Aspects of two-dimensional Conformal Field Theory

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Abstract

These notes constitute the write-up of a four hour lecture given at the First Modave Summer School on Mathematical Physics, held in Modave, 19-25 june 2005. It is intended to introduce basic tools of conformal field theory in two dimensions using as guideline the conformal Ward identities. Comments are welcome!

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Contents

1 Conformal transformations in two dimensions

Consider flat space-time in d dimensions, that is, $\mathbb{R}^d$ endowed with a flat metric $g_{\mu\nu}$ (which will be taken euclidian or lorentzian) and coordinates x^μ, $\mu = 0, \cdots, d-1$. First recall that *Poincaré transformations* are the set of transformations

$$x^\mu \to {x'}^\mu(x) \quad , \quad \mu = 0, \cdots, d-1 \tag{1}$$

leaving the components of the flat metric (with lorentzian signature) unchanged :

$$g_{\mu\nu} \to g'_{\mu\nu}(x'(x)) \stackrel{!}{=} g_{\mu\nu}(x) \quad . \tag{2}$$

This statement can also be expressed from the "active" point of view by demanding that the squared "Minkowskian norm" ds^2 of a vector with components dx^μ be preserved under (1) :

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \to ds'^2 = g_{\mu\nu}\, dx'^\mu dx'^\nu \stackrel{!}{=} ds^2 \quad , \tag{3}$$

that is

$$g_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} = g_{\alpha\beta} \quad . \tag{4}$$

Conformal transformations are defined as the set of transformations (1) leaving the components of the metric tensor invariant *up to a scale* :

$$g_{\mu\nu} \rightarrow g'_{\mu\nu}(x'(x)) = \Lambda(x) g_{\mu\nu} \quad . \tag{5}$$

Condition (5) expresses that the scalar product of basis vectors of the tangent space is conserved only up to a local scale factor (possibly depending on the point). In particular, angles are preserved by conformal transformations. Equivalently, conformal transformations are such that

$$ds^2 \rightarrow ds'^2 \overset{!}{=} \Lambda^{-1}(x) ds^2 \quad . \tag{6}$$

Let us focus on the two-dimensional case. We will be working with an euclidian metric, such that the line element is $(dx^1)^2 + (dx^2)^2$. We introduce complex coordinates

$$z = x^1 + ix^2 \quad \text{and} \quad \overline{z} = x^1 - ix^2 \quad , \tag{7}$$

in which the metric reads $ds^2 = dzd\overline{z}$. Notice it is only in 2 dimensions that the metric in complex coordinates factorizes in dz and $d\overline{z}$. Consequently, any change of coordinates

$$z \rightarrow f(z) \overset{\triangle}{=} z + \alpha(z) \quad , \quad \overline{z} \rightarrow \bar{f}(\overline{z}) \overset{\triangle}{=} \overline{z} + \bar{\alpha}(\overline{z}) \tag{8}$$

with f and $\bar{f}$ depending only upon z and $\overline{z}$ respectively[2] will satisfy (6), because

$$ds^2 \rightarrow ds'^2 = \left(\frac{df}{dz}\right)\left(\frac{d\bar{f}}{d\overline{z}}\right) ds^2 \quad . \tag{9}$$

Actually, these are the only conformal transformations in two dimensions. To verify this, one may consider infinitesimal transformations $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$ in (5), for infinitesimal $\varepsilon^{\mu}(x)$. With $g'_{\mu\nu} \overset{\triangle}{=} g_{\mu\nu} + f(x) g_{\mu\nu}$, one gets, to first order in $\varepsilon^{\mu}(x)$:

$$\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu} = -f(x) g_{\mu\nu} \quad . \tag{10}$$

By taking the trace of (10), one determines $f(x)$, so that we get

$$\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu} = \frac{2}{d}(\partial_{\alpha}\varepsilon^{\alpha}) g_{\mu\nu} \quad , \tag{11}$$

which are called *conformal Killing equations.* Their solution, in two dimensions, expresses that $\varepsilon^{z}(z,\overline{z}) = \varepsilon^{z}(z) \overset{\triangle}{=} \varepsilon(z)$ and $\varepsilon^{\overline{z}}(z,\overline{z}) = \varepsilon^{\overline{z}}(\overline{z}) \overset{\triangle}{=} \bar{\varepsilon}(\overline{z})$, see [3] for a detailed computation.
Now, suppose $\alpha(z)$ in (8) admits a Laurent expansion around say $z = 0$ (we only consider the "holomorphic" sector) :

$$\alpha(z) = \sum_{n=-\infty}^{+\infty} \alpha_n z^{n+1} \quad . \tag{12}$$

[2]It is a common abuse of language to call $f(z)$ (resp. $\bar{f}(\overline{z})$) a holomorphic (resp. anti-holomorphic) function, because it may in general possess singularities. One usually restricts to meromorphic functions, i.e. functions admitting a set of isolated poles

On functions (scalar fields) depending only upon one coordinate (say z), the mappings (8) are generated by differential operators. Indeed, by considering a transformation of the form $x \rightarrow x'$, $\phi(x) \rightarrow \phi'(x')$, the variation of a field is defined as :

$$\delta\phi(x) = \phi'(x) - \phi(x) \stackrel{\triangle}{=} \omega^a G_a \phi(x) \quad , \tag{13}$$

where $\{\omega^a\}$ is a set of infinitesimal parameters, and G_a are the symmetry generators[3]. By considering an infinitesimal version of (8), the effect on a function $f(z)$ is

$$\delta f(z) = -\alpha(z)\partial_z f(z) = \sum_n \alpha_n l_n f(z) \quad , \tag{14}$$

where we used (12) with infinitesimal parameters α_n, and defined the generators

$$l_n = -z^{n+1}\partial_z \quad . \tag{15}$$

They satisfy the following commutation relation

$$[l_n, l_m] = (n-m) l_{n+m} \quad , \tag{16}$$

with analogous definition and commutation relations for the $\bar{l}_n$'s. The infinite Lie algebra defined by (16) is known as the *Witt algebra*. Notice that the infinitesimal transformations resulting from (15) are the most general that are analytic (or holomorphic, in strict sense) *near* the point $z = 0$. They may introduce singularities at $z = 0$, but not branch cuts[4].
An important subset of the transformations (8) consists in transformations defining invertible mappings globally defined on $\mathbb{C} \cup \{\infty\}$, that is, the complex plane *plus* a point at infinity. This space is called the *Riemann sphere*, as it may be compactified (through stereographic projection) to the two-dimensional sphere S^2. These transformations are called *global conformal transformations*.
To determine them, a first argument is based on the form (15) of the generators. Clearly l_n is non-singular at $z = 0$ if $n \geq -1$. To investigate the behavior at ∞, consider the conformal map $z \rightarrow z' = \frac{1}{z}$. Under this transformation, the generator l_n transforms to

$$-z^{n+1}\partial_z \longrightarrow -z'^{-(n+1)} \left(\frac{\partial z'}{\partial z}\right) \partial_{z'} = z'^{(1-n)}\partial_{z'} \quad . \tag{17}$$

This operator is non-singular at $z' = 0$ if $n \leq 1$. The generators (15) are thus defined everywhere on the Riemann sphere if $-1 \leq n \leq 1$.
Let us see to what kind of transformations they correspond. For $n = -1$, the infinitesimal transformation is

$$z \rightarrow z' = z + \delta A \quad , \tag{18}$$

where the infinitesimal parameter α_{-1} is supposed to arise from the variation of a finite complex parameter A. The finite transformation follows from a trivial integration :

$$\frac{dz}{dA} = 1 \Rightarrow \int_z^{z'} dw = \int_0^A dA \Rightarrow z' = z + A \quad . \tag{19}$$

[3]As an example, for translations, the infinitesimal parameters are ω^μ, defined by $x'^\mu = x^\mu + \omega^\mu$ and $\phi'(x') = \phi(x)$, so that $G_\mu = -\partial_\mu$

[4]to admit a Laurent expansion around z_0, a function must be analytic in a crown centered at z_0. If the function has a branch cut at z_0 (like for instance $\log z$ at $z = 0$), then one cannot "close a contour" around z_0 and the Laurent expansion cannot be defined

Recalling the definition of z and going back to the original coordinates x^1 and x^2, this corresponds to

$$x^1 \to x^1 + \mathrm{Re}(A) \quad , \quad x^2 \to x^2 + \mathrm{Im}(A) \quad , \tag{20}$$

and hence to *translations* : $x^\mu \to x^\mu + a^\mu$. For $n = 0$, one finds, with $\alpha_0 = \delta T$:

$$\frac{dz}{dT} = z \Rightarrow \int_z^{z'} \frac{dw}{w} = \int_0^T dT \Rightarrow z' = Bz \quad , B = \mathrm{e}^T \quad . \tag{21}$$

Let us consider two cases : $B = \mathrm{e}^{i\theta}$ and $B = \lambda$. With the first one, we find that the effect on the coordinates (x^1, x^2) is

$$x^\mu \to \Lambda^\mu_\nu \, x^\nu \quad , \quad \Lambda \in SO(2) \quad , \tag{22}$$

thus corresponding to *rotations*. At this point, we just found the (euclidian) Poincaré group in two dimensions, which was expected to be a subgroup of the conformal group. With $B = \lambda$, we find that

$$x^\mu \to \lambda x^\mu \quad , \tag{23}$$

corresponding to *dilations*. Finally, for $n = 1$, we have

$$\frac{dz}{z^2} = -D \Rightarrow z' = \frac{z}{1 + Dz} \quad . \tag{24}$$

This transformation is called *special conformal transformation* (SCT), and can be seen as the combination of an inversion, a translation and an inversion (notice that inversions are conformal, even though they do not have an infinitesimal form).
By combining (19),(21) and (24), one obtains the so-called *global conformal transformations* in two dimensions :

$$z' = \frac{Bz + A}{1 + D(Bz + A)} = \frac{Bz + A}{(1 + BD) + BDz} \quad , \tag{25}$$

whose general form is

$$z' = \frac{az + b}{cz + d} \quad . \tag{26}$$

Notice that (26) has 4 complex parameters, while (25) has only 3. The excessive parameter is not relevant, since we could divide numerator and denominator of (26) by d. It is however common to use this parametrization for the two-dimensional global (or finite) conformal group, with additional condition

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \quad , \tag{27}$$

which leaves us with 3 complex parameters. The parametrization (26) is very convenient because successive transformations correspond to the product of the corresponding matrices. Indeed, it can be checked, by looking at successive transformations of the form

$$z_1 = \frac{a_1 z + b_1}{c_1 z + d_1} \quad , \quad z_2 = \frac{a_2 z_1 + b_2}{c_2 z_1 + d_2} \quad , \tag{28}$$

that the resulting transformation is

$$z_2 = \frac{a' z + b'}{c' z + d'} \quad , \tag{29}$$

where the parameters a',b',c' and d' are given by

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad . \tag{30}$$

This establishes that the global conformal group in two dimensions is equivalent to $\mathrm{Sl}(2,\mathbb{C})/\mathbb{Z}_2$, the quotient coming from the fact that A and $-A \in \mathrm{Sl}(2,\mathbb{C})/\mathbb{Z}_2$ correspond to the same conformal transformation[5].

At this level, it seems interesting to make a point about the status of the pair of variables $(z, \overline{z})$. It is customary to consider (x^1, x^2) as a point in $\mathbb{C}^2$, so that (7) is a mere change of coordinates , $\overline{z}$ being not the complex conjugate of z but rather a distinct complex coordinate. This constitutes a convenient simplification in some situations, where one could for instance completely forget about the $\overline{z}$ (sometimes called *anti-holomorphic*) sector and focus on the z (*holomorphic*) sector, because both sectors are regarded as independent. The global conformal transformations would then be given by

$$z \longrightarrow \frac{az+b}{cz+d} \quad , \quad \overline{z} \longrightarrow \frac{\bar{a}\overline{z}+\bar{b}}{\bar{c}\overline{z}+\bar{d}} \quad , \tag{31}$$

with 8 complex parameters subject to the constraints $ad-bc=1$ and $\bar{a}\bar{d}-\bar{b}\bar{c}=1$, thus corresponding to the group $\mathrm{Sl}(2,\mathbb{C})/\mathbb{Z}_2 \times \mathrm{Sl}(2,\mathbb{C})/\mathbb{Z}_2$. In the complex plane $\mathbb{C}^2$ parameterized by $(z, \overline{z})$, the surface where $\overline{z} = z^*$ will be called the *real surface*, because it corresponds to $(x^1, x^2) \in \mathbb{R}^2$, i.e. to the physical space. Furthermore, conformal transformations send points from the real surface on itself if the condition

$$f^*(z) = \bar{f}(\overline{z}) \tag{32}$$

is satisfied. Obviously, when restricted to the real surface the global conformal group reduces to $\mathrm{Sl}(2,\mathbb{C})/\mathbb{Z}_2$.

Note that in obtaining (26), we assumed that the parameters of the transformations, $\alpha_{-1}, \alpha_0, \alpha_1$, are complex, and we forgot about the $\overline{z}$ dependance (assuming we are on the real surface). An equivalent approach used in literature consists in keeping the $\overline{z}$ dependance, and *then* restrict the transformations

$$z \longrightarrow z + \sum_n \alpha_n z^{n+1} \quad , \quad \overline{z} \longrightarrow \overline{z} + \sum_n \bar{\alpha}_n \overline{z}^{n+1}, \tag{33}$$

with real parameters α_n and $\bar{\alpha}_n$ to the real surface $\overline{z} = z^*$. The transformations preserving the real surface ($\overline{z} = z^* \Rightarrow \overline{z}' = z'^*$) are generated by $(l_n + \bar{l}_n)$ and $i(l_n - \bar{l}_n)$. The global conformal group is thus generated by the 6 generators obtained by taking $n = -1, 0, 1$, which can be seen to satisfy the commutation relations of the $so(1,3)$ real Lie algebra. It can indeed by checked that $\partial_1 \stackrel{\triangle}{=} \partial_{x^1} = -(l_1 + \bar{l}_1)$, $\partial_2 = i(\bar{l}_1 - l_1)$ (translations), $x^2\partial_2 - x^2\partial_1 = -(\bar{l}_0 - l_0)$ (rotations), $x^1\partial_1 + x^2\partial_2 = -(l_0 + \bar{l}_0)$ (dilations), the remaining two corresponding to SCT.

Another way of finding the form of the global conformal transformations consists in looking at the finite transformations (8). To describe a bijective map on the Riemann sphere, the function $f(z)$ should not have any branch point nor any essential singularity. Indeed, around a branch point the map is not uniquely defined[6], whereas the image of a small neighborhood of an essential singularity

[5] One recognizes easily in (26) the different transformations : translations, $z' = z + b$ for $a = d = 1$ and $c = 0$, rotations, $z' = \mathrm{e}^{i\theta}z$ for $c = b = 0$, $a = \frac{1}{d} = \mathrm{e}^{i\theta/2}$, dilations, $z' = \mathrm{e}^{\lambda}z$ for $c = b = 0$, $a = \frac{1}{d} = \mathrm{e}^{\lambda/2}$ and SCT, $z' = \frac{z}{1+cz}$, for $a = d = 1$, $b = 0$

[6] Consider $\log z = \ln\rho + i\varphi$, for $z = \rho\mathrm{e}^{i\varphi}$. If one adds 2π to φ, this doesn't change z, but the imaginary part of $\log z$ gets modified, showing that $\log z$ is a "multivalued function" having distinct branches

under f is dense in $\mathbb{C}$ (Weierstrass' theorem). Thus f is not invertible in these cases, and the only acceptable singularities are poles. Then, f can be written as a ratio of polynomials without common zeros[7]

$$f(z) = \frac{P(z)}{Q(z)} \quad . \tag{34}$$

If $P(z)$ has several distinct zeros, then the inverse image of zero is not unique and f is not invertible. If $P(z)$ has a multiple zero z_0 of order $n > 1$, then the image of a small neighborhood of z_0 is wrapped n times around zero, and therefore f is not invertible, see Fig.1.

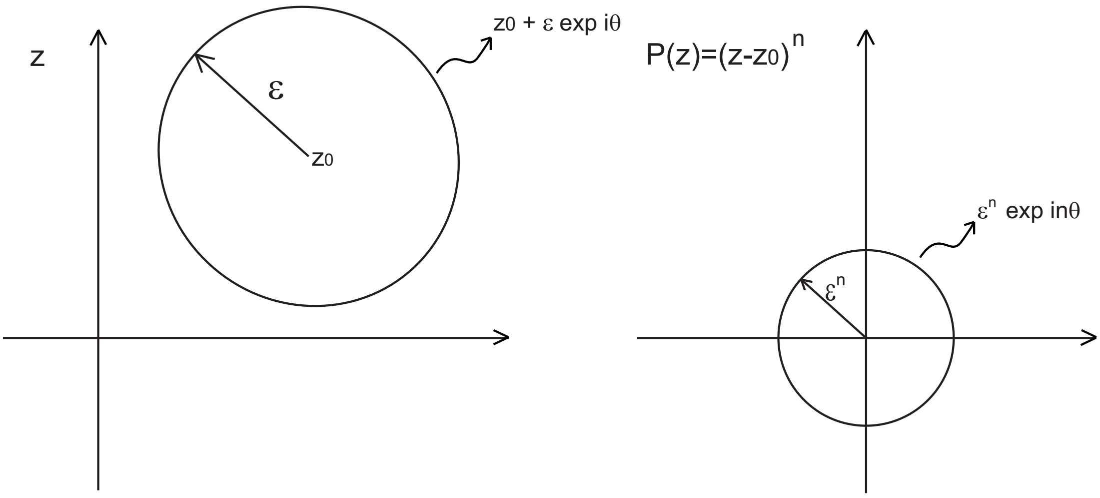

Figure 1

Thus $P(z)$ must be of the form $P(z) = az + b$. The same argument holds for $Q(z)$ when looking at the behavior of $f(z)$ near $z = \infty$. Thus

$$f(z) = \frac{az + b}{cz + d} \quad , \tag{35}$$

with $ad - bc \neq 0$ in order for the mapping to be invertible (this is the Jacobian of the transformation). Since an overall scaling of all coefficients does not change f, the normalization $ad - bc = 1$ has been adopted , so that we recover the $\mathrm{Sl}(2, \mathbb{C})$ global conformal group in two dimensions.

