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**Poincaré synchronization:
From the local time to the Lorentz group**

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1 Introduction

The importance of the works of Lorentz and Poincaré in the founding of the Theory of Special Relativity has been greatly underestimated in the past. But in the last ten years, several authors have denounced this injustice and have pleaded for a more balanced recognition of the contributions of the different actors in the field. Today, I would like to discuss the specific point of the role of Time and to describe the path that Poincaré followed, starting from his first interrogations on the meaning of Lorentz “local time” up to his discovery of the Lorentz group. It is not easy to trace back when and why Poincaré became interested in the meaning of Time, in its measurement and in its role in physical theories. From his education and his researches in Mechanics, he knew of course that Newton mechanics doesn’t allow much freedom in the choice of the time variable: a change of variable $t' = f(t)$ causes generally the appearance of “reaction terms” and the equations become more complicated. This might be at the origin of his idea that Time should be defined in such a way that the equations of Physics be as simple as possible. Alternatively, Peter Galison defended recently the thesis that Poincaré’s idea of synchronizing distant clocks by the exchange of electromagnetic signals finds its origin in his membership of the “Bureau des Longitudes”, where this synchronization of distant clocks was of great concern [1]. Also, the frequent meetings of Poincaré with philosophers as well as his own philosophical preoccupations did play some role. But it is certain that the physicist Poincaré, who followed closely the evolution of physical theories, was most interested by the Lorentz proposal of using an artificial, purely mathematical time, that he called “local time”.

2 Lorentz local time

In 1895, the great Dutch physicist Hendrik Antoon Lorentz who worked for more than twenty years on problems of electro-optical phenomena published a paper that everybody now calls by its first word “Versuch..”, just like for a papal bull [2]. This paper proposed a first and approximate solution to the numerous problems met in attempts to extend Maxwell-Lorentz electrodynamics to the case of moving bodies. This solution of Lorentz was only correct for velocities much smaller than the velocity of light (i.e. to first order in v/c). Lorentz showed that it was always possible to redefine the electromagnetic fields for moving bodies in such a way that all the electro-optical phenomena became just alike as for a body at rest in the ether. However, in order to obtain this compensation of the effects of motion, one had also to make use in the moving body of a new time variable, which depended on the position. Time became a kind of “local time”, like the local time that was in use in the past for distant towns. In order to be more specific, let us consider the case of a body in uniform motion with velocity v with respect to the ether, in a direction defined by some axis x . According to Lorentz (1895), observers bound to this body should not make use of the true universal time t of Newton, but of a “local time” t' which depends on the position through the formal relation:

$$t' = t - vx/c^2 , \tag{1}$$

(where one implicitly assumes that true time and local time be equal at the origin $x = 0$). In Mechanics, and more generally in Physics before the appearance of this “Versuch..”, the connexion between an observer at rest who used (x, y, z, t) coordinates and an observer bound to the moving body who used (x', y', z', t') coordinates, was given by the Galileo relation:

$$\begin{aligned}x' &= x - vt , \\y' &= y , \\z' &= z , \\t' &= t .\end{aligned}\tag{2}$$

(again with implicit conventions of equality at the origin of coordinates and at time zero). This equality of the time variables t' and t makes that if some event A occurs at point x_A on time t_A , it is perceived by the moving observer at point $x'_A = x_A - vt_A$, on the same time $t'_A = t_A$. If two events occur simultaneously at points x_A and x_B , they will be detected by the moving observer as simultaneous events at the points x'_A and x'_B . This common simultaneity is not true any more if one makes use of the Lorentz local time, for the difference of positions causes a time shift equal to :

$$t'_B - t'_A = v \cdot (x_A - x_B)/c^2 \approx v \cdot (x'_A - x'_B)/c^2 .\tag{3}$$

This rather strange situation didn't bother Lorentz who did not even mention it. For him, the introduction of the local time was only an ad hoc change of variable: this local time was not “real”, as was the “true” Newton time. It was only an artificial and purely mathematical trick, that helped to save the appearances.

During this same year 1895, Poincaré published a series of papers where he made a detailed and critical review of the different theories proposed to account for the electro-optical phenomena in moving bodies. On the basis of a few criteria that he considered as “necessary conditions” for the physical consistency of these theories, he concluded that no one of these theories was really satisfactory, reserving however a special attention to Lorentz electrodynamics:

“Il faut donc renoncer à développer une théorie parfaitement satisfaisante et s'en tenir provisoirement à la moins défectueuse de toutes qui paraît être celle de Lorentz.”

