}$ and substituting (32), one finds

$$\begin{aligned} \lim_{\Lambda \rightarrow \infty} 2 \frac{\partial m}{\partial m_R} \frac{m_R}{m} \Gamma_{\Delta, \Lambda}[\phi, m, g] &= \\ &= \left[m_R \frac{\partial g_R}{\partial m_R} \frac{\partial}{\partial g_R} - \int d^4x \left(1 + \frac{1}{2} m_R \frac{\partial \ln Z}{\partial m_R} + x \cdot \frac{\partial}{\partial x} \right) \phi_R(x) \frac{\delta}{\delta \phi_R(x)} \right] \Gamma_R[\phi_R, m_R, g_R]. \end{aligned} \quad (33)$$

To simplify the RHS, one defines

$$\beta(g_R) = m_R \frac{\partial g_R}{\partial m_R} = -\Lambda \frac{\partial g_R}{\partial \Lambda}, \quad \gamma(g_R) = \frac{1}{2} m_R \frac{\partial \ln Z}{\partial m_R} = -\frac{1}{2} \Lambda \frac{\partial \ln Z}{\partial \Lambda} \quad (34)$$

where the limit $\Lambda \rightarrow \infty$ is understood.

To understand the LHS, one couples to the starting point action $\phi^2(x)$ with an external source $K(x)$,

$$S_K = - \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{g}{4!} \phi^4 + \frac{1}{2} m^2 (1 + K(x)) \phi^2(x) \right] \quad (35)$$

which gives $\Gamma_{\Delta, \Lambda} = \frac{1}{2} m \frac{\partial \Gamma_\Lambda}{\partial m} = \int d^4x \frac{\delta \Gamma_{\Lambda, K}}{\delta K(x)} \Big|_{K=0}$. In the computation of $\Gamma^{(1)}[\phi]$, there is a new term because $V_K''[\phi(x)] = \frac{g}{2} \phi^2(x) + m^2 K(x)$. We thus have

$$\Gamma_{\infty, K}^{(1)} = \int d^4x -V_K''[\phi(x)] \frac{A}{2i} + (V_K''[\phi(x)])^2 \frac{B}{4i} \quad (36)$$

with A, B divergent constants, which gives

$$\frac{\delta \Gamma_{\infty, K}^{(1)}}{\delta K(x)} \Big|_{K=0} = -m^2 \frac{A}{2i} + gm^2 \phi^2(x) \frac{B}{4i}. \quad (37)$$

The first term can be absorbed by a term of the form $\int d^4x \nu K(x)$ in the Lagrangian. We suppose that one uses normalization conditions that guarantee the absence of this term in the renormalized effective action, cf. absence of tadpoles. The second term can be absorbed by a redefinition of the source $K(x)$: $K(x) = Z_K K_R(x)$ with $Z_K = (1 - \hbar g \frac{B}{4i})$. This implies that the part linear in $K(x)$ of $\Gamma_{\Lambda, K}[Z_K K_R(x), Z^{1/2} \phi_R(x), m(), g()] = \Gamma_{R, K}[K_R, \phi_R, m_R, g_R]$ is finite as well at 1-loop for $\Lambda \rightarrow \infty$. One then defines

$$\begin{aligned} \Gamma_{R, \Delta}[\phi_R, m_R, g_R] &= \int d^4x \frac{\delta \Gamma_{R, K}}{\delta K_R(x)} \Big|_{K_R=0} = \\ &= \int d^4x Z_K \frac{\delta \Gamma_{\Lambda, K}}{\delta K(x)} \Big|_{K=0} = Z_K \Gamma_{\Lambda, \Delta}[Z^{1/2} \phi_R, m(), g()], \end{aligned} \quad (38)$$

The LHS of (33) becomes $2 \frac{\partial m}{\partial m_R} \frac{m_R}{m} \frac{1}{Z_K} \Gamma_{R, \Delta}[\phi_R, m_R, g_R]$. If one defines

$$1 + \delta(g_R) = \frac{1}{Z_K} \frac{m_R}{m} \frac{\partial m}{\partial m_R} = -\frac{1}{Z_K} \frac{\Lambda}{m} \frac{\partial m}{\partial \Lambda} \quad (39)$$

equation (33) writes as

$$\begin{aligned} 2(1 + \delta(g_R)) \Gamma_{R, \Delta}[\phi_R, m_R, g_R] &= \\ &= \left[\beta(g_R) \frac{\partial}{\partial g_R} - \int d^4x \left(1 + \gamma(g_R) + x \cdot \frac{\partial}{\partial x} \right) \phi_R(x) \frac{\delta}{\delta \phi_R(x)} \right] \Gamma_R[\phi_R, m_R, g_R]. \end{aligned} \quad (40)$$

This is the Callan-Symanzik equation. The naïve relation is thus violated by the terms containing $\delta(g_R), \gamma(g_R), \beta(g_R)$ which are of order \hbar . They are due to the fact that the relation between renormalized and bare quantities, needed in order to absorb the divergences, involves the cut-off Λ .