2 Conformal Ward identities in two dimensions : part I

2.1 Preliminaries

The objective of a quantum field theory is to determine the scattering amplitudes between various asymptotic states (free particles). In practice these amplitudes are computed from the correlation

[7]Any meromorphic function, i.e. holomorphic except at isolated poles, can be expressed as the ratio of two holomorphic functions. These could be exponential functions or combination thereof, but we exclude them because they are periodic in the complex plane and thus not invertible. Furthermore, such functions are even not defined on the whole Riemann sphere, because $(x, y) = (0, \infty)$ and $(x, y) = (\infty, 0)$are supposed to represent the same point (the North Pole in the stereographic projection), while $\mathrm{e}^z = \mathrm{e}^x(\cos y + i \sin y)$ is not univocal at this point

functions (Green's functions) via the so-called reduction formulas. The quantum description of a physical system may be tackled using different methods. One of them, the *operator formalism*, consists in replacing classical quantities by operators acting on a vector space in which the states of the system reside. Another method is called *path integration* or *functional integration*, and can be related to the first one, see e.g. Appendix A of [11]. To begin, we will work in the second formalism (the operator formalism of two dimensional CFT will be discussed in Sect. 4), in which a euclidian correlation function of N fields $\phi_i(x_i)$ reads

$$\langle \phi_1(x1)\cdots\phi_N(x_N)\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\ \phi_1(x1)\cdots\phi_N(x_N)\ \exp(-S[\phi]) \quad , \tag{36}$$

where $Z = \int \mathcal{D}\Phi \exp(-S[\phi])$, Φ denotes the set of all functionally independent fields in the theory[8], and $S[\Phi, \partial_\mu\Phi] = \int d^2x\, L(\Phi(x), \partial_\mu\Phi(x))$ is the action.
Classically, the invariance of the action under a continuous symmetry[9] implies the existence of a conserved current via *Noether's theorem.* At the quantum level, a continuous symmetry of the action *and* of the functional measure leads to constraints on the correlation functions, which are expressed via the so-called *Ward identities.*
Before turning to their derivation in the context of 2D CFT, we define the notion of a *primary field.* Under a conformal map $z\longrightarrow w(z)$, $\overline{z}\longrightarrow\overline{w}(\overline{z})$, a primary field transforms as

$$\phi(z,\overline{z})\longrightarrow\phi'(w(z),\overline{w}(\overline{z})) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\bar{h}}\phi(z,\overline{z}) \quad , \tag{37}$$

where h (resp. $\bar{h}$) is called holomorphic (resp. anti-holomorphic) conformal dimension of the field. Eq. (37) states that a primary field of conformal dimensions $(h, \bar{h})$ transforms as the components of a tensor with h covariant indices z and $\bar{h}$ covariant indices $\overline{z}$. The conformal dimensions are related to the spin s and scaling dimension Δ of the field by $s = h - \bar{h}$ and $\Delta = h + \bar{h}$ [10].
These fields play a crucial rôle in conformal field theories, namely because correlation functions involving any field of the theory can be reduced to correlations functions involving only primary fields. This is one of the properties rendering conformal field theories in two dimensions solvable, where a theory is said to be solved when all correlation function can be written (at least in principle). Fields transforming like (37), but only under global conformal transformations (31) are called *quasi-primary.*

2.2 A derivation of the conformal Ward identities

I first would like to mention that there are quite a lot seemingly different derivations of the conformal Ward identity. In this section, I will present that of [1], which is to my opinion the shortest way to get to the goal and gives a good flavor of how things fit all together. Skeptical readers may wish to

[8]When one speaks of a field in CFT, it does not necessarily mean that this field figures independently in the functional integration measure. For instance, a single boson ϕ, its derivative $\partial_\mu\phi$ and a composite quantity such as the energy-momentum tensor are called fields, since they are local quantities with a coordinate dependence. However, only *some* fields are integrated over in the functional integral

[9]This means that under a transformation $x\longrightarrow x'$, $\phi(x)\longrightarrow\phi'(x')$, we have $S' = S$, with $S' = \int_{D'} d^dx'\, L(\Phi'(x'), \partial'_\mu\Phi'(x')) = \int_D d^dx\, L(\Phi'(x), \partial_\mu\Phi'(x))$, where D is the integration domain (often $D = \mathbb{R}^2$)

[10]The variation of a field under a transformation can be decomposed in $\delta\phi(x) = (\phi'(x) - \phi'(x')) + (\phi'(x') - \phi(x))$, where the second term encodes the spin and scaling properties of the field. For a rotation, $z\longrightarrow e^{i\theta}z$, we find $\phi'(x') = e^{i\theta(h-\bar{h})}\phi(x)$, while under a scaling $z\longrightarrow e^{\lambda}z$, we have $\phi'(x') = e^{-\lambda(h+\bar{h})}\phi(x)$

consult another derivation, maybe more precise, based on [2] and presented in Appendix C (see also [2] p123 for an alternative approach to global conformal Ward identities). Later on, I will return to the these identities in the context of the operator formalism (see Sect. 4.4 and 4.7).
To study the consequences of the consequences of the conformal symmetry on correlation functions, it is usually more convenient to consider infinitesimal transformations of the form (8), for small $\alpha(z)$. With $w(z) = z + \alpha(z)$, $\overline{w}(\overline{z}) = \overline{z} + \bar{\alpha}(\overline{z})$, one finds the infinitesimal transformation of a primary field:

$$\delta_\alpha \phi(z, \overline{z}) = -\left(h\alpha'(z) + \alpha(z)\partial + \bar{h}\bar{\alpha}'(\overline{z}) + \bar{\alpha}(\overline{z})\overline{\partial}\right) \phi(z, \overline{z}) \quad . \tag{38}$$

At this point, I would like to draw attention on the infinitesimal character of the transformation we are to use. As we found, general conformal transformations are given by (8), with $f(z)$ and $\alpha(z)$ depending only upon the variable z (I will sometimes omit the $\overline{z}$ dependence, being understood it is always present). This makes f and α holomorphic functions, *except at the location of possible singularities.* Now, it is clear that close to a singularity, the functions cannot remain infinitesimal, whatever the parameters α_n in (12). Furthermore, on the Riemann sphere, even globally holomorphic (analytic) functions (admitting everywhere a Taylor expansion), like polynomials, are unbounded, except the constant function (for another discussion on this point and another way round, see [4], p129).
To avoid this problem, we will define transformation (8) in a finite region D, say in the neighborhood of $z = 0$, with $\alpha(z)$ analytic in D, and put $\alpha(z) = 0$ outside D. This way of cutting $\alpha(z)$ will produce boundary terms which we will have to keep track of. The region D will be chosen so to as to contain the positions of all the fields inside the correlation function, see Fig. 2.

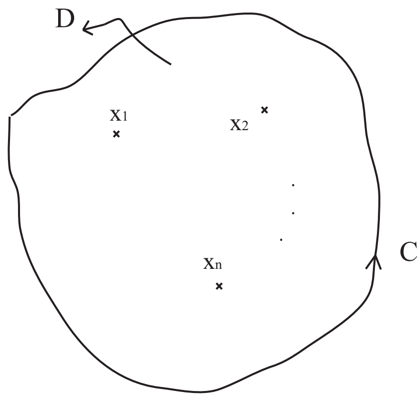

Figure 2

An analytic function in a region D containing the origin may be Taylor expanded around $z = 0$ as

$$\alpha(z) = \sum_{n=-1}^{\infty} \alpha_n z^{n+1} \quad . \tag{39}$$

Because $|z|^{n+1}$ is bounded in D, the coefficients $\{\alpha_n\}$ can always be chosen sufficiently small to render $\alpha(z)$ infinitesimal. These coefficients are the parameters of the corresponding conformal transformation, and their number is infinite expressing the infinite-dimensional character of the symmetry.

Let us consider a variation affecting both coordinates and fields[11] of the form

$$x \rightarrow x' \quad , \tag{40}$$

$$\phi(x) \rightarrow \phi'(x') \quad . \tag{41}$$

In the functional integral (36), let us make the following change of functional integration variable :

$$\phi(z,\overline{z}) \longrightarrow \phi'(z,\overline{z}) = \phi(z,\overline{z}) + \delta\phi(z,\overline{z}) \quad . \tag{42}$$

We will assume that the functional measure is invariant under conformal transformations , that is, that the Jacobian of this change of variable is equal to 1. This cannot be checked formally, but only on specific theories, so we will take this as a definition of a conformally invariant filed theory. With this assumption, $\mathcal{D}\phi = \mathcal{D}\phi$, and replacing (42) in (36) does not change the correlation function. With $X = \phi(x_1)\cdots\phi(x_N)$ $(x_i = (z_i, \overline{z}_i))$, we find that

$$\langle X\rangle = \frac{1}{Z}\int \mathcal{D}\Phi'\, \phi'(x_1)\cdots\phi'(x_N)\, \mathrm{e}^{-S[\Phi']} \tag{43}$$

$$= \int \mathcal{D}\Phi'\, (\phi_1(x_1) + \delta\phi_1(x_1))\cdots(\phi_N(x_N) + \delta\phi_N(x_N))\, \mathrm{e}^{-S[\Phi]-\delta S[\phi]} \tag{44}$$

$$= \langle X\rangle + \int \mathcal{D}\Phi\, \delta X \mathrm{e}^{-S[\Phi]} - \int \mathcal{D}\Phi\, X\delta S[\Phi]\mathrm{e}^{-S[\Phi]} \quad , \tag{45}$$

where $\delta X = \sum_i(\phi(x_1)\cdots\delta\phi(x_i)\cdots\phi(x_N)\ )$. Thus, we have

$$\delta\langle X\rangle \triangleq \langle \delta X\rangle = \langle \delta S\, X\rangle \quad . \tag{46}$$

When all fields in the correlation function are primary, we may use (38), so that the r.h.s. of (46) becomes

$$\delta_{\alpha,\bar{\alpha}}\langle X\rangle = -\sum_{i=1}^{N}[h_i\alpha'(z_i) + \alpha(z_i)\partial_i + \bar{h}_i\bar{\alpha}'(\overline{z}_i) + \bar{\alpha}(\overline{z}_i)\overline{\partial}_i]\langle X\rangle \quad , \tag{47}$$

assuming that all points x_i in the correlation function are distinct[12]. The l.h.s. can be rewritten by using the relation

$$\delta S = \oint_C ds^\mu \alpha^\nu T_{\mu\nu} \quad , \tag{48}$$

where the contour C is the boundary of D, and $T_{\mu\nu}$ is the canonical energy momentum tensor[13]

$$T_{\mu\nu}(\phi) = \delta_{\mu\nu}L(\phi,\partial_\mu\phi) - \frac{\partial L}{\partial\phi_{,\mu}}\partial_\nu\phi \quad , \tag{49}$$

[11]We consider the variation of all fields, even those not appearing in the functional integration

[12]Correlation functions typically become potentially singular when two or more points come to coincide, so the operation of "taking the derivative out of the correlation function" , $\langle\cdots\partial_i\phi(x_i)\cdots\rangle = \partial_i\langle\cdots\phi(x_i)\cdots\rangle$ may pose problems in this case. We will always assume $x_i \neq x_j$ for $i \neq j$.

[13]From now one, we consider for simplicity the situation where Φ reduces to a single field ϕ

with $\phi_{,\mu} = \partial_\mu \phi$. This is the conserved current associated with translation invariance of the action. Eq. (48) expresses that, although the action S is supposed to be conformally invariant, we find a boundary term because $\alpha(z)$ is discontinuous on C.
Let us derive (48), in the case where $\alpha^\nu = a^\nu = \text{cst}$. Let $S[\phi]$ be an action given by

$$S[\phi] = \int d^2x \, L(\phi(x), \phi_{,\mu}(x)) \quad , \tag{50}$$

where the lagrangian L is supposed to be local in $\phi(x)$ and $\phi_{,\mu}(x)$, as well as invariant under translations. This implies that

$$\delta S[\phi] = \int d^2x L(\phi'(x), \phi'_{,\mu}(x)) \; - \int d^2x L(\phi(x), \phi_{,\mu}(x)) \; = \; 0 \quad , \tag{51}$$

where the integration is over the whole space and $\phi'(x) = \phi(x) - a^\mu \partial_\mu \phi(x) = \phi(x - a)$. Now, when restricted to a finite region D, the transformation of the coordinates reads

$$x^\mu \quad \rightarrow \quad x'^\mu = x^\mu + a^\mu(x) \quad , \tag{52}$$

with

$$a^\mu(x) = \begin{cases} a^\mu & \text{if } x \in D\,, \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

and the variation of the field is

$$\phi'(x) = \begin{cases} \phi(x-a) & \text{if } x \in D\,, \\ \phi(x) & \text{otherwise} \end{cases} \tag{54}$$

A one-dimensional section of $\phi'(x)$ is given in Fig.3

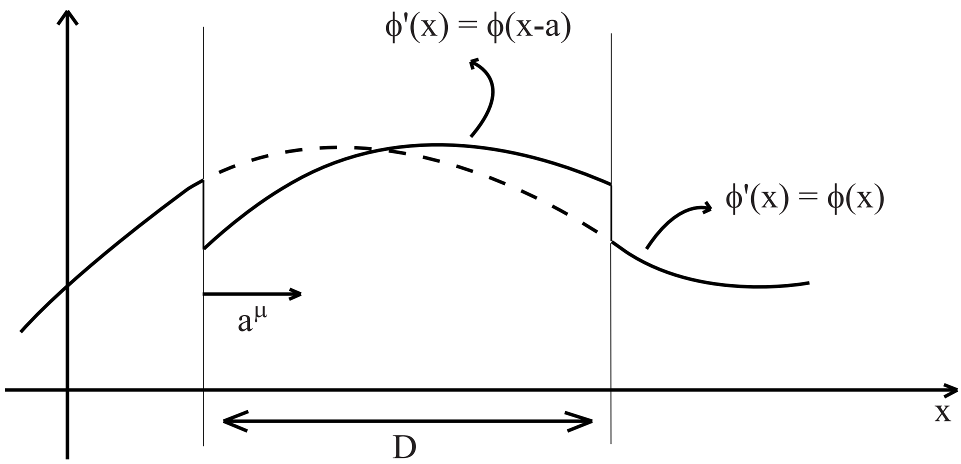

Figure 3

The variation of the action now becomes

$$\delta S[\phi] = \int\limits_D d^2x L(\phi'(x), \phi'_{,\mu}(x)) \; - \int\limits_D d^2x L(\phi(x), \phi_{,\mu}(x)) \quad , \tag{55}$$

where only the contribution of the part inside D appears. By restricting ourselves to a finite region, we broke translation invariance, in some sense. There are two contributions to $\delta S[\phi]$ in (55). To see this, we rewrite it as

$$\delta S[\phi] \;=\; \int\limits_D d^2x \left[\left(\frac{\partial L}{\partial \phi}\right) \delta\phi + \left(\frac{\partial L}{\partial \phi_{,\mu}}\right) \partial_\mu \delta\phi \right] \tag{56}$$

$$= \int\limits_D d^2x \left[\left(\frac{\partial L}{\partial \phi}\right) - \partial_\mu \left(\frac{\partial L}{\partial \phi_{,\mu}}\right) \right] \delta\phi \; + \int\limits_D d^2x \, \partial_\mu \left(\frac{\partial L}{\partial \phi_{,\mu}} \delta\phi \right) \tag{57}$$

$$\triangleq \; \delta S_{\text{reg}}[\phi] + \delta S_{\text{jump}}[\phi] \quad . \tag{58}$$

The first term corresponds to a "bulk term", while the second one can be rewritten using Stokes' theorem as a boundary term, whose contribution comes from the "jump" that ϕ experiences on $\partial D = C$:

$$\delta S_{\text{jump}}[\phi] \triangleq \int\limits_D d^2x \, \partial_\mu \left(\frac{\partial L}{\partial \phi_{,\mu}} \delta\phi \right) = \oint\limits_C \left(\frac{\partial L}{\partial \phi_{,\mu}} \delta\phi \right) ds^\mu \quad . \tag{59}$$

Thus, with $\delta\phi = -a^\nu \partial_\nu \phi$,we get

$$\delta S_{\text{jump}}[\phi] = -\oint\limits_C a^\nu \left(\frac{\partial L}{\partial \phi_{,\mu}} \right) \partial_\nu \phi \, ds^\mu \quad . \tag{60}$$

The analysis of the regular term is better performed starting directly from (55) and performing the change of variable $\tilde{x} = x - a$ in the first integral. So, the function to integrate becomes the same, the difference being the integration domains :

$$\delta S_{\text{reg}}[\phi] \;=\; \int\limits_{\tilde{D}} d^2\tilde{x} L(\phi(\tilde{x}), \phi_{,\mu}(\tilde{x})) \; - \int\limits_D d^2x L(\phi(x), \phi_{,\mu}(x)) \tag{61}$$

$$= \int\limits_{\delta D} d^2x L(\phi(x), \phi_{,\mu}(x)) \quad , \tag{62}$$

where the domain δD is represented in Fig.4 .
Because a^μ is infinitesimal, this integral may be transformed in a contour integral (on C or $\tilde{C}$, to first order), with

$$\int\limits_{\delta D} d^2x = \oint\limits_C \epsilon_{\mu\nu} a^\mu dx^\nu = \oint\limits_C ds^\mu a_\mu = \oint\limits_C ds^\mu a^\nu \delta_{\mu\nu} \quad . \tag{63}$$

Finally, with (60) ,(61) and (63), we find

$$\delta S[\phi] = \oint\limits_C ds^\mu a^\nu \left[\delta_{\mu\nu} L - \left(\frac{\partial L}{\partial \phi_{,\mu}} \right) \partial_\nu \phi \right] \quad , \tag{64}$$