Poincaré maintains however a relatively cautious attitude about this “least defectuous theory” that he considers as interesting but by far incomplete, in particular because it violates the Newtonian principle of equality of action and reaction:

“Il est à peine nécessaire de souligner que cette théorie, si elle peut nous rendre certains services pour notre objet, en fixant un peu nos idées, ne peut nous satisfaire pleinement, ni être regardée comme définitive.

Il me paraît bien difficile d'admettre que le principe de réaction soit violé, même en apparence, et qu'il ne soit plus vrai si l'on envisage seulement les actions subies par la matière pondérable et si on laisse de côté la réaction de cette matière sur l'éther.

Il faudra donc un jour ou l'autre modifier nos idées en quelque point important et briser le cadre où nous cherchons à faire entrer à la fois les phénomènes optiques et les phénomènes électriques. Mais même en se bornant aux phénomènes

optiques proprement dits, ce qu'on a dit jusqu'ici pour expliquer l'entraînement partiel des ondes n'est pas satisfaisant."

This rather severe judgement on all the existent theories was followed by a sentence which now appears as really premonitory, being formulated ten years before the discovery of special relativity:

"L'expérience a révélé une foule de faits qui peuvent se résumer dans la formule suivante: il est impossible de rendre manifeste le mouvement absolu de la matière, ou mieux le mouvement relatif de la matière par rapport à l'éther; tout ce qu'on peut mettre en évidence, c'est le mouvement de la matière pondérable par rapport à la matière pondérable."

The main objection of Poincaré against Lorentz theory can be sketched as follows. The ether of Lorentz has no inertial mass and ordinary matter has no mechanical action on it. On the other hand, because of the retardation in the propagation of the electromagnetic fields, the reaction of another distant material body is retarded, so that during some time interval, which can be arbitrary long, the Newtonian principle of reaction is violated. Poincaré will now concentrate his efforts on this difficulty and, a few years later (1900), he will show that the solution goes through the acceptance of Lorentz local time and the attribution of an inertial mass to the electromagnetic field, according to a rule which will become famous some years later: $m = E/c^2$.

3 La mesure du temps

Going on with Poincaré's walking through the study of Time, I would like to quote some sentences of his famous philosophical paper entitled "La mesure du temps", published in 1898 [7]. This paper reveals the already reached depth of his thoughts on the notion of "time" and on its "measurement". For this reason alone, it would be worth a separate communication, but I shall limit myself to a few sentences which are more specifically related to my subject:

"De sorte que la définition implicitement adoptée par les astronomes peut se résumer ainsi: Le temps doit être défini de telle façon que les équations de la mécanique soient aussi simples que possible.";

"Quand un astronome me dit que tel phénomène stellaire, que son télescope lui révèle en ce moment, s'est cependant passé il y a cinquante ans, je cherche à comprendre ce qu'il veut dire et pour cela, je lui demande d'abord comment il le sait, c'est-à-dire comment il a mesuré la vitesse de la lumière. Il a commencé par admettre que la lumière a une vitesse constante, et en particulier que sa vitesse est la même dans toutes les directions. C'est là un postulat sans lequel aucune mesure de cette vitesse ne pourrait être tentée. Ce postulat ne pourra jamais être vérifié directement par l'expérience; il pourrait être contredit par elle, si les résultats des diverses mesures n'étaient pas concordants. (...). Le postulat, en tout cas, conforme au principe de la raison suffisante, a été accepté par tout le monde; ce que je veux retenir c'est qu'il nous fournit une règle nouvelle pour la recherche de la simultanéité, ..." ;

"Il est difficile de séparer le problème qualitatif de la simultanéité du problème quantitatif de la mesure du temps; soit qu'on se sert d'un chronomètre, soit qu'on ait à tenir compte d'une vitesse de transmission, comme celle de la lumière, car on ne saurait mesurer une pareille vitesse sans mesurer un temps.";

"La simultanéité de deux événements, ou l'ordre de leur succession, l'égalité

de deux durées, doivent être définis de telle sorte que l'énoncé des lois naturelles soit aussi simple que possible.”.

These sentences look significant enough for the subsequent history of synchronization and of simultaneity. No doubt that they did catch the attention of some physicists, for the paper is explicitly mentioned in the book “La Science et l’Hypothèse ” published in 1902, a book that was then read by many physicists.