2.3 High energy behavior and massless theory

In terms of Fourier transforms, the Callan-Symanzik reads

$$2(1 + \delta(g_R))\tilde{\Gamma}_{R,\Delta}(0; p_1, \dots, p_n) = [\beta(g_R)\frac{\partial}{\partial g_R} - n(1 + \gamma(g_R)) - \sum_{k=1}^n p_k \cdot \frac{\partial}{\partial p_k}]\tilde{\Gamma}_R(p_1, \dots, p_n). \quad (41)$$

At tree level, one can check that the terms involving m^2 vanish on their own. Because

$$\tilde{\Gamma}^{(1)}(p_1, \dots, p_{2n}) \sim \tilde{\gamma}^{(n)}(p_1 + p_2, \dots, p_{2n-1} + p_{2n}), \quad (42)$$

$$\tilde{\Gamma}_{\Delta}^{(1)}(0; p_1, \dots, p_{2n}) \sim \tilde{\gamma}^{(n+1)}(0; p_1 + p_2, \dots, p_{2n-1} + p_{2n}), \quad (43)$$

one can then show that, if all external momenta grow large together, $p \rightarrow \lambda p$,

$$\tilde{\Gamma}^{(1)}(\lambda p_1, \dots, \lambda p_{2n}) = O\left(\frac{1}{\lambda^{2(n-1)+4}}\right), \quad (44)$$

$$\tilde{\Gamma}_{\Delta}^{(1)}(0; \lambda p_1, \dots, \lambda p_{2n}) = O\left(\frac{1}{\lambda^{2n+4}}\right). \quad (45)$$

Hence, the LHS of (41) decreases faster than the RHS. Because this property is not affected by renormalization, one concludes, by denoting by $\tilde{\Gamma}_R^{as}(p_1, \dots, p_n)$ the dominant behavior of $\tilde{\Gamma}_R(p_1, \dots, p_n)$ at high energy, that

$$[\beta(g_R)\frac{\partial}{\partial g_R} - n(1 + \gamma(g_R)) - \sum_{k=1}^n p_k \cdot \frac{\partial}{\partial p_k}]\tilde{\Gamma}_R^{as}(p_1, \dots, p_n) = 0. \quad (46)$$

Because usual dimensional analysis applies to the renormalized theory, scaling $p \rightarrow \lambda p$ in $\Gamma_R^{as}(p_i; g_R, m_R)$ is equivalent, up to an overall factor, to scaling $m_R \rightarrow \lambda^{-1}m_R$. One can then consider the Green's functions $\Gamma_R^{as}(p_i; g_R)$ solutions of (46) as defining the Green's functions of the massless theory.

2.4 Running coupling constant

Forgetting about the subscript R and introducing

$$\lambda \frac{d}{d\lambda} g(\lambda) = \beta(g(\lambda)), \quad g(1) = g \iff \lambda = \exp \int_g^{g(\lambda)} \frac{dg'}{\beta(g')}, \quad (47)$$

$$\lambda \frac{d}{d\lambda} z(\lambda) = \gamma(g(\lambda))z(\lambda), \quad z(1) = 1 \iff z(\lambda) = \exp \int_1^\lambda \frac{d\lambda' \gamma(g(\lambda'))}{\lambda'} = \exp \int_g^{g(\lambda)} \frac{dg' \gamma(g')}{\beta(g')}, \quad (48)$$

equation (46), which is valid for all values of the external momenta and of g , can be rewritten as

$$\lambda \frac{d}{d\lambda} [\lambda^{-n} (z(\lambda))^{-n} \tilde{\Gamma}^{as}(\lambda^{-1} p_i; g(\lambda))] = 0, \quad (49)$$

which is equivalent to

$$\tilde{\Gamma}^{as}(\lambda p_i; g) = \lambda^{-n} (z(\lambda))^{-n} \tilde{\Gamma}^{as}(p_i; g(\lambda)). \quad (50)$$

Starting from $x' = \lambda^{-1}x$, $\phi'(x') = \lambda\phi(x)$, one finds in terms of the Fourier transform, $p' = \lambda p$, $\tilde{\phi}'(p') = \lambda^{-3}\tilde{\phi}(p)$. At the quantum level, one defines $\tilde{\phi}'(p') = \lambda^{-3}z(\lambda)\tilde{\phi}(p)$, justifying the name ‘‘anomalous dimension’’ for $z(\lambda)$. In terms of the functional