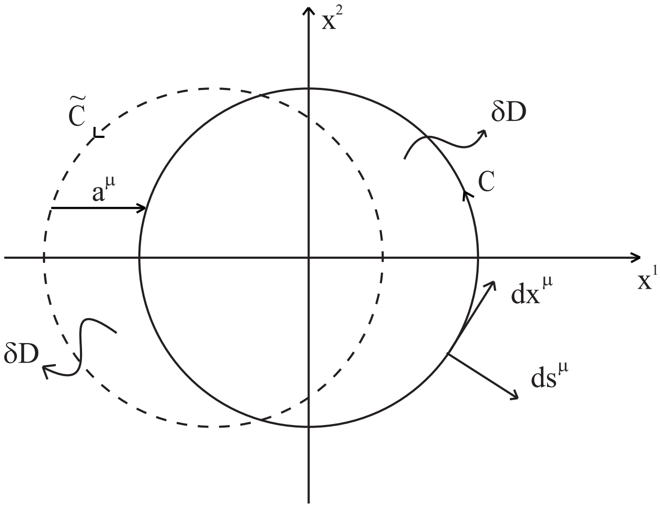

Figure 4

and hence Eq. (48). The same argument may still be applied if α^μ is replaced by a conformal Killing vector $\alpha^\mu(x)$. Eq. (46) becomes, using (48)

$$\delta\langle X\rangle = -\frac{1}{2\pi}\oint_c ds^\mu \alpha^\nu \langle T_{\mu\nu} X\rangle \quad , \tag{65}$$

where the factor of $-\frac{1}{2\pi}$ is placed for convenience and corresponds to a normalization choice for $T_{\mu\nu}$ or the action. Let us now express (65) in complex coordinates. With $z = x^1 + ix^2$, $\overline{z} = x^1 - ix^2$ and $(dx^\mu) = (dx^1, dx^2)$, $(ds^\mu = \epsilon^\mu_\nu dx^\nu) = (dx^2, -dx^1)$, we find that $ds^z \triangleq ds = -idz$ and $ds^{\overline{z}} \triangleq d\bar{s} = id\overline{z}$, as well as $\alpha^z = \alpha(z)$, $\alpha^{\overline{z}}(\overline{z})$ (with the help of the conformal Killing equations (11)), so that (65) becomes

$$\begin{aligned}\delta_{\alpha,\bar{\alpha}}\langle X\rangle = &-\frac{1}{2\pi i}\oint_C dw\,[\alpha(w)\langle T_{zz}(w,\overline{w})X\ \rangle + \bar{\alpha}(\overline{w})\langle T_{z\overline{z}}(w,\overline{w})X\rangle] \\ &+\frac{1}{2\pi i}\oint_C d\overline{w}\,[\bar{\alpha}(\overline{w})\langle T_{\overline{z}\overline{z}}(w,\overline{w})X\ \rangle + \alpha(w)\langle T_{\overline{z}z}(w,\overline{w})X\rangle] \quad .\end{aligned} \tag{66}$$

This is a first version of the ward identity, in integral form. It is possible to go a step further to obtain it in local form. We are first going to simplify (66) by analyzing two special cases, that is (1) $\alpha(z) = a = cst.$ and (2) $\alpha(z) = bz$, and by applying (66) to an arbitrary domain D^* which does not contains the points x_i (see Fig 5).
In this way, the l.h.s. vanishes, because now $\delta\phi(x_i) = 0$, as everywhere outside the original domain D. With the first case, we find

$$0 = -\oint_{C^*} dw\langle T_{zz}(w,\overline{w})X\ \rangle + \oint_{C^*} d\overline{w}\langle T_{\overline{z}z}(w,\overline{w})X\rangle \quad , \tag{67}$$

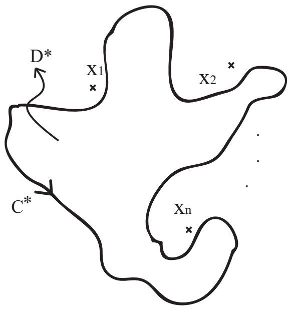

Figure 5

The two contour integrals may be rewritten using Stokes's theorem $\int_M d^2x\partial_\mu F^\mu = \int_{\partial M} ds^\mu F_\mu$, which implies in particular

$$\int_D d^2x\,\partial f(z,\overline{z}) = -\frac{1}{2i}\oint_C d\overline{z} f(z,\overline{z}) \quad \text{and} \quad \int_D d^2x\,\overline{\partial} f(z,\overline{z}) = \frac{1}{2i}\oint_C dz f(z,\overline{z}) \quad . \tag{68}$$

Eq. (67) can then be rewritten

$$\int_{D^*} d^2x\ \left[\partial_{\overline{w}}\langle T_{zz}(w,\overline{w})X\ \rangle + \partial_w\langle T_{\overline{z}z}(w,\overline{w})X\ \rangle\right] = 0 \quad . \tag{69}$$

Because this holds for any domain D^* (excluding the points x_i), this implies

$$\partial_{\overline{w}}\langle T_{zz}(w,\overline{w})X\ \rangle + \partial_w\langle T_{\overline{z}z}(w,\overline{w})X\ \rangle = 0 \quad , \quad (w,\overline{w}) \neq x_i \quad . \tag{70}$$

With the second case, we find

$$0 = -\oint_{C^*} dw\, w\,\langle T_{zz}(w,\overline{w})X\ \rangle + \oint_{C^*} d\overline{w}\, w\,\langle T_{\overline{z}z}(w,\overline{w})X\ \rangle \quad , \tag{71}$$

and by using Stokes' theorem

$$\int_{D^*} d^2x\,\left[w\partial_{\overline{w}}\langle T_{zz}(w,\overline{w})X\ \rangle\ + w\partial_w\langle T_{\overline{z}z}(w,\overline{w})X\ \rangle\ + \langle T_{\overline{z}z}(w,\overline{w})X\ \rangle\right] \quad , \tag{72}$$

where the first two terms of the integrand vanish because of (70). Thus (72) implies

$$\langle T_{\overline{z}z}(w,\overline{w})X\ \rangle = 0 \quad , (w,\overline{w}) \neq x_i \quad . \tag{73}$$

From (69) and (73), we find that

$$\partial_{\overline{w}}\langle T_{zz}(w,\overline{w})X\ \rangle = 0 \quad ,(w,\overline{w}) \neq x_i \quad . \tag{74}$$

Similar relations hold for $\langle T_{\overline{z}z}(w,\overline{w})X\ \rangle$ and $\partial_w\langle T_{\overline{z}\overline{z}}(w,\overline{w})X\ \rangle$, so that we end up with

$$\partial_{\overline{w}}\langle T_{zz}(w,\overline{w})X\ \rangle = \partial_w\langle T_{\overline{z}\overline{z}}(w,\overline{w})X\ \rangle = 0 \qquad \text{if}\,(w,\overline{w}) \neq (w_i,\overline{w}_i) \tag{75}$$

and

$$\langle T_{\overline{z}z}(w,\overline{w})X\ \rangle = \langle T_{z\overline{z}}(w,\overline{w})X\ \rangle = 0 \qquad \text{if}\,(w,\overline{w}) \neq (w_i,\overline{w}_i) \quad . \tag{76}$$

Eq. (75) shows that inside correlation functions T_{zz} (resp. $T_{\overline{z}\overline{z}}$) does not depend on $\overline{z}$ (resp. z), and is said to be an *holomorphic operator* (resp. *anti-holomorphic operator*), except at the points x_i. We set

$$T_{zz} \stackrel{\triangle}{=} T(z) \quad \text{and} \quad T_{\overline{z}\overline{z}} \stackrel{\triangle}{=} \bar{T}(\overline{z}) \quad . \tag{77}$$

Furthermore, $T^\mu_\mu = g^{\mu\nu}T_{\mu\nu} = T_{11} + T_{22} = \frac{1}{2}(T_{z\overline{z}} + T_{\overline{z}z})$, so (76) expresses the vanishing of the trace of the energy-momentum tensor inside a correlation function except at coincident points[14][15] Finally, eq. (66) reduces, with (75), (76) and (77), to[16]

$$\delta_{\alpha,\bar{\alpha}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dw\alpha(w)\langle T(w)X\ \rangle + \frac{1}{2\pi i}\oint_C d\overline{w}\bar{\alpha}(\overline{w})\langle\bar{T}(\overline{w})X\ \rangle \quad . \tag{78}$$

Because of the form of (47) and (78), we observe that (78) splits into two independent equations (recall we may consider z and $\overline{z}$ as two independent variables, as well as $\alpha(z)$ and $\bar{\alpha}(\overline{z})$ as two independent functions). Let us focus on the z-part ("holomorphic" part), and consider a correlation function involving *primary fields* :

$$-\frac{1}{2\pi i}\oint_C dw\alpha(w)\langle T(w)X\ \rangle = -\sum_i(\alpha'(z_i)h_i + \alpha(z_i)\partial_i)\langle X\rangle \quad . \tag{79}$$

Using the residue theorem[17], the r.h.s. may be rewritten as

$$-\frac{1}{2\pi i}\sum_{i=1}^N\oint_{C_i} dw\alpha(w)\left(\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i}\partial_i\right)\langle X\rangle \quad , \tag{80}$$

because $\alpha(w)$ is analytic in D and in particular in any of the contours C_i represented in Fig.6 .

[14]Classically, the canonical energy-momentum tensor of a system invariant under rotations can be made symmetric on-shell, and put in the so-called Belinfante form (see [2], p47). An alternative definition of the energy-momentum tensor is given by the functional derivative of the action w.r.t. the metric, evaluated in flat space : $T^{\mu\nu} = \frac{\delta S}{\delta g_\mu{}^\nu}$, which makes it identically symmetric. For a system invariant under dilations, it can be shown that in $d > 2$ and assuming a technical ("virial") condition that the canonical energy-momentum tensor can be made symmetric on-shell ([2], p102, [4] p122). In $d = 2$, this is assumed to hold.

[15]The tracelessness of the energy-momentum tensor is sufficient to guarantee the invariance of the action under conformal transformations . Indeed, under $x^\mu \to x^\mu + \varepsilon^\mu$, the action varies as $\delta S = \int d^dx\, T^{\mu\nu}\partial_\mu\varepsilon_\nu$. If ε_ν is a conformal Killing vector, then (see 11) $\delta S = \int d^dx\, T^\mu_\mu\partial^\rho\varepsilon_\rho$, which vanishes if $T^\mu_\mu = 0$. The converse is not true, since $\partial^\rho\varepsilon_\rho$ is not an arbitrary function.

[16]We may use (75) and (76) because the contour C does not pass exactly through the points x_i

[17]$\oint_C dz\,\frac{\phi(z)}{(z-z_0)^n} = 2\pi i\frac{\phi^{(n-1)}}{(n-1)!}$, if $\phi(z)$ is analytic in C

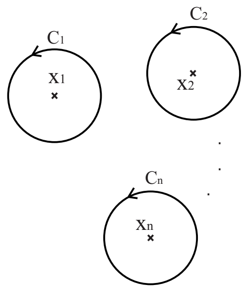

Figure 6

For the same reason (and because $\langle X\rangle$ does not depend on z!), the contours C_i may be deformed into the contour C, see Fig.2 . Eq. (79) then becomes

$$\oint_C dw\alpha(w)\left[\langle T(w)X\rangle - \sum_{i=1}^{N}\left(\frac{h_i}{(w-z_i)^2}+\frac{1}{w-z_i}\partial_i\right)\langle X\rangle\right]=0 \quad . \tag{81}$$

This is an equation of the form $\oint_C dz\,\alpha(z)f(z)=0$, for all analytical function $\alpha(z)$. Thus $f(z)$ has to be holomorphic inside C and may not contain singularities. Therefore we write

$$\langle T(w)X\rangle = \sum_{i=1}^{N}\left(\frac{h_i}{(w-z_i)^2}+\frac{1}{w-z_i}\partial_i\right)\langle X\rangle + \text{reg} \quad , \tag{82}$$

where "reg" stands for a holomorphic function of z inside C, regular in particular as $w\to z_i$.

3 Operator product expansion and conformal transformation of fields

The main result of the preceding section is encapsulated in (82) (when X contains only primary fields) and in (78), which are called the Ward identities of two-dimensional conformal field theories. These will be of great importance in what follows. We saw that the z and $\overline{z}$ parts of the transformations split in two parts, so in the following we will only consider the z part, keeping in mind that the $\overline{z}$ part is always present.
Eq. (82) delivers an important information. It yields the singular behavior of the correlator of the field $T(z)$ with the primary fields $\phi_i(z_i,\overline{z}_i)$, as $z\to z_i$. For a single primary field of holomorphic conformal dimension h, this is usually written as

$$T(z)\phi(w,\overline{w}) \sim \frac{h}{(z-w)^2}\,\phi(w,\overline{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\overline{w}) \quad . \tag{83}$$

The precise meaning of this expression is the following : when $z{\to}w$, the (correlation) function $\langle T(z)\phi(w,\overline{w})Y\rangle$ behaves as the function $\frac{h}{(z-w)^2}\langle\phi(w,\overline{w})Y\rangle + \frac{1}{z-w}\partial_w\langle\phi(w,\overline{w})Y\rangle$, where $Y = \phi_1(w_1,\overline{w}_1)\cdots\phi_n(w_n,\overline{w}_n)$, none of the points being coincident in $\langle T(z)\phi(w,\overline{w})Y\rangle$. An expression like (83) is called (the singular part of) the *operator product expansion* (or *OPE*) of the energy-momentum tensor with a primary field. As we stressed, it only makes sense inside a correlation function. The symbol " $\sim$ " means equality up to terms which are regular as $z{\to}w$. A general OPE of two fields takes the form

$$A(z)B(w) = \sum_{n=-\infty}^{N}\frac{\{AB\}_n(w)}{(z-w)^n} \sim \sum_{n=1}^{N}\frac{\{AB\}_n(w)}{(z-w)^n} \quad , \tag{84}$$

where the composite fields $\{AB\}_n(w)$ are non-singular at $w = z$. For instance, $\{T\phi\}_1(w) = \partial_w\phi(w)$. Let us now return to eq. (78), focusing on the z part :

$$\delta_\alpha\langle X\rangle = -\frac{1}{2\pi i}\oint_C dw\,\alpha(w)\langle T(w)X\rangle \quad , \tag{85}$$

where $\delta_\alpha\langle X\rangle$ stands for $\langle(\delta_\alpha\phi_1)\phi_2\cdots\phi_N\rangle + \cdots + \langle\phi_1\cdots(\delta_\alpha\phi_N)\rangle$ as before. This Ward identity can be interpreted as defining the transformation of the fields within the correlation function. The variation of the *function* on the l.h.s. is by definition that produced by the variation of the *operators*. Now remember the contour C encircles the positions of all operators. This contour may be "distributed" on the operators $\phi_1,\cdots,\phi_N$ as in Fig.6, because the only singular points of $\langle T(w)X\rangle$ are located at $w = z_i$. Thus

$$\delta_\alpha\langle\phi_1\cdots\phi_N\rangle \triangleq \sum_{k=1}^{N}\langle\phi_1\cdots\delta_{\alpha(z_k)}\phi_k(z_k)\cdots\phi_N\rangle \tag{86}$$

$$= \sum_{k=1}^{N}\frac{1}{2\pi i}\oint_{\mathbb{C}_k} dw\,\alpha(w)\langle T(w)\phi_1\cdots\phi_N\rangle \quad . \tag{87}$$

Each integral in (87) corresponds to the variation of a field in (86). So, the variation of a single field is represented by the integral

$$\delta_{\alpha(z_i)}\phi(z_i) = \frac{1}{2\pi i}\oint_{C_i} dw\,\alpha(w)T(w)\phi_i(z_i) \quad , \tag{88}$$

where C_i encircles only z_i, see Fig.6. Consequently, because $\alpha(z)$ is analytic inside C_i, we conclude that *the transformation of a field under* $z{\to}z{+}\alpha(z)$ *is completely determined by the singular terms of the OPE of this field with the holomorphic part* $T(z)$ *of the energy-momentum tensor, and vice-versa.* This explains the important rôle of OPE's in conformal field theories.
We now concentrate on the transformation properties of the holomorphic energy-momentum tensor, $T(z)$. From its definition, it seems natural that it should transform as a rank 2 covariant tensor,

$$T(z){\longrightarrow}T'(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) \tag{89}$$

under $z{\to}w(z)$, thus as a primary field of conformal weight $h = 2, \bar{h} = 0$ (see eq.(37)) (similarly, $\bar{T}(\overline{z})$ should have $h = 0, \bar{h} = 2$. Actually, it turns out, in passing to the quantum theory, that the situation is complicated by the addition of a term in the transformation law (89), so that the latter becomes in general

$$T(z)\longrightarrow T'(w)=\left(\frac{dw}{dz}\right)^{-2}\left[T(z)-\frac{c}{12}\{w;z\}\right] \quad , \tag{90}$$

where $c$is a numerical constant characterizing the theory and where

$$\{w;z\}\triangleq\frac{d^3w/dz^3}{dw/dz}-\frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2 \tag{91}$$

is called the *Schwartzian derivative.* The additional term in (90) is called *Schwinger term* and is related, as we will see in Sect. 4.2, to the *conformal anomaly.* It can be shown (see for instance [4], pp134-141) that, requiring that (90) defines a group law and that (91) be a local functional (i.e. containing derivatives of $w(z)$ only up to finite order and degree), the form of the functional (91) is completely fixed up to a constant, hence the term $\frac{c}{12}$[18]. Notice that the Schwartzian derivative of the global conformal map

$$w(z)=\frac{az+b}{cz+d} \quad , \quad ad-bc=1 \tag{92}$$

vanishes, making $T(z)$ a quasi-primary field of conformal weight $h=2$. The infinitesimal variation $\delta_\alpha T(z)=T'(z)-T(z)$ can be obtained from (91) by considering $\{z+\alpha(z),z\}$ to first order in α :

$$\{z+\alpha(z),z\}=\frac{\partial^3\alpha}{1+\partial\alpha}-\frac{3}{2}\left(\frac{\partial^2\alpha}{1+\partial\alpha}\right)^2\sim\partial^3\alpha \tag{93}$$

then (90) gives

$$\begin{aligned} T'(z+\alpha) &= (1-2\partial\alpha(z))\left[T(z)-\frac{c}{12}\partial^3\alpha\right] \\ &= T'(z)+\alpha(z)\partial T(z) \quad , \end{aligned} \tag{94}$$

hence

$$\delta_\alpha T(z)=T'(z)-T(z)=-\frac{c}{12}\partial^3\alpha-2\partial\alpha(z)T(z)-\alpha(z)\partial T(z) \quad . \tag{95}$$

An alternative way of proceeding is to consider the OPE of $T(z)$ with itself which, according to the discussion of preceding section and eq. (88), should encode (95). It again turns out that, for a general CFT, this OPE has the form

$$T(z)T(w)\sim\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w} \quad . \tag{96}$$

Given a specific CFT, (96) is generally easier to obtain by standard techniques than (95), as we will see on some particular examples in the exercises (see also [2] pp128-135, p138). The last two terms of (96) again reflect the fact that $T(z)$ is a quasi-primary field of conformal dimension $h=2$ (see (83)). The term $\frac{c/2}{(z-w)^4}$ is actually the only addition compatible with the scaling transformation of (96). Let

[18]The group property to be verified is that the result of two successive transformations $z\to w\to u$ should coincide with what is obtained from the single transformation $z\to u$, that is $T''(u)=\left(\frac{du}{dw}\right)^{-2}\left[T'(w)-\frac{c}{12}\{u;w\}\right]=\left(\frac{du}{dw}\right)^{-2}\left[\left(\frac{dw}{dz}\right)^{-2}(T(z)-\frac{c}{12}\{w;z\})-\frac{c}{12}\{u;w\}\right]\overset{!}{=}\left(\frac{du}{dz}\right)^{-2}\left[T(z)-\frac{c}{12}\{u;z\}\right]$. The last equality requires the following relation between the Schwartzian derivatives : $\{u;z\}=\{w;z\}+\left(\frac{dw}{dz}\right)^{-2}\{u;w\}$, which is indeed satisfied.

us see that (95) and (96) really expresses the same thing. The relation between the variation of a field and the OPE is given by (88). By plugging (96) in (88), we get

$$\begin{aligned} \delta_\alpha T(z) &= -\frac{1}{2\pi i}\oint_{C_z} dw\,\alpha(w)T(w)T(z) \\ &= -\frac{1}{2\pi i}\oint_{C_z} dw\,\alpha(w)\left[\frac{c/2}{(w-z)^4}+\frac{2T(z)}{(w-z)^2}+\frac{\partial T(z)}{w-z}\right] \quad , \end{aligned} \tag{97}$$

so that after performing the integration, we effectively recover (95).