4 The Leyden lecture of 1900

Let us now come back on the famous Poincaré’s lecture given at Leyden on December 1900 at the occasion of the 25th anniversary of the Thesis of Lorentz [8]. In my opinion, this is one of the most important milestones on the way to the discovery of Relativity. It is also one of the most difficult papers for today’s readers and this might be the reason why it is so seldom correctly quoted by physics historians, who are not always physics experts. I shall only quote the following sentence, where Poincaré clarifies the physical meaning of Lorentz local time. This sentence is of uttermost importance for it defines, for the first time, the principle of synchronization of distant clocks at rest in moving bodies:

“Pour que la compensation se fasse, il faut rapporter les phénomènes, non pas au temps vrai t , mais à un certain temps local t' défini de la façon suivante. Je suppose que des observateurs placés en différents points, règlent leurs montres à l’aide de signaux lumineux; qu’ils cherchent à corriger ces signaux du temps de la transmission, mais qu’ignorant le mouvement de translation dont ils sont animés et croyant par conséquent que les signaux se transmettent également vite dans les deux sens, ils se bornent à croiser les observations en envoyant un signal de A en B , puis un autre de B en A . Le temps local t' est le temps marqué par les montres ainsi réglées. Si alors c est la vitesse de la lumière, et v la translation de la terre que je suppose parallèle à l’axe des x positifs, on aura:

$$t' = t - vx/c^2 .”$$

Poincaré didn’t give any detail about this calculation. Perhaps did he think that it was by far too simple to be worth mentioning to the prestigious audience of the meeting. History has alas proven that he was wrong in doing so, because he then let the door open to the possibility that any one else would publish it later under his own name. Poincaré repeated his argumentation a few years later at the St-Louis Conference where he also enounced the Principle of Relativity, for the first time in the history of physics (September 1904). However, in order to take the latest results of Lorentz into account, he completed his previous reasoning by including the Lorentz-Fitzgerald contraction as a complementary hypothesis:

“The most ingenious idea has been that of local time. Imagine two observers who wish to adjust their watches by optics signals; they exchange signals, but as they know that the transmission of light is not instantaneous, they take care to cross them. When the station B perceives the signal from the station A , its clock should not mark the same hour as that of station A at the moment of sending the signal, but this hour augmented by a constant representing the duration of the transmission. Suppose, for example, that the station A sends its signal when its clock marks the hour 0, and that the station B perceives it when its clock marks the hour t . The clocks are adjusted if the slowness equal to t represents

the duration of the transmission, and to verify it the station B sends in turn a signal when its clock marks 0; then the station A should perceive it when its clock marks t . The time pieces are then adjusted. And in fact, they mark the same hour at the same physical instant, but on one condition, namely, that the two stations are fixed. In the contrary case the duration of the transmission will not be the same in the two senses, since the station A , for example, moves forward to meet the optical perturbation emanating from B , while the station B flies away before the perturbation emanating from A . The watches adjusted in that manner do not mark, therefore, the true time; they mark what one may call the local time, so that one of them goes slow on the other. It matters little, since we have no means of perceiving it. All the phenomena which happen at A , for example, will be late, but all will be equally so, and the observer who ascertains them will not perceive it, since his watch is slow; so, as the principle of relativity would have it, he will have no means of knowing whether he is at rest or in absolute motion. Unhappily, that does not suffice, and complementary hypotheses are necessary; it is necessary to admit that the bodies in motion undergo a uniform contraction in the sense of the motion. One of the diameters of the earth, for example, is shrunk by $1/200.000.000$ in consequence of the motion of our planet, while the other diameter retains its normal length. Thus, the last differences find themselves compensated. And then there still is the hypothesis about forces. (...).”

We meet here one of the favorites of Poincaré’s detractors: they claim that he didn’t really understand Relativity for he didn’t realize that the contraction can be deduced from the Principle of Relativity and therefore that it is in no way a complementary hypothesis (cf. f.i. [5, 13]). We shall show in a moment that this criticism doesn’t take the historical circumstances of early presentations of the theory into account, and also not some pedagogical reasons required in later presentations.