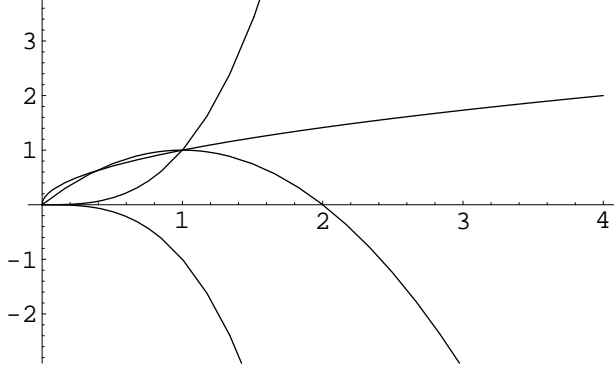
$$\tilde{\Gamma}^{as}[\tilde{\phi}; g] = \sum_{n=2} \frac{1}{n!} \int \frac{d^4 p_1}{(2\pi)^4} \dots \int \frac{d^4 p_n}{(2\pi)^4} \tilde{\phi}(-p_1) \dots \tilde{\phi}(-p_n) \tilde{\Gamma}^{as}(p_1, \dots, p_n; g), \quad (51)$$

equation (50) reads as

$$\tilde{\Gamma}^{as}[\tilde{\phi}'; g] = \tilde{\Gamma}^{as}[\tilde{\phi}'; g(\lambda)]. \quad (52)$$

We have thus shown that dilatation invariance of the massless theory can be restored at the quantum level by the introduction of an anomalous dimension for the field and a coupling ‘‘constant’’ that varies with energy.

The behavior of the coupling constant as a function of the energy is determined by the β -function:



- **Singularity at finite energy:** $\beta(g) > 0$ increases fast enough so that $\int_g^\infty \frac{dg'}{\beta(g')} < \infty$, for instance $\beta(g) = g^3$. In this case, $g(\lambda)$ increases from 0 to ∞ and $g = \infty$ is reached for a finite value of the energy, $\lambda_\infty = \exp \int_g^\infty \frac{dg'}{\beta(g')}$.
- **Singularity at infinite energy:** $\beta(g) > 0$ and $\int_g^\infty \frac{dg'}{\beta(g')} \rightarrow \infty$, for instance, $\beta(g) = \sqrt{g}$. In this case, $g(\lambda)$ increases from 0 to ∞ and reaches $g = \infty$ for an infinite value of the energy, $\lambda_\infty = \infty$.
- **Ultraviolet fixed point:** $\beta(g) > 0$ for $0 < g < g^*$, $\beta(g^*) = 0$, $\beta(g) < 0$ for $g^* < g$, for instance, $\beta(g) = -g(g - 2)$. In this case, $g(\lambda)$ increases for $0 < g < g^*$ and $g(\lambda)$ decreases for $g^* < g$. Whatever side of g^* one starts off, $g(\lambda) \rightarrow g^*$. If the zero of $\beta(g)$ is simple, $\beta(g) \rightarrow a(g - g^*)$ for $g \rightarrow g^*$ with $a > 0$. The solution of $\lambda \frac{d}{d\lambda} g = a(g - g^*)$ is $g(\lambda) - g^* = (g(1) - g^*)\lambda^{-a}$. Assuming that $\gamma(g) \rightarrow \gamma(g^*)$, the anomalous dimension at high energy becomes $z(\lambda) = \lambda^{\gamma(g^*)}$.
- **Asymptotic freedom:** $\beta(g) = -bg^n$, $b > 0$, $n > 2$ for g small. Integrating the differential equation, one finds $g(\lambda) = g[1 + g^{n-1}(n-1)b \ln \lambda]^{-\frac{1}{n-1}}$. We thus have $g(\lambda) \rightarrow 0$ for $\lambda \rightarrow \infty$. In this case, the assumption on which the perturbation theory is based, namely that g is small, is valid at high energy.