4 Operator formalism

Throughout the previous sections, all our manipulations were assumed to hold inside correlation functions. The consequences of conformal symmetry on two-dimensional field theories were embodied in constraints imposed on these correlation functions via the Ward identities (78) (we will elaborate on this point in Sect 4.4). These Ward identities were most easily expressed in the form of an OPE of the energy-momentum tensor with local fields (see (83) and (96)). Up to now, we only used the path-integral representation of the theory in which all correlation functions could in principle be obtained. We would now like to give an operator interpretation in terms of states in a Hilbert space.

4.1 Radial quantization

The operator formalism distinguishes a time direction from a space direction. This in natural in Minkowski space-time, but somewhat arbitrary in euclidian space. This allows to choose as time direction the radial direction form the origin, the space direction being orthogonal to it. This choice leads to the so-called *radial quantization* of two-dimensional conformal field theories.
We may start from a two-dimensional Minkowski space with coordinates t and σ. One usually takes the space direction σ to be periodic, $\sigma \in [0, L]$, defining this way the theory on a cylinder[19]. We continue to euclidian space, $t = -i\tau$, and then perform the conformal transformation

$$z = \mathrm{e}^{(\tau+i\sigma)\frac{2\pi}{L}} \quad , \quad \bar{z} = \mathrm{e}^{(\tau-i\sigma)\frac{2\pi}{L}} \quad , \tag{98}$$

which maps the cylinder onto the complex plane $\mathbb{C} \cup \{\infty\}$, topologically a sphere (the Riemann sphere). Surfaces of equal euclidian time τ on the cylinder will become circles of equal radii on the complex plane. This means that the infinite past ($\tau = -\infty$) gets mapped onto the origin of the plane ($z = 0$) and the infinite future becomes $z = \infty$, see Fig.7. Time reversal becomes $z \to 1/z^*$ ($= \mathrm{e}^{-\tau+i\sigma}$) on the complex plane, and parity is $z \to z^*$. The operator $(l_0 + \bar{l}_0)$ (preserving the real surface, see below eq. (33)) generates dilations in the complex plane : $z \to \mathrm{e}^\lambda z, \bar{z} \to \mathrm{e}^\lambda \bar{z}$ which, according to (98), corresponds to (euclidian) time translations on the cylinder. Consequently, this operator represents the *hamiltonian* of our system (since we are working in euclidian space, a more appropriate name would be *transfer operator*, which upon Wick rotation becomes the hamiltonian; similarly, the exponential of the transfer operator gives the transfer matrix, which would become

[19] This is natural from the closed-string theory point of view, where the two-dimensional space describes the world-sheet of the string. Also, this procedure allows to avoid infrared divergences, e.g. as the one present in the vacuum functional of a free boson (see [2], p142). L can be chosen very large and taken to infinity at the end of the calculation.

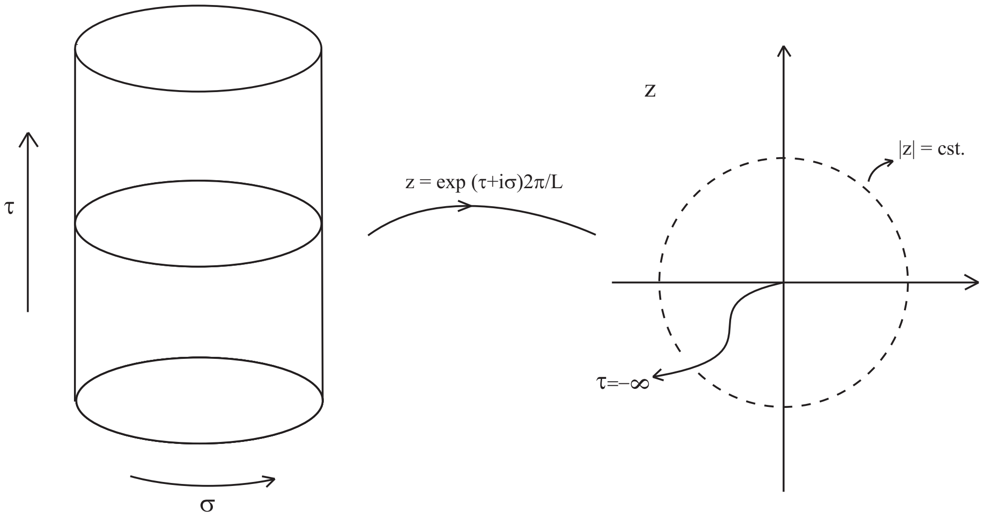

Figure 7

the time evolution operator upon Wick rotation). Finally, *an integral over the space direction σ, at fixed τ, will become a contour integral on the complex plane.* This enables us to use all the powerful techniques of complex analysis. We would now like to translate a relation like (85) :

$$\delta_{\alpha(z)}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dw\,\alpha(w)\langle T(w)X\rangle$$

in the operator formalism language. Expressions like $\langle\phi(z_1,\overline{z}_1)\phi(z_2,\overline{z}_2)\rangle$ which were understood in the path integral formalism (see (36)) must now be interpreted as

$$\langle\, 0\,|T(\phi(\tau_1,\sigma_1)\phi(\tau_2,\sigma_2)|\,0\,\rangle \quad , \tag{99}$$

where $|\,0\,\rangle$ is the vacuum state of the Hilbert space of states $\mathcal{H}$ (see Sect. 4.3) and where the fields become operators acting on $\mathcal{H}$. As is known (see Appendix B for a reminder), in ordinary QFT insertion of operators in the path integral represents time-ordered matrix elements of the corresponding operators. The *time-ordering*, defined by

$$T\left[\phi(\tau_1,\sigma_1)\phi(\tau_2,\sigma_2)\right] = \begin{cases} \phi(\tau_1,\sigma_1)\phi(\tau_2,\sigma_2) & \text{if } \tau_1 > \tau_2\,, \\ \varepsilon\phi(\tau_2,\sigma_2)\phi(\tau_1,\sigma_1) & \text{if } \tau_2 > \tau_1 \end{cases} \tag{100}$$

where $\varepsilon = 1$ for bosons and -1 for fermions, appears naturally by the nature of the path integral, which builds amplitudes from successive time steps. In radial quantization, this notion obviously generalizes to the *radial-ordering*

$$R\left[\phi(z_1,\overline{z}_1)\phi(z_2,\overline{z}_2)\right] = \begin{cases} \phi(z_1,\overline{z}_1)\phi(z_2,\overline{z}_2) & \text{if } |z_1| > |z_2|\,, \\ \varepsilon\phi(z_2,\overline{z}_2)\phi(z_1,\overline{z}_1) & \text{if } |z_2| > |z_1| \end{cases} \tag{101}$$

Since all fields within correlation functions must be radially ordered, so must be the l.h.s. of any OPE if it is to have an operator meaning. In particular, all OPE's written previously have an operator meaning if $|z| > |w|$.

Recall that in (88), C is a contour encircling all points z_i appearing in $X = \phi_1 \cdots \phi_N$. Let us consider the variation of a single operator, $X = \phi(z, \overline{z})$:

$$\delta_{\alpha(z)}\phi(z,\overline{z}) = -\frac{1}{2\pi i}\oint_C dw\,\alpha(w)R\left[T(w)\phi(z,\overline{z})\right] \tag{102}$$

where C is represented in Fig.8 . We will forget about the $\overline{z}$ dependence in what follows. As we

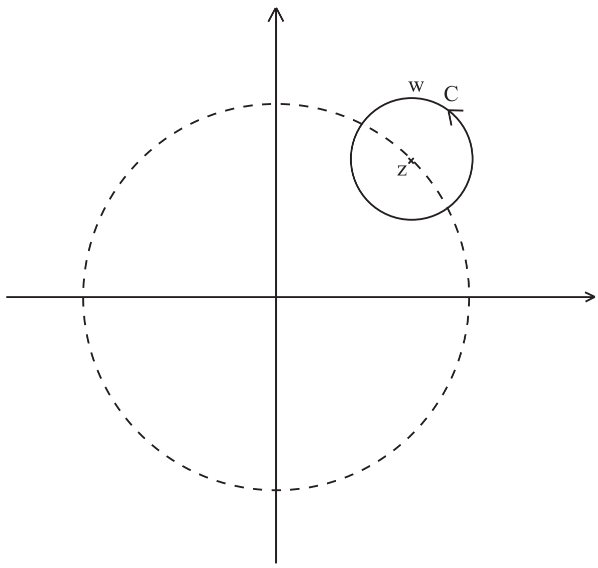

Figure 8

already pointed out, it is expected that a correlation function involving the product of operators $T(w)\phi(z)$ will potentially exhibit a singularity as $w \to z$ (see e.g. (83)). We will assume that $w = z$ is the *only* singularity in the product. Because of the radial ordering, the integrand of (102) is to be changed along the path C. To avoid this, we deform C in C', as shown in Fig. 9. This can be done safely, because there are no other singularities. The integral can further be rewritten as

$$\oint_C = \oint_{C'} = \oint_{C_1} - \oint_{C_2} - \oint_{C_\varepsilon} \quad , \tag{103}$$

see Fig.10, and where the last integral vanishes because the contour C_ε does not contain any singularity. Thus, (102) can be rewritten

$$\delta_{\alpha(z)}\phi(z) \quad = \quad -\left[\frac{1}{2\pi i}\oint_{C_1} dw\,\alpha(w)T(w)\phi(z) - \frac{1}{2\pi i}\oint_{C_2} dw\,\alpha(w)\phi(z)T(w)\right] \tag{104}$$

$$\triangleq \quad -[Q_\alpha, \phi(z)] \quad , \tag{105}$$

where

$$Q_\alpha \triangleq \frac{1}{2\pi i}\oint dw\,\alpha(w)T(w) \quad . \tag{106}$$

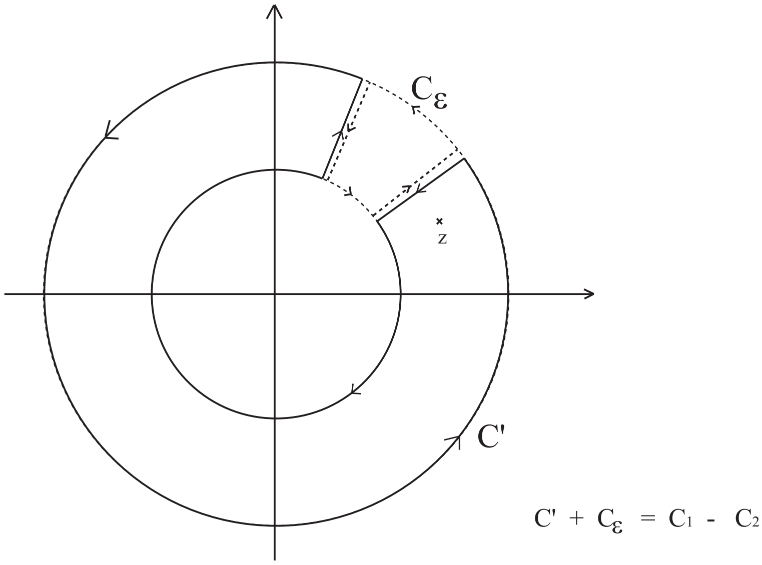

Figure 9

The integrals over C_1 and C_2 correspond to integrals over the space direction, at fixed time, and there is no ordering ambiguity in them. Note that to allow an arbitrary number of fields to lie beside $\phi(z)$ (recall an operator relation involving a field ϕ is obtained by considering a string of the form $X = \phi\phi_1 \cdots \phi_N$), the contours C_1 and C_2 have to be chosen appropriately. Indeed, the decomposition into two contours is valid if $\phi(z)$ is the only field of the string having a singular OPE with $T(w)$ between the two circles C_1 and C_2. Therefore these circles are taken to have radii respectively equal to $|z| + \varepsilon$ and $|z| - \varepsilon$, with $\varepsilon \rightarrow 0$. The commutator (105) may then be seen as an *equal time commutator*[20].

The operator (106) is called *conformal charge* and represents the generator of conformal transformations $z \longrightarrow z + \alpha(z)$. This situation is familiar from QFT. We know in this context from Noether's theorem that to every continuous symmetry of the action one may associate a current j_a^μ (where "a" stands for a collection of indices coming from the set $\{\omega_a\}$ of infinitesimal parameters of the symmetry transformation) that is classically conserved :

$$\partial_\mu j_a^\mu \approx 0 \tag{107}$$

where $\approx$ stands for equality when the equations of motion hold (on-shell). One may define the conserved charge associated with j_a^μ, as the integral over the spatial dimensions of the time component of the current :

$$Q_a = \int d^{d-1}x \, j_a^0 \quad , \tag{108}$$

whose time derivative vanishes : $\dot{Q}_a = 0$ (provided the spatial components of the current vanish sufficiently rapidly at infinity). This charge is also seen to generate the infinitesimal symmetry transformations in the operator formalism :

$$[Q_a, \phi] = G_a \phi \quad , \tag{109}$$

[20]It seems to me that this construction does not work if there is a point z_i in the correlator such that $|z| = |z_i|$ because then, we cannot avoid it yo lie between to two contours

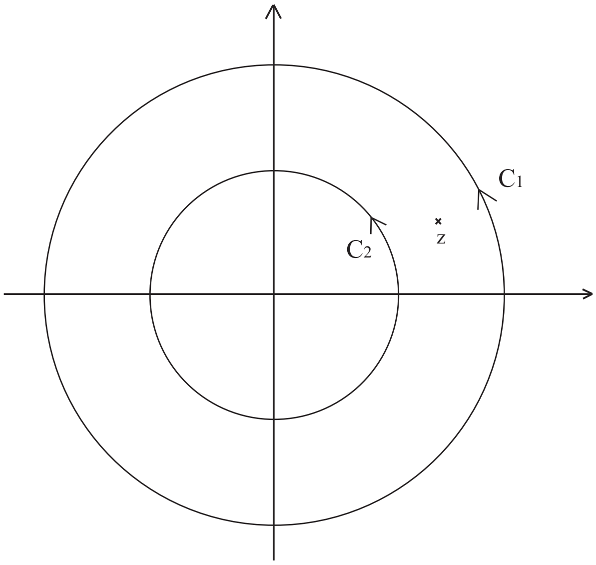

Figure 10

where the generators G_a are defined in (13). This justifies the appelation "conformal charge" for Q_α as well as the usual expression *"the energy-momentum tensor $T(z)$ generates conformal transformations"*.
More generally, if $a(w)$ and $b(z)$ denote two holomorphic fields (i.e. depending only upon one complex coordinate), then

$$\oint_{C_z} dw\, a(w)b(z) = \oint_{C_1} dw\, a(w)b(z) - \oint_{C_2} dw\, b(z)a(w) \tag{110}$$

$$\triangleq [A, b(z)] \quad , \tag{111}$$

where the contours are as in Fig.10, and the operator A is the integral over space at fixed time of $a(w)$: $A = \oint a(w)dw$ (compare to (106)). In the l.h.s. of (110), the radial-ordering symbol R is understood, as everywhere where products of fields are involved. The commutator of two operators, each the integral of a holomorphic field, is obtained by integrating (111) over z :

$$[A, B] = \oint_{C_0} dz \oint_{C_z} dw\, a(w)b(z) \quad , \tag{112}$$

where $A = \oint a(w)dw$ and $B = \oint b(w)dw$. This commutator can also be seen as

$$[A, B] = \left[\oint_{C_1} dw \oint_{C_2} dz - \oint_{C_1} dz \oint_{C_2} dw \right] R[b(z)a(w)] \quad . \tag{113}$$

Indeed, the difference of contours can be deformed as shown in Fig.11, by fixing for instance z and then deforming the contour integral on w as in Fig.10 . We then recover the same integrations as in (112). We will extensively use these relations in the next section. Notice that we have similar

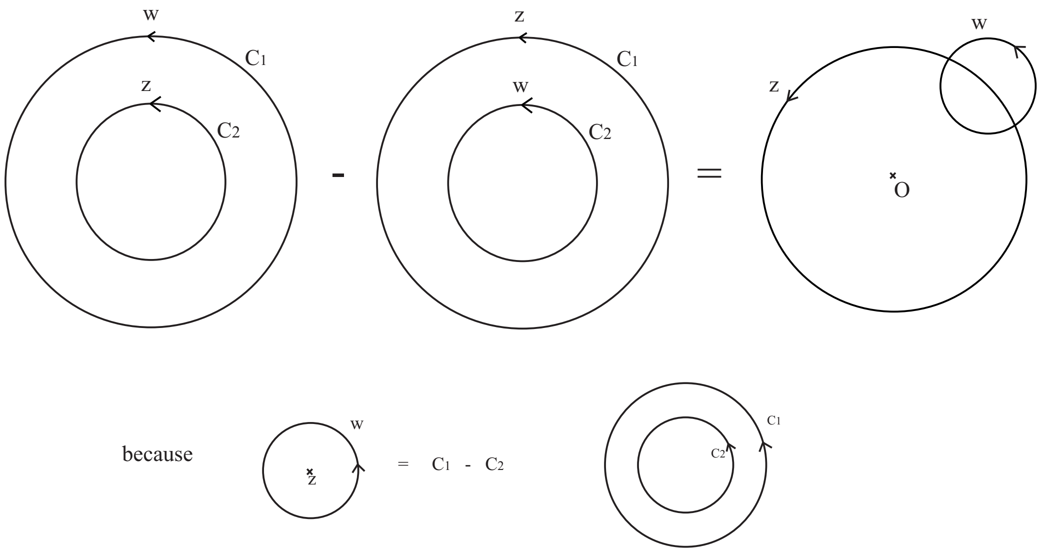

Figure 11

relations for operators defined by a contour integration around an arbitrary point : $A(z) = \oint_z a(w)dw$. This integral does not represent a fixed-time integral, but the commutation of two such operators can still be defined through (112) and (113) by replacing the point $z = 0$ by an arbitrary z.
From now on, whenever a contour integral is written without a specific contour, it is understood that we integrate at fixed time, i.e. along a circle centered at the origin. Otherwise the relevant points surrounded by the contours are indicated below the integral sign.
As a conclusion, formulas (111) and (112) are important because they allow to relate OPE's to commutation relations, as we will see in the next section.

4.2 Mode expansions and Virasoro algebra

A conformal field (primary or quasi-primary) of dimensions $(h, \bar{h})$ may be mode expanded as follows:

$$\phi(z, \overline{z}) = \sum_{m,n\in\mathbb{Z}} z^{-m-h}\overline{z}^{-n-\bar{h}}\, \phi_{m,n} \quad , \tag{114}$$

with

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz\, z^{m+h-1}\phi(z,\overline{z})\,\frac{1}{2\pi i}\oint d\overline{z}\,\overline{z}^{m+\bar{h}-1}\phi(z,\overline{z}) \quad . \tag{115}$$

With this definition, the modes transform under scaling $z{\to}\lambda z$ as $\phi_n{\to}\phi'_n = \lambda^n\phi_n$, thus having scaling dimension n. Eq. (114) corresponds to a Laurent expansion around $z = 0$. Later on we will use a development around a arbitrary point w :

$$\phi(z, \overline{z}) = \sum_{m,n\in\mathbb{Z}} (z-w)^{-m-h}(\overline{z}-\overline{w})^{-n-\bar{h}}\, \phi_{m,n}(w,\overline{w}) \quad , \tag{116}$$

with

$$\phi_{m,n}(w,\overline{w}) = \frac{1}{2\pi i}\oint dz\,(z-w)^{m+h-1}\phi(z,\overline{z})\,\frac{1}{2\pi i}\oint d\overline{z}\,(\overline{z}-\overline{w})^{m+\bar{h}-1}\phi(z,\overline{z}) \quad . \tag{117}$$

We set hereafter $\phi_{m,n}(0,0) \stackrel{\triangle}{=} \phi_{m,n}$.
For the energy-momentum tensor, we have

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad , \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n \quad , \tag{118}$$

where the modes are given by the contour integrals

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z), \quad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{m+1}\bar{T}(\bar{z}) \quad . \tag{119}$$

Considering an infinitesimal conformal transformation $z \rightarrow z + \alpha(z)$, with $\alpha(z) = \sum_n z^{n+1}\alpha_n$, the conformal charge (106) can be reexpressed as

$$Q_\alpha = \frac{1}{2\pi i}\sum_n \oint dw\, \alpha_n w^{n+1} T(w) = \sum_n \alpha_n L_n \quad . \tag{120}$$

The variation of a field then reads with (105)

$$\delta_{\alpha(z)}\phi(z) = -\sum_n \alpha_n [L_n, \phi(z)] \quad . \tag{121}$$

In particular , for a transformation $z \rightarrow z + \alpha_n z^{n+1} \stackrel{\triangle}{=} z + \alpha_n(z)$, we have

$$\delta_{\alpha_n(z)}\phi(z) = -\alpha_n [L_n, \phi(z)] \quad . \tag{122}$$

The mode operators L_n (and similarly $\bar{L}_n$, which we will omit) are the generators of the local conformal transformations on the Hilbert space, exactly like l_n (and $\bar{l}_n$) (see (15)) are the generators of conformal mappings on the space of functions[21]. Using (111), the algebra they satisfy may be derived, with the help of the OPE (96) :

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz\, z^{n+1} w^{m+1}\, T(z)T(w) \\
&= \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz\, z^{n+1} w^{m+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right] \\
&= \frac{1}{2\pi i}\oint_0 dw\, w^{m+1}\left[\frac{c/2}{3!}\partial^3(z^{n+1}) + 2T(w)\partial(z^{n+1}) + \partial T(w) z^{n+1}\right]_{|z=w} \\
&= \frac{1}{2\pi i}\oint_0 dw\, \left[\frac{c}{12} w^{m+n-1} n(n^2-1) + 2T(w)(n+1)w^{m+n+1} + w^{m+n+2}\right] \\
&= n(n^2-1)\frac{c}{12}\delta_{m+n,0} + 2(n+1)L_{m+n} \\
&+ \frac{1}{2\pi i}\left[\oint_0 dw \partial(w^{m+n+2}T(w)) - \oint_0 dw T(w)(m+n+2)w^{m+n+1}\right] \quad ,
\end{aligned} \tag{123}$$

[21] Recall the L_n's generate the conformal transformations of the operators ϕ

hence

$$[L_n, L_m] = \frac{c}{12}n(n^2-1)\delta_{m+n,0} + (n-m)L_{m+n} \quad . \tag{124}$$

A similar relation holds for the $\bar{L}_n$'s and $[L_n, \bar{L}_m] = 0$. This is the *Virasoro algebra*. It can be seen as a central extension of the Witt algebra (16). It is clear that the central term proportional to c is due to the presence of the Schwinger term in (90) and (96). We note that the generators L_{-1},L_0 and L_1 form a Sl$(2,\mathbb{C})$ subalgebra identical to that generated by the l_n's for $n = -1, 0, 1$:

$$[L_{-1}, L_0] = -L_{-1}\,,\ [L_1, L_0] = L_1\,,\ [L_1, L_{-1}] = 2L_0\,. \tag{125}$$

Thus, only the global subgroup Sl$(2,\mathbb{C})$ is not affected by the conformal anomaly. This will be reflected when returning to the Ward identities in Sect. 4.4, where we will see that *the correlation functions are invariant only under the global conformal group.*

4.3 The Hilbert space and representations of the Virasoro algebra

To describe the Hilbert space $\mathcal{H}$, we will use the standard formalism of "in" and "out" states of quantum field theory. We first have to assume the existence of a vacuum state $|\,0\,\rangle$ upon which $\mathcal{H}$ is constructed by application of creation operators (or their likes). In free field theories (in the QFT sense), the vacuum may be defined as the state annihilated by the positive frequency part of the field. For an interacting field, we assume following the usual procedure of QFT that the Hilbert space is the same as for a free field, except that the actual energy eigenstates are different. We suppose that the ic interaction is attenuated as $t \rightarrow \pm\infty$ and that asymptotic fields are free.
In a two-dimensional CFT on the complex plane, the in-states are defined as

$$|\,\phi_{in}\,\rangle = \lim_{z,\overline{z}\rightarrow 0} \phi(z,\overline{z})|\,0\,\rangle \quad , \tag{126}$$

for some quasi-primary field $\phi(z,\overline{z})$. On $\mathcal{H}$, we must also define a bilinear product, which we do indirectly by defining asymptotic out-states together with the action of Hermitian conjugation on conformal fields. One sets

$$\langle\,\phi_{out}\,| = |\,\phi_{in}\,\rangle^+ \quad , \tag{127}$$

with

$$[\phi(z,\overline{z})]^+ \triangleq \overline{z}^{-2h} z^{-2\bar{h}}\, \phi(\frac{1}{\overline{z}}, \frac{1}{z}) \quad , \tag{128}$$

where ϕ is a quasi-primary field of dimensions $(h, \bar{h})$. In Minkowski space, Hermitian conjugation does not affect the space-time coordinates, while in euclidian space, $\tau = it$ must be reversed $(\tau \rightarrow -\tau)$ upon Hermitian conjugation if t is to be left unchanged. So this corresponds to the mapping $z \rightarrow \frac{1}{z^*} = \frac{1}{\overline{z}}$ on the real surface (see Sect. 4.1 and (98)) and explains the interchange $z \rightarrow \frac{1}{\overline{z}}$, $\overline{z} \rightarrow \frac{1}{z}$ in (128). The additional factors on the r.h.s. of (128) may be justified by demanding that the inner product $\langle\,\phi_{out}\,|\phi_{in}\rangle$ is well-defined[22]. Let us focus on the holomorphic part of the stress-energy tensor. By taking the adjoint of the mode expansion (118), one finds on the one hand

$$[T(z)]^+ = \sum_n \overline{z}^{-n-2} L_n^+ \quad , \tag{129}$$

[22] $\langle\,\phi_{out}\,|\phi_{in}\rangle = \lim_{z,\overline{z},w,\overline{w}\rightarrow 0} \langle\,0\,|\phi(z,\overline{z})\phi(w,\overline{w})|\,0\,\rangle = \lim \overline{z}^{-2h} z^{-2\bar{h}} \langle\,0\,|\phi(1/\overline{z},1/z)\phi(w,\overline{w})|\,0\,\rangle = \lim_{\xi,\bar{\xi}\rightarrow 0} \langle\,0\,|\phi(\bar{\xi},\xi)\phi(w,\overline{w})|\,0\,\rangle \xi^{2\bar{h}}\bar{\xi}^{2h}$. We will see in Sect 4.4 that the 2-point functions are completely fixed by conformal invariance, and have the form $\langle\phi(\bar{\xi},\xi)\phi(w,\overline{w})\rangle = \frac{C}{\bar{\xi}^{2h}\xi^{2\bar{h}}}$, so that the product is well defined because the factors cancel out. See also [5] p52, [4] p147.

and on the other hand, from (128)

$$[T(z)]^{+} = \overline{z}^{-4} T(1/\overline{z}) = \overline{z}^{-4} \sum_{n} \overline{z}^{n+2} L_n = \sum_{m} \overline{z}^{-m-2} L_{-m} \quad . \tag{130}$$

These expressions are compatible if

$$L_n^{+} = L_{-n} \quad \text{, and similarly} \quad \bar{L}_n^{+} = \bar{L}_{-n} \quad . \tag{131}$$

We may also determine the action of the L_n's on $\mathcal{H}$. With (117) and (119), we have

$$L_n(z)\phi(z,\overline{z}) \triangleq (L_n\phi)(z,\overline{z}) = \frac{1}{2\pi i} \oint_z dw\,(w-z)^{n+1}\,T(w)\phi(z,\overline{z}) \quad . \tag{132}$$

Consider a primary field $\phi(z)$ of conformal weight h (leaving the anti-holomorphic part aside) and the asymptotic state

$$|\,h\,\rangle \triangleq \phi(0)|\,0\,\rangle \quad . \tag{133}$$

Acting with the L_n's on this state yields

$$(L_n\phi)(0) \triangleq L_n|\,h\,\rangle = \frac{1}{2\pi i} \oint_0 dw\, w^{n+1} T(w)\phi(0)|\,0\,\rangle \quad . \tag{134}$$

Using the OPE (83), one gets

$$L_n|\,h\,\rangle = \frac{1}{2\pi i} \oint_0 dw\, w^{n+1} \left[\frac{h}{w^2}\phi(0) + \frac{1}{w}\partial\phi(0) + reg\right] |\,0\,\rangle \tag{135}$$

$$= \frac{1}{2\pi i} \oint_0 dw\, \left[h w^{n-1}\phi(0) + w^n \partial\phi(0) + w^{n+1}.reg\right] |\,0\,\rangle \quad . \tag{136}$$

For $n \geq -1$, none of the regular terms of the OPE will contribute the integral. We thus find that

$$L_n|\,h\,\rangle = 0 \quad \text{if} \quad n > 0 \quad , \quad L_0|\,h\,\rangle = h|\,h\,\rangle \tag{137}$$

Remark we also have following the same computation that $(L_n\phi)(w) = 0$ for $n > 0$ and $(L_0\phi)(w) = h\phi(w)$.
Eq. (137) defines a *highest weight representation* of the Virasoro algebra (124)[23] of , just like the ones one usually encounters in the $su(2)$ case (see the Lectures on Lie algebras).The operator L_0 measures the conformal dimension of the state, while the L_n's play the rôle of the raising and lowering operators, taking one state to another into a representation. The major difference between the Virasoro (est arrivéééé) representations and those of $su(2)$ is of course that the Virasoro representations are infinite-dimensional. Although the algebra is infinite-dimensional, the Cartan subalgebra (i.e. the maximal set of commuting Hermitian generators) contains just the identity operator and L_0 (from (124), we

[23]Representations of the anti-holomorphic counterpart of (124) are constructed by the same method. Since the two parts of the overall algebra decouples ($[L_n, \bar{L}_m] = 0$), representations of the latter are obtained simply by taking tensor products

see that no pair of generators commute, so we may choose L_0 which, from (137), is diagonal in the representation space). The operator L_0 is used to label the states, and from (124) we find

$$[L_0, L_{-n}] = nL_{-n} \quad , \tag{138}$$

so the state $L_{-n}|\,h\,\rangle$ has eigenvalue $h+n$ under L_0 ($n>0$).
We see that the Virasoro algebra $\mathcal{V}$ plays an important rôle in conformal field theory. The states of the theory span a set of infinite-dimensional representations of $\mathcal{V}$. Eqs. (133) to (137) exhibit what is generally referred to as *field-state* correspondence : the primary fields of the theory are in one-to-one correspondence with the highest-weight states (h.w.s.) of $\mathcal{V}$. Then, using (138), one obtains excited states by successive applications of the raising operators on $|\,h\,\rangle$:

$$L_{-k_1}\cdots L_{-k_n}|\,h\,\rangle \quad . \tag{139}$$

When the L_{-k_i} appear in increasing order of the k_i's, $1 \leq k_1 \leq \cdots \leq k_n$, the states (139) provide a complete basis for the representation descended from $|\,h\,\rangle$, because a different ordering can always be brought into a linear combination of the well-ordered states (139) by applying the commutation rules (124) as necessary. The state (139) is an eigenstate of L_0 with eigenvalue

$$h' = h + k_1 + \cdots + k_n \stackrel{\triangle}{=} h + N \quad . \tag{140}$$

The states (139) are called *descendants* of the asymptotic state $|\,h\,\rangle$ and the integer N is called the *level* of the descendant. The number of distinct, linearly independent states at level N is the number $p(N)$ of partitions of the integer N. The complete set of states generated by the h.w.s. $|\,h\,\rangle$ and its descendants is closed under the action of the Virasoro generators and forms a representation of $\mathcal{V}$, called a *Verma module*. A Verma module in conformal field theory is characterized by the central charge c and the dimension h of the h.w.s., and is denoted by $V(c,h)$. The Verma modules for the anti-holomorphic part are denoted by $\bar{V}(\bar{c},\bar{h})$. Now recall that the hamiltonian of the system was identified with the operator $L_0 + \bar{L}_0$, so that the energy eigenstates belong to the tensor product $V \otimes \bar{V}$. In general ,the total Hilbert space is a direct sum of such tensor products, over all conformal dimensions of the theory

$$\mathcal{H} = \sum_{h,\bar{h}} V(c,h) \otimes \bar{V}(\bar{c},\bar{h}) \quad , \tag{141}$$

where the number of terms in this sum may finite or infinite (we will see that in the simple case of a free boson, this number is infinite).
To close this section, we notice that there is a very special Verma module formed from the identity operator $I = I(z,\overline{z})$. The h.w.s. is just the vacuum $|\,h\,\rangle$, having $h=0$. Clearly $L_1|\,0\,\rangle = 0$ because the vacuum is a h.w.s. The Virasoro algebra tells us that $L_1(L_{-1}|\,0\,\rangle) = 0$, so $L_{-1}|\,0\,\rangle$ is also a h.w.s., with $h=1$. It is common to require that $L_{-1}|\,0\,\rangle = 0$, so that the vacuum is $\mathrm{Sl}(2,\mathbb{C})$ invariant (recall L_{-1}, L_0, L_1 generate a $\mathrm{Sl}(2,\mathbb{C})$ subalgebra). This originates from the following. In a theory, the vacuum is generally taken to be invariant under the underlying symmetry algebra. In principle, we should thus take

$$L_n|\,0\,\rangle = 0 \quad , \forall n \tag{142}$$

but this would imply

$$[L_n, L_m]|\,0\,\rangle = 0 \quad , \forall n,m \quad , \tag{143}$$

which is not true because of the central term in (124) (take e.g. $n = m = -2$). So, the best we can do is

$$L_n|\,0\,\rangle = 0 \quad , n \geq -1 \quad . \tag{144}$$

This condition can be recovered by requiring that $T(z)|\,0\,\rangle$ be well-defined as $z \to 0$:

$$\lim_{z\to 0} T(z)|\,0\,\rangle = \lim_{z\to 0} z^{-n-2} L_n|\,0\,\rangle \tag{145}$$

is well defined if $L_n|\,0\,\rangle = 0$ for $n \geq -1$, which includes in particular invariance under the global subgroup.