For the scalar field with $V[\phi] = \frac{g}{4!}\phi^4$ in 4 dimensions, (15) and the definition of the β function (34) imply that

$$\beta(g_R) = -\Lambda \frac{d}{d\Lambda} g_R = \hbar \frac{3}{16\pi^2} g_R^2 + O(\hbar^2). \quad (53)$$

We are thus in case 1. This also applies to quantum electrodynamics where

$$e_R = e - \hbar \frac{e^3}{12} \ln(\Lambda/\mu) + O(\hbar^2), \quad (54)$$

with μ an infrared cut-off and thus

$$\beta(e_R) = \hbar \frac{e_R^3}{12\pi^2} + O(\hbar^2). \quad (55)$$

For non abelian Yang-Mills theories based on the gauge group $SU(N)$, however, one can show that

$$g_R = g + \hbar \frac{g^3}{4\pi^2} \left(\frac{11}{12} N - \frac{n_f}{6} \right) \ln(\Lambda/\mu) + O(\hbar^2), \quad (56)$$

where n_f is the number of fermions. One thus finds

$$\beta(g_R) = \hbar \frac{g_R^3}{4\pi^2} \left(\frac{n_f}{6} - \frac{11}{12} N \right) + O(\hbar^2). \quad (57)$$

For $n_f \leq 5N$, $\beta(g_R) < 0$ and the case 4 of asymptotic freedom applies. In particular, this holds for quantum chromodynamics, where $n_f = 6$ and $N = 3$. Since $g_R(\lambda) \rightarrow 0$ for $\lambda \rightarrow \infty$, quarks behave as free particles at high enough energies.

3 Quantum gauge fields

The rest of the lectures was devoted to the derivation of (56).

After shortly discussing the classical action for quantum chromodynamics, it was pointed out that the quadratic part of the gluon action is not invertible as a consequence of gauge invariance.

The discussion of gauge fixing and BRST invariance was done in the context of the Batalin-Vilkovisky approach: after introducing ghosts, antifields and an antibracket, one first constructs the proper minimal solution to the Batalin-Vilkovisky master equation. After introducing a non minimal sector that contains the antighosts, auxiliary fields and their antifields, gauge fixing amounts to performing an anticanonical transformation generated by a gauge fixing fermion in such a way that the classical master equation still holds. Formal gauge independence was then shown to be a consequence of this master equation. This approach is reviewed for instance in [7, 8]. Its relation to the more standard BRST approach is reviewed briefly in appendix 2.A of [9].

The background field gauge was then introduced as in chapter 17 of [3]. The computation of the renormalized coupling constant was performed following the same reference: in the background field gauge, the renormalization of the coupling constant and the background field are related due to background gauge invariance. Hence, in the presence of the background fields, the computation can be done by computing vacuum amplitudes. Furthermore, it can be reduced to straightforward matrix algebra by using constant background fields.

References

- [1] C. Itzykson and J. B. Zuber, “Quantum field theory,”. New York, USA: McGraw-hill (1980) 705 P.(International Series In Pure and Applied Physics).
- [2] J. Zinn-Justin, *Quantum field theory and critical phenomena*. International series of monographs on physics. Oxford, UK: Clarendon, third ed., 1996.
- [3] S. Weinberg, “The Quantum Theory of Fields. vol. 2: Modern Applications,”. Cambridge, UK: Univ. Pr. (1996) 489 p.
- [4] L. F. Abbott, “Introduction to the background field method,” *Acta Phys. Polon.* **B13** (1982) 33.
- [5] G. Barnich, “Méthodes de la théorie quantique des champs.” Lecture notes Université Libre de Bruxelles, available online at <http://homepages.ulb.ac.be/%7Egbarnich/methodes.pdf>, 2005.
- [6] C. Schombld, “Théorie quantique des champs.” Lecture notes Université Libre de Bruxelles, available online at <http://homepages.ulb.ac.be/%7Ecschomb/intfonc+QED+QCD.pdf>, 2005.
- [7] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*. Princeton University Press, 1992.
- [8] J. Gomis, J. París, and S. Samuel, “Antibracket, antifields and gauge theory quantization,” *Phys. Rept.* **259** (1995) 1–145, [hep-th/9412228](http://arxiv.org/abs/hep-th/9412228).
- [9] G. Barnich, F. Brandt, and M. Henneaux, “Local BRST cohomology in gauge theories,” *Phys. Rept.* **338** (2000) 439–569, [hep-th/0002245](http://arxiv.org/abs/hep-th/0002245).
- [10] S. Weinberg, “The Quantum Theory of Fields. vol. 1: Foundations,”. Cambridge, UK: Univ. Pr. (1995) 609 p.
- [11] G. Serman, “An introduction to quantum field theory,”. Cambridge, UK: Univ. Pr. (1993) 572 p.
- [12] B. L. Voronov, P. M. Lavrov, and I. V. Tyutin, “Canonical transformations and the gauge dependence in general gauge theories.,” *Sov. J. Nucl. Phys.* **36** (1982) 292.
- [13] P. Di Francesco, P. Mathieu, and D. Senechal, “Conformal field theory,”. New York, USA: Springer (1997) 890 p.
- [14] M. Henneaux, “Physique mathématique II.” Notes de cours, 2ième licence en sciences physique, 1993/94.