4.4 Ward identities strike back : global Ward identities

The operator formalism provides us with a useful framework to investigate how the conformal symmetry puts constraints on correlation functions. In this formalism, a generic N-point function of primary fields is written

$$\langle\,0\,|\phi_1(z_1)\cdots\phi_N(z_N)|\,0\,\rangle \quad , \quad \text{with}\,|z1| \geq \cdots \geq |z_N| \quad . \tag{146}$$

From (142), we know that the vacuum satisfies $L_n|\,0\,\rangle = 0$ for $\forall n \geq -1$. This also implies $(L_n|\,0\,\rangle)^+ = \langle\,0\,|L_n^+ = 0\,\forall n \geq -1$, by using (131). Thus, we find that

$$\langle\,0\,|L_i = L_i|\,0\,\rangle = 0 \quad \text{for} \quad i = -1, 0, +1 \quad , \tag{147}$$

that is for the generators of the global subgroup (that are not affected by the conformal anomaly in (124)). Therefore, we may write

$$\begin{aligned} 0 &= \alpha_i \langle\,0\,|L_i\phi_1(z_1)\cdots\phi_N(z_N)|\,0\,\rangle \\ &= \alpha_i \sum_j \langle\,0\,|\phi(z_1)\cdots\phi(z_{j-1})[L_i,\phi(z_j)]\phi(z_{j+1})\cdots\phi(z_N)|\,0\,\rangle + \alpha_i\langle\,0\,|\phi_1\cdots\phi_N L_i|\,0\,\rangle \end{aligned} \tag{148}$$

where the last term vanishes because of (147) and where α_i is an infinitesimal parameter. But, according to (122),

$$\delta_{\alpha_i(z)}\phi(z) = -\alpha_i[L_i,\phi(z)] \quad , \tag{149}$$

and (148) becomes

$$\sum_j \langle\,0\,|\phi_1(z_1)\cdots\delta_{\alpha_i(z_j)}\phi_j(z_j)\cdots\phi_N(z_N)|\,0\,\rangle = 0 \tag{150}$$

that is, by definition

$$\delta_{\alpha_i}\langle X\rangle = 0 \quad , \quad i = -1, 0, 1 \quad , \tag{151}$$

for $X = \phi_1\cdots\phi_N$. Let us analyze the consequences of (151). Recall that

$$\delta_{\alpha(z)}\phi(x) = -(h\alpha'(z) + \alpha(z)\partial_z)\phi(z) \quad , \tag{152}$$

where $\alpha(z) = \sum_n \alpha_n z^{n+1} \triangleq \sum_n \alpha_n(z)$. We now develop (151) explicitly. For the 1-point function of a primary field, this yields the following set of equations:

$$\begin{cases} \partial_z\langle\phi(z)\rangle = 0\,, \\ (h + z\partial_z)\langle\phi(z)\rangle = 0 \\ (2zh + z^2\partial_z)\langle\phi(z)\rangle = 0 \end{cases} \quad , \tag{153}$$

which imply that $\langle \phi(z) \rangle = cst.$, where the constant may be different from zero only if $h = 0$.
A 2-point function $G_{12}(z_1, z_2)$, with $z_1 \neq z_2$ has to satisfy

$$\langle \delta_{\alpha_i(z_1)} \phi_1(z_1) \phi_2(z_2) \rangle + \langle \phi_1(z_1) \delta_{\alpha_i(z_2)} \phi_2(z_2) \rangle = 0 \tag{154}$$

and thus

$$[h_1 \alpha_i'(z_1) + \alpha_i(z_1) \partial_1 + h_2 \alpha_i'(z_2) + \alpha_i(z_2) \partial_2] \, G_{12}(z_1, z_2) = 0 \quad . \tag{155}$$

For $i = -1$, one gets

$$(\partial_1 + \partial_2) G(z_1, z_2) = 0 \tag{156}$$

and hence

$$G(z_1, z_2) = \mathcal{G}(x) \quad , \quad x = z_1 - z_2 \quad , \tag{157}$$

as is natural from translation invariance. The equation for $i = 0$ then reduces to

$$(h_1 + h_2 + x \partial_x) \mathcal{G}(x) = 0 \quad , \tag{158}$$

with solution

$$\mathcal{G}(x) = C x^{-(h_1 + h_2)} \quad . \tag{159}$$

Finally, by substituting in the equation for $i = 1$, we find

$$(h_1 - h_2) \, x \, \mathcal{G}(x) = 0 \quad , \tag{160}$$

leading to the result

$$G(z_1, z_2) = \begin{cases} C_{12} (z_1 - z_2)^{-2h} & \text{if } h_1 = h_2 = h \,, \\ 0 & \text{otherwise} \end{cases} \quad . \tag{161}$$

Taking the anti-holomorphic part into account would have given rise to an additional factor of $(\overline{z}_1 - \overline{z}_2)^{-2\bar{h}}$.
A similar reasoning also completely fixes the form of the three-point functions (see [3] p43, [2]p105, and [6] p89-97 for an explicit computation). The result is

$$G_{123}(z_1, z_2, z_3) = C_{123} \, z_{12}^{h_3 - h_1 - h_2} \, z_{23}^{h_1 - h_2 - h_3} \, z_{31}^{h_2 - h_3 - h_1} \quad , \tag{162}$$

with $z_{ij} = z_i - z_j$.
As a consequence, the coordinate dependance all n-point functions up to $n = 3$ are completely fixed (up to multiplicative constant) by the (global) conformal symmetry. This can be understood as follows. We have three complex transformations at our disposal ($i = -1, 0, 1$). Using them, we can always bring three variables z_1, z_2, z_3 to three arbitrary fixed points α_1, α_2, α_3, by means of

$$\frac{a z_1 + b}{c z_1 + d} = \alpha_1 \quad , \quad \frac{a z_2 + b}{c z_2 + d} = \alpha_2 \quad , \quad \frac{a z_3 + b}{c z_3 + d} = \alpha_3 \ . \tag{163}$$

These 3 equations for the four complex variables a, b, c, d, $ad - bc = 1$ always have a solution if all the z_i's are different. One usually chooses $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = \infty$. Hence, the entire answer is determined if we know the 3-point function in just three points.
This doesn't work any longer for n-point functions, $n \geq 4$. The 4-point function takes the following form (see e.g. [6], Exercises 3,4,8) :

$$G(z_1, z_2, z_3, z_4) = f\left(\frac{z_{12} z_{34}}{z_{13} z_{24}}\right) \prod_{i<j} z_{ij}^{-h_i - h_j + h/3} \quad , \tag{164}$$

where $h = \sum_{i=1}^{4} h_i$.

4.5 Descendant fields

We found in Sect. 4.3 that a highest weight state $|\, h\,\rangle$ of the Virasoro algebra is obtained by applying a primary field of conformal dimension h to the vacuum state $|\, 0\,\rangle$, see (133), (137). This state is the source of an infinite tower of descendant states of higher conformal dimension (see (139), (140)). Under a conformal transformation, i.e. by application of L_n's, the state $|\, h\,\rangle$ and its descendants tranform among themselves.

Each descendant state $L_{-n}|\, h\,\rangle$ can be viewed as the result of the application on the vacuum of a *descendant field*. Indeed, as seen in (134) :

$$L_{-n}|\, h\,\rangle = L_{-n}\phi(0)|\, 0\,\rangle \triangleq (L_{-n}\phi)(0)|\, 0\,\rangle = \frac{1}{2\pi i}\oint dz\, z^{-n+1}T(z)\phi(0)|\, 0\,\rangle \quad . \tag{165}$$

The descendant field associated with the state $L_{-n}|\, h\,\rangle$ is denoted (see (132))

$$\phi^{(-n)}(w) \triangleq (L_{-n}\phi)(w) = \frac{1}{2\pi i}\oint_w dz\, (z-w)^{-n+1}T(z)\phi(w) \quad . \tag{166}$$

With (166), we may write the complete OPE of $T(z)$ with a primary field (*including all regular terms*) as

$$T(z)\phi(w) = \sum_{k\geq 0}(z-w)^{k-2}\phi^{(-k)}(w) \quad . \tag{167}$$

Because we know the singular part of the OPE (see (83)), we have that

$$\phi^{(0)}(w) = h\phi(w) \quad \text{and} \quad \phi^{(-1)}(w) = \partial\phi(w) \quad , \tag{168}$$

see also below eq. (137). The other descendant fields are defined by the regular part of the OPE, and can be computed explicitly once the regular part is known (see Sect. 4.6)

By taking $\phi(w) = I$, the identity field, we obtain

$$I^{(-n)}(w) = \frac{1}{2\pi i}\oint_w dz\, z^{-n+1}T(z) \quad , \tag{169}$$

so $I^{(-n)}(w) = 0$ for $n \leq 1$, and

$$I^{(-n)}(w) = \frac{1}{(n-2)!}\partial^{(n-2)}T(w) \quad \text{for} \quad n \geq -2 \quad . \tag{170}$$

In particular, this shows for $n = 2$ that *the energy-momentum tensor can be seen as a descendant field of the identity operator (field).*

A remarkable property of these fields is that their correlation functions may be derived from those of their corresponding primary field. Consider the correlator

$$\langle\phi^{(-n)}(w)X\rangle \quad , \tag{171}$$

with $X = \phi_1(w_1)\cdots\phi_N(w_N)$ a string of primary fields of conformal dimensions h_i :

$$\langle\phi^{(-n)}(w)X\rangle = \frac{1}{2\pi i}\oint_w dz\, (z-w)^{-n+1}\langle T(z)\phi(w)X\rangle \quad , \tag{172}$$

where the contour circles w only, excluding the positions of the other fields. This can further be expressed, by "reversing" the contour (see Fig.12), deforming it in a sum over contours circling only one point w_i and by considering, in each contour, the OPE (83) of $T(z)$ with the relevant field $\phi_i(w_i)$:

$$\begin{aligned}
\langle\phi^{(-n)}(w)X\rangle &= -\frac{1}{2\pi i}\sum_i \oint_{w_i} dz\,(z-w)^{-n+1}\langle\phi(w)\phi_1(w_1)\cdots T(z)\phi_i(w_i)\cdots\phi_N(w_N)\rangle \\
&= -\frac{1}{2\pi i}\sum_i \oint_{w_i} dz\,\left[\frac{h_i(z-w)^{-n+1}}{(z-w_i)^2}+\frac{(z-w)^{-n+1}}{z-w_i}\partial_{w_i}\right]\langle\phi(w)X\rangle \\
&= -\sum_i \left[h_i(1-n)(w_i-w)^{-n}+(w_i-w)^{-n+1}\partial_{w_i}\right]\langle\phi(w)X\rangle \\
&\equiv \mathcal{L}_{-n}\langle\phi(w)X\rangle \quad , \quad n\geq 1
\end{aligned}$$

with the differential operator

$$\mathcal{L}_{-n}=\sum_i \left[h_i(1-n)(w_i-w)^{-n}+(w_i-w)^{-n+1}\partial_{w_i}\right] \quad . \tag{173}$$

We have thus reduced the evaluation of a correlation function containing a descendant field to that

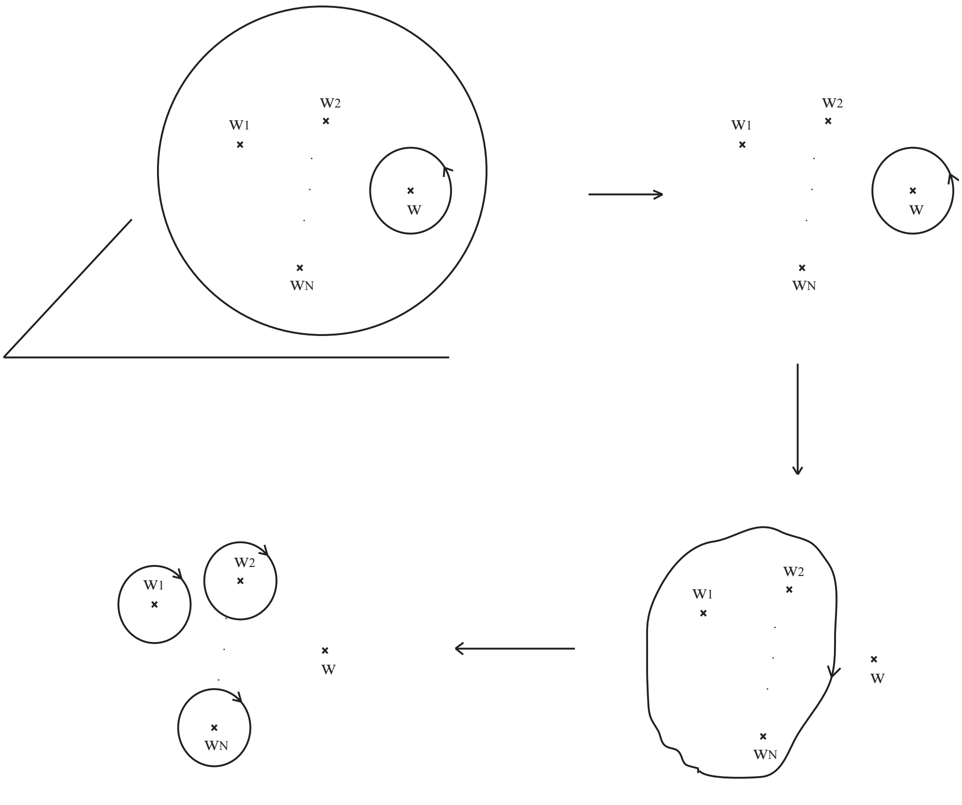

Figure 12

of a correlator of primary fields, on which we must apply a differential operator. Because

$$(\partial_w + \sum_i \partial_{w_i})\langle\phi(w)X\rangle = 0 \tag{174}$$

for any correlator as a consequence of translation invariance (see (150) and (152) for $i = -1$) and $\mathcal{L}_{-1} = -\sum_i \partial_{w_i}$, we have

$$\mathcal{L}_{-1}\langle\phi(w)X\rangle = \partial_w\langle\phi(w)X\rangle = 0 \quad . \tag{175}$$

A general descendant field, corresponding to the state (139), has the form

$$\phi^{(-k_1,\cdots,\kappa_N)}(w) \triangleq (L_{-k_1}\cdots L_{-k_N}\phi)(w) \quad . \tag{176}$$

These field are defined recursively, for instance

$$\begin{aligned} \phi^{(-k,-n)}(w) &= (L_{-k}L_{-n}\phi)(w) \\ &= \frac{1}{2\pi i}\oint_w dz\,(z-w)^{1-k}T(z)(L_{-n}\phi)(w) \quad , \end{aligned} \tag{177}$$

and so on. In particular,

$$\phi^{(0,-n)}(w) = (h+n)\phi^{(-n)}(w) \quad \text{and} \quad \phi^{(-1,-n)}(w) = \partial_w\phi^{(-n)}(w) \quad . \tag{178}$$

The first relation is clear from the Virasoro algebra:

$$\phi^{(0,-n)}(w) = (L_0L_{-n}\phi)(w) = (L_{-n}L_0\phi)(w) + n(L_{-n}\phi)(w) = (h+n)(L_{-n}\phi)(w) \quad . \tag{179}$$

The second expresses the transformation of the fields under translations. The variation of $\phi^{(-n)}(w)$ under translations ($z\longrightarrow z+\alpha_{-1}$, $\Phi(z)\longrightarrow\Phi'(z') = \Phi(z)$) is given by

$$\delta_{\alpha_{-1}}\phi^{(-n)}(w) = -\alpha_{-1}\partial\phi^{(-n)}(w) \tag{180}$$

as for any field. This can also be expressed from (122) and (119), for $n = -1$:

$$\delta_{\alpha_{-1}}\phi^{(-n)}(w) = -\frac{1}{2\pi i}\alpha_{-1}\oint_z dw\,T(w)\phi^{(-n)}(w) \quad . \tag{181}$$

But the r.h.s. of (181) is just $-\alpha_{-1}\phi^{(-1,-n)}(w)$, see (177), and thus with (180) we find the second relation.
Finally, it can be shown that

$$\langle\phi^{(-k_1,\cdots,\kappa_N)}(w)X\rangle = \mathcal{L}_{-k_1}\cdots\mathcal{L}_{-k_N}\langle\phi(w)X\rangle \quad , \tag{182}$$

that is, we simply need to apply the differential operators in succession. We may also consider correlators containing more than one descendant field, but at the end the result is the same : *correlation functions of descendant fields may be reduced to correlation functions of primary fields.* This is why primary fields are of prime interest in CFT's.
The set comprising a primary field ϕ and all of its descendants is called a *conformal family*, and is denoted by

$$[\phi] = \{\phi, (L_{-n}\phi), \cdots, (L_{-k_1}\cdots L_{-k_N})\} \quad , n > 0, k_i > 0 \quad . \tag{183}$$

The members of a family transform amongst themselves under a conformal transformation. This means that the OPE of $T(z)$ (which generates conformal transformations , see (106), (120)) with any member of the family will be composed solely of other members of the same family (conformal fields have an anti-holomorphic part as well, so there will also be descendants of the field through the action of the anti-holomorphic generators $\bar{L}_{-n}$).

4.6 Operator algebra

We have seen in section **4.4** that conformal invariance takes us a step towards solving a given conformal field theory (i.e. to be able, at least in principle, to write down all correlation functions of all the fields present in the theory), by completely fixing the coordinate dependence of the two- and three-point functions. To fix the numerical coefficients C_{ijk} in (162) and to go beyond three-point functions, some additional information is needed (namely the complete OPE of all primary fields with each other). Besides conformal invariance, another basic property of two-dimensional CFT's is that operator products can be defined for arbitrary pair of fields. That is, there is a closed operator product algebra among all the fields (called the *operator algebra* in short). The operator involved is indeed a product in the sense that it is associative. It is expected that this structure is a direct consequence of the fundamental properties of quantum field theory [24] (see e.g. [8] p. 21), but are usually postulates the existence of the operator algebra as a separate input (this is called the *bootstrap approach*, where the set of OPE's is treated as the fundamental information of the theory, for which the whole theory can be reconstructed). In general, the operator algebra is expressed as

$$O_i(x)\ O_j(y) = \sum_k C_{ij}{}^k(x-y)\ O_k(y), \tag{184}$$

where the sum runs over all fields present in the theory. (184) is to be understood as constraints on correlations functions. In particular, if $\phi(y)$ is some field in the QFT,

$$\langle \phi(z) O_i(x)\ O_j(y) \rangle = \sum_k C_{ij}^k(x-y) \langle \phi(z) O_k(y) \rangle \quad , \tag{185}$$

and hence (184) allows to compute all correlation functions of the theory recursively, reducing ultimately to the 2-point functions, which are known. Let us make this more precise. We have seen that the 2-point function vanishes if the conformal dimensions of the two fields are different (161). We are free to choose a basis of primary fields such that

$$\langle \phi_1(w,\bar{w}) \phi_2(z,\bar{z}) \rangle = \begin{cases} \frac{1}{(w-z)^{2h}(\bar{w}-\bar{z})^{2\bar{h}}} & \text{if } h_1 = h_2 = h \text{ and } \bar{h}_1 = \bar{h}_2 = \bar{h}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{186}$$

it is a simple matter of normalization.

Thus, primary fields ϕ_1 and ϕ_2 with different conformal dimensions are orthogonal in the sense of the two-point functions. This is also the case for all the descendants fields of ϕ_1 and ϕ_2, because as we have seen in section **4.5**, correlation functions of descendant fields can be reduced to correlation functions of primary fields. Thus, if $h_1 \neq h_2$, then the conformal families $[\phi_1]$ and $[\phi_2]$ are orthogonal ($\langle \phi_1^{(-n_1,\ldots,-n_h)} \phi_2^{-l_1,\ldots,-l_n)} \rangle = 0$ if $h_1 \neq h_2$). The same is true for the corresponding Verma modules. Let

[24] A heuristic argument goes as follows . According to the OPE assumption, in any QFT, the product of local operators acting at points that are sufficiently close to each other may be expanded in terms of the local fields of the theory: $\phi_i(x)\phi_j(y)\ \sum_k C_{ij}{}^k(x-y)\phi_k(y)$, where $C_{ij}{}^k(x-y)$ are (possibly singular for $x \to y$) functions. This is usually an asymptotic statement in QFT. In QFT though it is believe it becomes an exact statement because scale invariance (dilatations) prevents the appearance of any length l in the theory. So, there is no parameter to control the expansion, and thus no terms like $e^{l(x-y)}$ which would the breakdown of the exactness of the asymptotic expansion above.

$$|h_1,\bar{h}_1\rangle = \phi_1(0,0)|0\rangle \quad (187)$$
$$|h_2,\bar{h}_2\rangle = \phi_2(0,0)|0\rangle, \quad (188)$$

see (126). Then

$$\begin{aligned}\langle h_1,\bar{h}_1|h_2,\bar{h}_2\rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0}\langle 0|\phi_1^+(z,\bar{z})\phi_2(w,\bar{w})|0\rangle \\ &= \lim_{z,\bar{z},w,\bar{w}\to 0}\bar{z}^{-2h_1}z^{-2\bar{h}_1}\langle 0|\phi_1(\frac{1}{\bar{z}}\frac{1}{z})\phi_2(w,\bar{w})|0\rangle \\ &= \lim_{\rho,\bar{\rho}\to 0}\bar{\rho}^{2h_1}\rho^{2\bar{h}_1}\langle 0|\phi_1(\rho,\bar{\rho})\phi_2(0,0)|0\rangle \\ &= \begin{cases} 1 & \text{if } h_1 = h_2 = h \ , \\ 0 & \text{if } h_1 \neq h_1 \ . \end{cases} \end{aligned} \quad (189)$$

using (186).
This is also evident from the fact that two eigenspaces of a Hermitian operator (here L_0) having different eigenvalues are orthogonal. Furthermore, the othogonality of the h.w.s. implies the othogonality of the Verma modules associated to the two fields (this is also as consequence of the orthogonality of the conformal families, because descendant states can be seen as created for the vacuum by the application of the descendant fields, see (165)). The operator algebra takes the general form

$$\begin{aligned}\phi_1(z_1,z_2)\phi_2(0,0) = &\sum_p C_{12}^p\ z^{h_p-h_1-h_2}\ \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2}\times[\phi_p(0,0)+z\beta_p^{(-1)}\phi_p^{(-1)}(0,0)+\bar{z}\bar{\beta}_p^{(\bar{-1})}\phi_p^{(\bar{-1})}(0,0) \\ &+z\bar{z}\beta_p^{(-1)}\bar{\beta}_p^{(\bar{-1})}\phi_p^{(-1,\bar{-1})}(0,0)+z^2(\beta_p^{(-1,1)}\phi_p^{(-1,1)}(0,0)+\beta_p^{(-2)}\phi_p^{(-2)}(0,0)) \\ &+\bar{z}^2(...)+...]\end{aligned} \quad (190)$$

where the sum runs over all conformal dimensions present in the theory, $\phi_p^{(-1)} = (L_{-1}\phi_p)$, $\phi_p^{(\bar{-1})} = (\bar{L}_{-1}\phi_p)$, $\phi_p^{(-1)(\bar{-1})} = (L_{-1}\bar{L}_{-1}\phi_p), ...$ as usual, and the coefficients β_p are constants defining how the descendants of a given family $[\phi_p]$ contribute to the OPE, which can be determined as functions of the central charge and of the conformal dimensions by requiring that both sides of (190) behave identically upon conformal transformations (for an example of such a computation, see [2] p. 181 and [1] p. 51).

We now focus on the z-part of (190):

$$\phi_1(z)\phi_2(0) = \sum_p C_{12}^p z^{h_p-h_1-h_2}[\phi_p(0)+z\beta_p^{(-1)}\phi_p^{(-1)}(0)+z^2(\beta_p^{(-1,1)}\phi_p^{(-1,1)}(0)+\beta^{(-2)}\phi_p^{(-2)}(0))+.. \quad (191)$$

The prefactor $z^{h_p-h_1-h_2}$ ensures invariance under scaling transformations. Indeed, consider $z\to\lambda z$, $\phi(z)\to\phi'(z')=\lambda^{-h}\phi(z)$. Applying this to both sides, we have for the l.h.s. :

$$\phi_1(z)\phi_2(0)\to\lambda^{-h_1-h_2}\phi_1(z)\phi_2(0), \quad (192)$$

while for the first term on the r.h.s.:

$$z^{h_p-h_1-h_2}\phi_p(0) \to \lambda^{-h_1-h_2}\ \lambda^{h_p}\ z^{h_p-h_1-h_2}\ \lambda^{-h_p}\ \phi_p(0). \tag{193}$$

Thus both sides get multiplied by the same factor $\lambda^{-h_1-h_2}$. For the descending fields, one needs additional factors, because $L_0\phi_p^{(-1)} = \phi_p^{(0,-1)} = (h_p+1)\phi_p^{(-1)}$, see (178), which translates into the fact that $\phi_p^{(-1)}(w)$ transforms under dilations (L_0) as a primary field of conformal dimension (h_p+1). So,

$$z^{h_p-h_1-h_2}z\phi_p^{(-1)}(0) \to \lambda^{h_p+1}\lambda^{-h_1-h_2}\lambda^{-h_p-1}\phi_p^{(-1)}(0), \tag{194}$$

and so on for the other descendants at higher levels.
In conclusion, the complete operator algebra may be deduced form the conformal symmetry, the only necessary ingredients being the central charge, the conformal dimensions of the primary fields and the coefficients C_{ij}^k. The latter can be obtained from another source, through a procedure called the conformal bootstrap, consisting essentially in implementing the constraint of associativity of the operator product algebra (crossing symmetry) (see [2]Sections 6.6.4 and 6.6.5,[3] p49,[7] p16,[8] p42,[9] p22). Thus, any n-point function can in principle be calculated from the operator algebra by successive reduction of the products of primary fields. The correlations of descendant fields obtained can be expressed in terms of primary field correlators, and so on, up to two-point functions which are known. So the theory is solved, in principle!

4.7 Ward identities: final chapter

We discussed Ward identities at different places in these notes, using two different formalisms of quantum field theory. Within the path integral representation, we found that the variation of a correlation function $\langle\phi(x_1)...\phi(x_n)\rangle = \langle X\rangle$ under an arbitrary conformal transformation $z \to z + \alpha(z)$, $\phi(z) \to \phi'(z+\alpha(z))$ is given by

$$\delta_\alpha\langle X\rangle = -\frac{1}{2\pi i}\oint_C d\xi\alpha(\xi)\langle T(\xi)X\rangle \tag{195}$$

where C encircles the position of all fields in X. On the other side, in the operator formalism, we observed in section **4.4** that for $\alpha(z)$ corresponding to a global conformal transformation

$$\alpha_g(z) = \alpha_{-1} + \alpha_0 z + \alpha_{-1}z^2 \quad , \tag{196}$$

this variation vanishes:

$$\delta_{\alpha_g}\langle X\rangle = 0. \tag{197}$$

We would now like to verify that (195) effectively vanishes for $\alpha(z)$ of the form (196). This may be done using the operator algebra (191). Let us work out explicitly a simple example. Consider the case $X = \phi_1(z_1)\phi_2(0)$. This product can be expanded using (191). Now, recall from (169) that $T(z)$ belongs to the family of the identity operator I (with $h = 0$), and that conformal families corresponding to different primary fields are orthogonal. Thus, when considering $\langle T(\xi)\phi_1(z_1)\phi_2(0)\rangle$ and expanding $\phi_1(z_1)\phi_(0)$ using (191), the only family involved in the expansion will be that of the identity operator (which is the only (one) primary field with $h = 0$). Then,

$$\langle T(\xi)X\rangle = \langle T(\xi)z_1^{-h_1-h_2}(I + z_1\beta_0^{-1}I^{(-1)}(0) + z_1^2(\beta_0^{(-1,-1)}I^{(-1,-1)}(0) + \beta_0^{(-2)}I^{(-2)}(0)) + ...) \tag{198}$$

But, from (232), we know that $\langle T(\xi)\rangle = 0$, and $I^{(-1)} = (L_{-1}I) = 0$, because the vacuum is $SL(2,\mathbb{C})$ invariant ($L_{-1}|0\rangle = 0$). We have, because $I^{(-n)}(w) = \frac{1}{(n-2)!}\partial^{(n-2)}T(w)$, see (165),

$$\langle T(\xi)X\rangle = z_1^{-h_1-h_2+2}\langle T(\xi)T(0)\rangle\beta_0^{-2} + \text{terms involving}\langle T(\xi)\partial^n T(0)\rangle, n \geq 1 \tag{199}$$

In (195), note that we can send the contour (to infinity, because all operators are already inside C. This way, the integrand behaves as

$$\alpha(\xi)\langle T(\xi)X\rangle \underset{\xi\to\infty}{\sim} \alpha(\xi)\langle T(\xi)T(0)\rangle \propto \frac{\alpha(\xi)}{\xi^4}, \tag{200}$$

where we used (233). Thus for $\alpha(\xi)$ given by (196), the integral in (195) vanishes. We recover the invariance of $\langle X\rangle$ w.r.t. to transformations of the global conformal group. For local transformations it is relation (195) that holds, instead of the expected invariance. Returning to the derivation of section **4.4**, we may see that this originates from the apparition of the central charge c, which prevents to apply the same procedure, because we don't have $\langle 0|L_n = L_n|0\rangle$ $\forall n$, but only for $i = -1, 0, 1$ corresponding to the global subgroup.

4.8 A short story on normal ordering

In this section, I will introduce the notion of *conformal normal ordering*, and show that it reduces in some situations to the more familiar operator normal ordering, consisting in moving annihilator operators to the right. We saw that the OPE of two fields can be written in general as

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{201}$$

The conformal normal ordering of the two fields, denoted by $:AB:$, is by definition

$$:AB:(w) \triangleq \{AB\}_0(w), \tag{202}$$

that is, the term of order $(z-w)^0$ in the OPE. This is a particular prescription, which in general differs from the definition

$$:A_m B_n: = \begin{cases} A_m B_n & \text{for } n > 0\,, \\ B_n A_m & \text{for } n \leq 0\ . \end{cases} \tag{203}$$

where

$$\begin{aligned} A_m &:= A_m(0) = \frac{1}{2\pi i}\int dz z^{m+h_A-1}A(z) \\ B_n &:= \frac{1}{2\pi i}\int dz z^{m+h_B-1}B(z) \end{aligned} \tag{204}$$

are the modes of the fields $A(z)$ and $B(z)$ (see (115)). We will return to this.

The singular terms in an operator product are called *contraction* and are denoted by

$$\underbrace{A(z)B(w)} \triangleq \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}. \tag{205}$$

Again this definition is in general different from the QFT one, where contraction generally means propagator (two-point function).
This is well illustrated in the case of the field $T(z)$. Indeed, with (205) and (96), one has

$$\underbrace{T(z)T(w)} = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)},, \tag{206}$$

while

$$\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4}. \tag{207}$$

From (202), the conformal normal ordering can be represented using contour integration by

$$:AB:(w) = \frac{1}{2\pi i}\int \frac{dz}{(z-w)} A(z)B(w) \tag{208}$$

where, as usual radial ordering is understood in the r.h.s.

The OPE (201) is then

$$A(z)B(w) = \underbrace{A(z)B(w)} + :AB:(w) + O(z-w) \tag{209}$$

Notice that, when the contraction coincides with the propagator, this formula is exactly the one we get in usual QFT, where the ordering is the familiar operator normal ordering (203). A field for which the contraction contains only one singular term (necessarily coinciding with the 2-point function) is called a *free field*. In this situation, both orderings are equivalent, and Wick's theorem may be applied as such. When the field are not free, Wick's theorem must be adapted (see [2] p188).

5 The free boson

Consider the following action for a massless scalar field in two dimensions

$$S = \frac{1}{8\pi} \int d^2x \, \partial_\mu \phi \partial^\mu \phi \quad , \tag{210}$$

or in the usual complex coordinates $z = x^1 + ix^2$, $\partial = \partial_z, \cdots$:

$$S = \frac{1}{4\pi} \int d^2z \, \partial\phi \overline{\partial}\phi \quad , \tag{211}$$

where $d^2z = dzd\overline{z} = 2dx^1dx^2$. This action is invariant under scalings $x \longrightarrow \lambda x$, $\phi(x) \longrightarrow \phi'(x') = \lambda^{-\Delta}\phi(x)$ if $\Delta = 0$.
The equations of motion are

$$\frac{\delta S}{\delta \phi} = 0 \Rightarrow \partial\overline{\partial}\phi(z, \overline{z}) = 0 \quad . \tag{212}$$

The energy-momentum tensor (see (49)) is given by

$$T_{\mu\nu} = \frac{1}{4\pi}(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi) \tag{213}$$

and thus ($g_{z\overline{z}} = 1/2, g_{zz} = g_{\overline{z}\overline{z}} = 0$)

$$T_{zz} = \frac{1}{4\pi}\partial\phi\partial\phi \quad , \quad T_{\overline{z}\overline{z}} = \frac{1}{4\pi}\overline{\partial}\phi\overline{\partial}\phi \tag{214}$$

Later on we will use the following convenient normalization

$$T(z) = -2\pi T_{zz} \quad , \quad \bar{T}(\overline{z}) = -2\pi T_{\overline{z}\overline{z}} \quad . \tag{215}$$

The primary input we need is the OPE of the fields appearing in the action. This can be obtained by the following method. Consider the variation

$$\delta\phi(x) = \varepsilon(x) \tag{216}$$

and its effect on correlation functions of the form $\langle\phi(x_1)\cdots\phi(x_N)\rangle \triangleq \langle X\rangle$. The variation of $\langle X\rangle$ is given by (46), with

$$\delta S = \frac{1}{4\pi} \int d^2x \partial_\mu\varepsilon\partial^\mu\phi \quad . \tag{217}$$

Thus, we have

$$\sum_i \varepsilon(x_i)\langle\phi(x_1)\cdots\phi(\hat{x_i})\cdots\phi(x_N)\rangle = \frac{1}{4\pi} \int d^2x \, \partial_\mu\varepsilon(x)\partial^\mu\langle\phi(x)X\rangle \quad , \tag{218}$$

where $\hat{}$ stands for omission. Integrating the r.h.s. by parts and discarding the boundary term, one has

$$-\frac{1}{4\pi} \int d^2x \, \varepsilon(x)\partial_\mu\partial^\mu_{|x} \langle\phi(x)X\rangle \quad . \tag{219}$$

Therefore, (218) implies that

$$-\frac{1}{4\pi}\Box_x\langle\phi(x)X\rangle = \sum_i \delta^2(x-x_i)\,\langle\phi(x_1)\cdots\hat{\phi(x_i)}\cdots\phi(x_N)\rangle \quad . \tag{220}$$

This can be solved as

$$\langle\phi(x)X\rangle = \sum_i \ln|z-z_i|^{-2}\langle\phi(x_1)\cdots\hat{\phi(x_i)}\cdots\phi(x_N)\rangle + \langle\, :\phi(x)\phi(x_1)\cdots\phi(x_N): \rangle \quad , \tag{221}$$

where $|z-w|^2 = (z-w)(\overline{z}-\overline{w})$. This is understood as follows. The first term originates from the fact that $\Box_x = 4\partial\overline{\partial}$ and that

$$\partial\overline{\partial}\ln|z-w|^{-2} = -\pi\delta^2(x-y) \quad , \tag{222}$$

$x=(z,\overline{z})$, $y=(w,\overline{w})$ and $\int d^2x\,\delta^2(x)f(z) = f(0)$. Eq. (222) follows from the following representation of the δ function :

$$\delta^2(x) = \frac{1}{\pi}\partial_z\frac{1}{\overline{z}} = \frac{1}{\pi}\partial_{\overline{z}}\frac{1}{z} \quad , \tag{223}$$

see e.g. [2] p.119. The second term of the r.h.s. of (221) actually *defines* $:\phi(x)\phi(x_1)\cdots\phi(x_N):$, where the *function* $\langle\, :\phi(x)\phi(x_1)\cdots\phi(x_N):\rangle$ satisfies

$$\partial\overline{\partial}\langle\, :\phi(x)\phi(x_1)\cdots\phi(x_N): \rangle = 0 \quad , \tag{224}$$

i.e. it is a harmonic function. It is thus locally the sum of an analytic and anti-analytic function, and can be Taylor expanded. In particular, *it has no pole as* $x\rightarrow x_i$. Eq. (221) gives us the behavior of the function $\langle\phi(x)\phi(x_1)\cdots\phi(x_N)\rangle$ as $x\rightarrow x_i$, i.e. by definition the OPE of $\phi(x)$ with itself. This is written

$$\phi(z,\overline{z})\phi(z_i,\overline{z}_i) \sim \ln|z-z_i|^{-2} \quad , \tag{225}$$

where this relation is to be understood inside a correlation function with other fields, and $\sim$ means equality modulo terms regular as $x\rightarrow x_i$. If there are no other fields, this gives us the propagator

$$\langle\phi(z,\overline{z})\phi(z_i,\overline{z}_i)\rangle = \ln|z-z_i|^{-2} \quad , \tag{226}$$

up to a harmonic function (regular as $z\rightarrow z_i$). This shows, as expected, that $\phi(z,\overline{z})$ is a *free field*, as defined in the preceding section.
Finally, eq. (221) is usually rewritten as

$$\begin{aligned}\phi(z,\overline{z})\phi(z_i,\overline{z}_i) &= \ln|z-z_i|^{-2} + :\phi(z,\overline{z})\phi(z_i,\overline{z}_i): \\ &= \langle\phi(z,\overline{z})\phi(z_i,\overline{z}_i)\rangle + :\phi(z,\overline{z})\phi(z_i,\overline{z}_i): \quad ,\end{aligned} \tag{227}$$

which is the familiar relation between time(radial)-ordering and normal ordering in QFT.
This can be brought in the form (209) by Taylor expanding the whole series of regular terms $:\phi(z,\overline{z})\phi(z_i,\overline{z}_i):$ as

$$:\phi(z,\overline{z})\phi(z_i,\overline{z}_i): \;=\; :\phi\,\phi:(z_i,\overline{z}_i) + (z-z_i):\partial\phi\,\phi:(z_i,\overline{z}_i) + (\overline{z}-\overline{z}_i):\overline{\partial}\phi\,\phi:(z_i,\overline{z}_i)+\cdots \quad , \tag{228}$$

from which we deduce

$$\phi(z,\overline{z})\phi(z_i,\overline{z}_i) = \ln|z-z_i|^{-2} + :\phi\,\phi:(z_i,\overline{z}_i) + (z-z_i):\partial\phi\,\phi:(z_i,\overline{z}_i) + (\overline{z}-\overline{z}_i):\overline{\partial}\phi\,\phi:(z_i,\overline{z}_i)+\cdots \quad . \tag{229}$$

Focusing on the z sector, we are left with

$$\phi(z)\phi(w) = -\ln(z-w) + :\phi\,\phi:(w) + (z-w) :\partial\phi\,\phi:(w) + \cdots \quad , \tag{230}$$

which coincides exactly with (209). We may also write

$$:\phi\,\phi:(w,\overline{w}) = \lim_{z,\overline{z}\to w,\overline{w}} \left[\phi(z,\overline{z})\phi(w,\overline{w}) - \langle\phi(z,\overline{z})\phi(w,\overline{w})\rangle\right] \tag{231}$$

6 Exercises

6.1 Operator formalism

1. Show that (144) implies that

$$\langle T(z)\rangle = \langle\, 0\,|T(z)|\,0\,\rangle = 0 \tag{232}$$

2. Compute the two-point function of $T(z)$ and show that

$$\langle T(z)T(w)\rangle = \langle\, 0\,|T(z)T(w)|\,0\,\rangle = \frac{c/2}{(z-w)^4} \quad , \tag{233}$$

by using (144), (131) and (124).
This can also be inferred from (96) and (232)

3. Compute the norm of the state $L_{-2}|\,0\,\rangle$ and show that a necessary condition for the theory to be unitary is $c > 0$.

4. Compute the singular part of the OPE of $\phi^{(-1)}(w)$ with $T(z)$, and deduce from this the transformation law of the descendant field $\phi^{(-1)}(w)$. For this, use the expansion $T(z) = \sum_k (z-w)^{-k-2} L_k(w)$ and the Virasoro algebra (124).
(R : $T(z)\phi^{(-1)}(w) \sim \frac{2h\phi(w)}{(z-w)^3} + (h+1)\frac{\phi^{(-1)}(w)}{(z-w)^2} + \frac{\partial\phi^{(-1)}(w)}{z-w}$ and $\delta_{\alpha(z)}\phi^{(-1)}(z) = -h\alpha''(z)\phi(z) - (h+1)\alpha'(z)\phi^{(-1)}(z) - \alpha(z)\partial\phi^{(-1)}(z)$, where h is the conformal weight of the primary field $\phi(z)$ from which the field descends.)

6.2 Free boson

1. From (225), write the OPE of $\partial_z\phi(z,\overline{z})\partial_w\phi(w,\overline{w})$ and $\partial_{\overline{z}}\phi(z,\overline{z})\partial_{\overline{w}}\phi(w,\overline{w})$.

(R : $\partial_z\phi(z,\overline{z})\partial_w\phi(w,\overline{w}) \sim -\frac{1}{(z-w)^2}$ and $\partial_{\overline{z}}\phi(z,\overline{z})\partial_{\overline{w}}\phi(w,\overline{w}) \sim -\frac{1}{(\overline{z}-\overline{w})^2}$. Thus everything splits in a holomorphic and anti-holomorphic part. In the following, we will focus on the holomorphic field $\partial\phi \stackrel{\triangle}{=} \partial_z\phi$.)

2. The energy-momentum tensor (214) is not well defined because $\partial\phi(z)\partial\phi(w)$ have singularities as $z\to w$. Therefore the quantum version of (215) is

$$T(z) = -\frac{1}{2} :\partial\phi\,\partial\phi: \quad , \quad \bar{T}(\overline{z}) = -\frac{1}{2} :\overline{\partial}\phi\,\overline{\partial}\phi: \quad . \tag{234}$$

that is, with (231)

$$T(z) = -2\pi \lim_{w\to z} \left[\partial\phi(z)\,\partial\phi(w) - \langle\partial\phi(z,\overline{z})\,\partial\phi(w,\overline{w})\rangle\right] \quad . \tag{235}$$

Compute the OPE $T(z)\partial\phi(w)$. What does this show? You could need the following Wick's rules:

$$T(\phi_1\phi_2\phi_3) = :\!\phi_1\phi_2\phi_3\!: + :\!\phi_1\underbrace{\phi_2\phi_3}\!: + :\!\underbrace{\phi_1\phi_2}\phi_3\!: + :\!\underbrace{\phi_1\phi_2\phi_3}\!: \quad , \tag{236}$$

with similar relations holding when some factors are already normal-ordered in the the T-product, but with omission of contractions inside a same normal ordering. For instance, $T(\,:\!\phi_1\phi_2\!:\,:\!\phi_3\phi_4\!:\,)$ *does not* contain the term $\langle\phi_1\phi_2\rangle\,:\!\phi_3\phi_4\!:$.
(R : $T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w}$)

3. Compute the OPE $T(z)T(w)$. Check it has the form (96) and deduce the central charge of the system.

4. Consider the following field, called a *vertex operator* :

$$V_\alpha(w) = :\!e^{i\alpha\phi(w)}\!: \quad . \tag{237}$$

a. Compute its OPE with $\partial\phi(z)$
(R: $\partial\phi(z)V_\alpha(w) \sim -\frac{i\alpha}{z-w}\,V_\alpha(w)$)
b. Compute its OPE with $T(z)$ and show this operator is primary with conformal weight $\alpha^2/2$.

5. Compute the OPE $:\!(\partial\phi(z))^k\!:\,:\!V_\alpha(w)\!:$. For this, compute the OPE $:\!A^k\!:\,:\!e^B\!:$, for two general operators A and B.

6. Using the result of Ex.5, show that $:\!e^A\!:\,:\!e^B\!: \,=\, e^{\langle AB\rangle}\,:\!e^A\,e^B\!:$. From this, deduce the correlation function $\langle V_\alpha(z)V_\beta(w)\rangle$ (the correlator being given by the most singular part of the OPE).
(R : See [6], p47)

6.3 Ghost system

The $b-c$ ghost system is defined by

$$\frac{1}{2\pi}\int d^2x\, b_{\mu\nu}\partial^\nu c^\nu \quad , \tag{238}$$

where the field $b_{mu\nu}$ is a traceless symmetric tensor, and where both c^μ and $b_{\mu\nu}$ are fermions. Going to complex coordinates, with $c^z = c$, $c^{\bar{z}} = \bar{c}$, $b_{zz} = b$ and $b_{\bar{z}\bar{z}} = \bar{b}$, the action is rewritten ($d^2x = \frac{1}{2}dzd\bar{z} = \frac{1}{2}d^2z$)

$$\frac{1}{2\pi}\int d^2z\,(b\bar{\partial}c + \bar{b}\partial\bar{c}) \quad . \tag{239}$$

From the equations of motion the fields c and b are holomorphic. They satisfy the following OPE

$$b(z)c(w) \sim \frac{1}{z-w} \quad . \tag{240}$$

The normal-ordered holomorphic energy-momentum is obtained as (see e.g. [2], p133)

$$T(z) = :\!2\partial c\, b + c\,\partial b\!: \quad . \tag{241}$$

1. Compute the OPE's $T(z)c(w)$ and $T(z)b(w)$. Deduce the conformal weights of c and b.

2. Compute the (most singular part of the) OPE $T(z)T(w)$. What is the central charge of this system?

3. This system appears in the covariant quantization of the bosonic string through the gauge-fixing of two dimensional gravity (see [10], Chapter 3). In this context, each space-time coordinate is represented by a free boson. How many free bosons do you need in order for the central charge of the system "D free bosons + ghosts" to vanish? You just found one way of computing the critical dimension of bosonic string theory.

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Appendix A : Another derivation of the conformal Ward identities

This section is mainly based on [2]. Therefore I won't go through all calculations in details, since they may be found in the reference.

Let us start with an action $S[\phi] = \int d^dx\, L(\phi(x), \phi_{,\mu}(x))$, and consider the following infinitesimal transformations

$$x^\mu \longrightarrow x'^\mu = x^\mu + \omega^a \frac{\delta x^\mu}{\delta \omega^a} \quad , \tag{242}$$

$$\phi(x) \longrightarrow \phi'(x') = \phi(x) + \omega^a \frac{\delta \mathcal{F}}{\delta \omega^a}(x) \quad . \tag{243}$$

which are supposed to leave the action S invariant when the infinitesimal parameters $\{\omega^a\}$ are *constant* (the subscript "a" indicating as before a collection of indices). Noether's theorem states that to every continuous symmetry of a field theory corresponds a conserved current, and hence a conserved charge. To determine this current, one may use the "Noether's trick", which consists in looking at the variation of the action under the same transformation, but with ω^a depending on the position. This doesn't leave the action invariant anymore, and the variation may be written in general as

$$S' = S + \text{terms involving derivatives of } \omega^a(x) + \text{terms without derivatives of } \omega^a(x) \quad . \tag{244}$$

Because $S' = S$ when the ω^a's are constants, the last contributions must sum up to zero. Then the variation of the action can be found to be

$$\delta S = \int d^dx\, j^\mu_a\, \partial_\mu \omega_a \quad , \tag{245}$$

with

$$j_a^\mu = \left\{ \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta_\nu^\mu L \right\} \frac{\delta x^\nu}{\delta \omega^a} - \frac{\partial L}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{F}}{\delta \omega^a} \quad . \tag{246}$$

If $\omega^a(x) \rightarrow 0$ at infinity, this relation can be integrated by parts, to yield

$$\delta S = \int d^d x \, \partial_\mu j_a^\mu \, \omega_a \quad . \tag{247}$$

Now, if the field configuration satisfies the classical equations of motion, the action is stationary against any variation of the fields, and δS should vanish for any position-dependent parameters $\omega^a(x)$. This implies

$$\partial_\mu j_a^\mu \approx 0 \quad . \tag{248}$$

This is *Noether's theorem* : every continuous symmetry implies the existence of a current, conserved on-shell.
We may now return to the Ward identity in field theory. We follow the same procedure which led us to (46). In the path integral, we make the following change of variable :

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) - i\omega^a(x) G_a \phi(x) \quad , \tag{249}$$

where the parameters depend on x, and hence δS is not 0 and given by (247). Notice that the variation of a field is here defined by $\delta\phi(x) = -i\omega^a G_a \phi(x)$ and differs from (13) by a factor $-i$. We keep this factor to be coherent with [2] but it is unimportant. We may now rewrite (46) using (247) and (249) as

$$-i \sum_{i=1}^{n} \omega^a(x_i) \langle \phi_1 \cdots G_a \phi_i \cdots \phi_n \rangle = \int d^d x \, \omega^a(x) \, \partial_\mu \langle j_a^\mu(x) \phi_1 \cdots \phi_n \rangle \quad , \tag{250}$$

where $\omega^a(x)$ are arbitrary functions. In the l.h.s., only the values of $\omega^a(x_i)$, $i = 1, \cdots, n$ matter. This equation is thus of the form $\int f(x) h(x) dx = f(y) \phi(y) \, \forall f(x)$, which necessarily implies $h(x) = \phi(x) \delta(x - y)$. From (250), we thus conclude that

$$\partial_\mu \langle j_a^\mu(x) \phi_1(x_1) \cdots \phi_n(x_n) \rangle = -i \sum_{j=1}^{n} \delta(x - x_j) \langle \phi_1 \cdots G_a \phi_j(x_j) \cdots \phi_n \rangle \quad . \tag{251}$$

This is the *Ward identity* for the current j_a^μ. Applying this to translations, we find from (246) that

$$j^{\mu\nu} = -\eta^{\mu\nu} L + \frac{\partial l}{\partial(\partial_\mu \phi)} \partial^\nu \phi \triangleq T_c^{\mu\nu} \quad , \tag{252}$$

and recover the fact that the conserved current associated to translation invariance is the canonical energy-momentum tensor. This also shows that the energy-momentum tensor is conserved inside correlation functions *except at coincident points*.
Notice that the form of the current may be modified from its canonical definition (246) without affecting the Ward identity. We may indeed freely add to it the divergence of an antisymmetric tensor without affecting its conservation. In the case of a system invariant under Lorentz transformations (rotations), it can be shown that there always exist an antisymmetric tensor $B^{\mu\nu\rho}$, with $B^{\rho\mu\nu} = -B^{\mu\rho\nu}$ such that the tensor

$$T_B^{\mu\nu} \triangleq T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} \tag{253}$$

is symmetric *on-shell*, that is $T_B^{[\mu\nu]} \approx 0$. This new tensor is called the *Belinfante tensor* (see [2]p46 for its construction). Furthermore, much in the same way, the energy-momentum tensor of a theory with scale invariance can be made traceless on-shell, under certain certain conditions which are generally assumed to be fulfilled (see [2] p.102). By using this suitably modified energy-momentum tensor, the Ward identities associated with translation, rotation and dilation invariance can be obtained from (251). In two dimensions, the result is (see [2] p106,118,123)

$$\partial_\mu \langle T^\mu_\nu(x) X \rangle = -\sum_{i=1}^n \delta(x - x_i)\partial_{x_i^\nu} \langle X \rangle \tag{254}$$

$$\epsilon_{\mu\nu} \langle T^{\mu\nu}(x) X \rangle = -i \sum_{i=1}^n s_i \delta(x - x_i) \langle X \rangle \tag{255}$$

$$\langle T^\mu_\mu(x) X \rangle = -\sum_{i=1}^n \Delta_i \delta(x - x_i) \langle X \rangle \tag{256}$$

Here X stands for a string of primary fields. In two dimensions, the spin generators are given by $S^{\mu\nu} = s\epsilon^{\mu\nu}$, and Δ stands for the scaling dimensions of the fields. These relations show that, inside correlations functions, the energy-momentum tensor is conserved, symmetric and traceless, except at coincident points. This is the quantum counterpart of the classical statement that the energy-momentum tensor is conserved, symmetric and traceless on-shell. A derivation of these identities which does not require the hypothesis that the energy-momentum can be made classically traceless can be found in [2], p123.

We may now turn to the conformal Ward identity. Let $\varepsilon^\mu(x)$ be (the components of) a conformal Killing vector, satisfying (11), and defining an infinitesimal conformal transformation. We may write

$$\begin{aligned} \partial_\mu(\varepsilon_\nu T^{\mu\nu}) &= \varepsilon_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) T^{\mu\nu} + \frac{1}{2}(\partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu) T^{\mu\nu} & (257) \\ &= \varepsilon_\nu \partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\rho \varepsilon^\rho) T^\mu_\mu + \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha \varepsilon_\beta \epsilon_{\mu\nu} T^{\mu\nu} \quad , & (258) \end{aligned}$$

where we used

$$\begin{aligned} \frac{1}{2}(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) &= \frac{1}{2}(\partial_\rho \varepsilon^\rho)\eta_{\mu\nu} & (259) \\ \frac{1}{2}(\partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu) &= \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha \varepsilon_\beta \epsilon_{\mu\nu} \quad . & (260) \end{aligned}$$

Remark that $\partial_\mu(\varepsilon_\nu T^{\mu\nu}) \approx 0$, therefore $\varepsilon_\nu T^{\mu\nu}$ is sometimes called the *conformal current*. Let us compute

$$\int_M d^2x \, \partial_\mu \langle \varepsilon_\nu(x) T^{\mu\nu}(x) X \rangle \quad , \tag{261}$$

where M is a domain including all insertion points x_i. We will also assume that inside this domain, the functions $\varepsilon(z)$ and $\bar\varepsilon(\bar z)$ have no singularities (this can always be realized if ε and $\bar\varepsilon$ are regular at $x = x_i$). By using (258) and eqs. (254) to (256), as well as $\epsilon^{z\bar z} = -2i$, $h_i = \frac{1}{2}(\Delta_i + s_i)$, $\bar h_i = \frac{1}{2}(\Delta_i - s_i)$, we get, using complex coordinates

$$\begin{aligned} \int_M d^2x \, \partial_\mu \langle \varepsilon_\nu(x) T^{\mu\nu}(x) X \rangle &= -\sum_i (\varepsilon(z_i)\partial_{z_i} + \bar\varepsilon(\bar z_i)\partial_{\bar z_i} + h_i \partial_{z_i}\varepsilon(z_i) + \bar h_i \partial_{\bar z_i}\bar\varepsilon(\bar z_i)) \langle X \rangle \\ &\triangleq \delta_{\varepsilon,\bar\varepsilon} \langle X \rangle \quad , & (262) \end{aligned}$$

where $\varepsilon^z = \varepsilon(z)$, $\varepsilon^{\overline{z}} = \bar{\varepsilon}(\overline{z})$. The l.h.s. can further be expanded with the relation $\int_M d^2x\, \partial_\mu F^\mu = \frac{i}{2}\int_{\partial M}[-dzF^{\overline{z}} + d\overline{z}F^z]$, applied to $F^\mu = \varepsilon_\nu(x)T^{\mu\nu}(x)$. The fact that $T^{\mu\nu}$ is conserved, symmetric and traceless except at coincident points implies that

$$T_{zz}(z,\overline{z}) \;=\; T_{zz}(z) \stackrel{\triangle}{=} -\frac{1}{2\pi}T(z) \tag{263}$$

$$T_{\overline{z}\overline{z}}(z,\overline{z}) \;=\; T_{\overline{z}\overline{z}}(\overline{z}) \stackrel{\triangle}{=} -\frac{1}{2\pi}\bar{T}(\overline{z}) \tag{264}$$

$$T_{z\overline{z}}(z,\overline{z}) \;=\; T_{\overline{z}z}(z,\overline{z}) = 0 \quad , \quad \text{for} \quad (z,\overline{z}) \neq (z_i,\overline{z}_i) \quad , \tag{265}$$

so that (262) finally reduces, because the domain M strictly contains all insertion points (and so the contour $C = \partial M$ does not go through the insertion points), to

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dw\varepsilon(w)\langle T(w)X\ \rangle + \frac{1}{2\pi i}\oint_C d\overline{w}\bar{\varepsilon}(\overline{w})\langle \bar{T}(\overline{w})X\ \rangle \quad . \tag{266}$$

In deriving this relation, we have used the property that the fields in the set X are primary, through the Ward identities (254)-(256) (see their derivation p123 of [2]). However, eq. (266) may be taken as a definition of the effect of conformal transformations on an arbitrary local field within a correlation function, see [2], p122.

References

[1] Vl. Dotsenko, *Série de Cours sur la Théorie Conforme*, http://cel.ccsd.cnrs.fr/cours/cel-2/cel-2.html

[2] Ph. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Springer

[3] A.N. Schellekens , *Introduction to conformal field theory*, Fortsch. Phys, 1996, http://staff.science.uva.nl/~jdeboer/stringtheory/CFT.ps

[4] Y. Saint-Aubin, *Phénomènes critiques en deux dimensions et invariance conforme*, Lecture Notes, Université de Montreal (1987)

[5] E. Kiritsis,*Introduction to Superstring Theory*,hep-th/9709062

[6] C.J. Efthimiou, D.A. Spector, *A Collection of Exercises in Two-Dimensional Physics, Part 1*, hep-th/0003190

[7] J. Fuchs, *Lectures on conformal field theory and Kac-Moody algebras*, hep-th/9702194

[8] P.Ginsparg, *Applied Conformal Field Theory*, in "Fields, Strings and Critical Phenomena, (Les Houches, Session XLIX, 1988) ed. by E. Brézin and J. Zinn Justin, 1989", hep-th/9108028

[9] J. Bagger, *Basic Conformal Field Theory*, http://ccdb3fs.kek.jp/cgi-bin/img_index?200031975

[10] M.Green,J.Schwartz, E. Witten, *Superstring theory*, Cambridge University Press, 1987

[11] J.Polchinsky, *String theory*, Vol.1 ,Cambridge Univ. Pr. (1998)