5 Poincaré’s local time

Now, I would like to present the calculations that Poincaré implicitly proposed in his explanation of the physical meaning of Lorentz local time. There is no doubt that Poincaré has made such calculations, although as I said, he never cared to publish them. In 1900, Poincaré finds in Lorentz work the following formula’s that relate a frame at rest in the ether where one uses the “true time” (i.e. frame x, t) to a moving frame of velocity v (i.e. frame x', t'):

$$x' = x - vt, \tag{4}$$

$$t' = t - vx/c^2. \tag{5}$$

Formula (4) is clearly the ordinary Galileo transformation formula for such a moving frame. Formula (5) is the Lorentz definition of local time. Both formula’s are supposed to be used if the motion remains very slow (i.e. to first order in v/c). Some years later (i.e. in 1904), Lorentz proposed new formula’s that he claimed to be exact for any velocity smaller than c (i.e. to all order in v/c). These “exact” formula’s take now the Lorentz-Fitzgerald’s contraction into account. According to these authors, a solid body of length L when at rest in the ether, is shortened by a factor g along its direction of motion when it

moves with velocity v in the ether; this contracting factor is equal to :

$$g = (1 - v^2/c^2)^{1/2} .$$

Its real length, when measured in the ether frame is therefore gL . This contraction has to be taken into account in the Galileo transformation (4) which becomes:

$$x' = g^{-1}(x - vt) , \tag{6}$$

Simultaneously, but without any further justification, Lorentz introduced the same factor g^{-1} in the definition of his local time:

$$t' = g^{-1}(t - vx/c^2) . \tag{7}$$

Formula (7) is the new Lorentz local time (1904) that Poincaré justifies at the St-Louis Conference by his synchronization procedure completed by the contraction hypothesis. Formula's (4)-(5) and (6)-(7) are essentially alike, except for the factor g . We can consider g as some common arbitrary factor which, once introduced in (4) or (6), must also be present in (5) or (7). Therefore, let us simply show that, if we accept (4) or (6) as it is, fully justified as corresponding to the Galileo transformation complemented by some arbitrary change of scale g , Poincaré's synchronization gives automatically (5) or (7). We are looking for the coefficients a and b of a linear form:

$$t' = at + bx , \tag{8}$$

as proposed by Lorentz. Remember that the rules of the game, as defined by Poincaré are the following:

- Observers A and B are at rest in the moving frame and they are not aware that this frame is moving in the ether. Therefore, they reasonably assume that the velocity of light is the same in both directions. We place A at the origin and B at some distance L along the x -axis. The distance AB is equal to L when measured in the moving frame (because all sticks, including the measuring sticks are contracted in the same manner), but it appears to be gL when measured in the ether.
- When the watch of A points to $t'_A = 0$, this observer sends a signal to B and A asks B to set his watch at $t'_B = L/c$ at reception.
- Reciprocally, and in order to control the procedure, B sends a signal to A when his watch points to some time $t'_B = t'_0$ and he asks A to check that his watch points to time $t'_A = t'_0 + L/c$ at reception. When this is done, the watches are "synchronized" just in the same way as for a frame at rest in the ether, and both observers can remain convinced that they are at rest in the ether.

Let us show that these two operations define two equations which determine the coefficients a and b of equation (7). We have simply to conduct the reasoning by following the different operations from the point of view of an external observer which is really at rest in the ether and which therefore uses the true Newton time t . The signal sent by A reaches B after some true time t_1 that corresponds to crossing some length D_1 equal to the distance AB as measured in the ether

(i.e. gL) increased by the displacement vt_1 of the moving frame during this time interval t_1 :

$$D_1 = gL + vt_1 = ct_1 . \quad (9)$$

This relation gives t_1 and the distance D_1 that is also the abscissa of B in the ether at the instant of reception of the signal:

$$t_1 = \frac{gL}{c-v} , \quad (10)$$

$$x_B(t_1) = gL \frac{c}{c-v} . \quad (11)$$

The B watch points then to $t'_1 = L/c$. Putting this back in equation (8), one gets a first relation between the coefficients a and b :

$$\frac{L}{c} = (a+bc) \frac{gL}{c-v} . \quad (12)$$

We do just the same for the crossed signal, taking into account that the “local time” $t'_B = t'_0$ has to be converted into the “true time” t_0 by means of equation (8):

$$t'_0 = at_0 + b(gL + vt_0) , \quad (13)$$

and also that the light signal and the frame are now running in opposite directions, so that equation (10) becomes:

$$t_2 - t_0 = \frac{gL}{c+v} . \quad (14)$$

When the signal reaches A , the true time is t_2 , the local time at A is $t'_0 + L/c$ and the coordinate of A in the ether is vt_2 . Therefore, equation (8) now gives:

$$t'_0 + \frac{L}{c} = at_2 + bvt_2 , \quad (15)$$

Substitution of the values of t'_0 and t_2 [given by (13) and (14)] gives us the second relation that we need in order to calculate the unknown coefficients a and b :

$$\frac{L}{c} = (a-bc) \frac{gL}{c+v} . \quad (16)$$

Solving equations (12) and (16) for a and b , we obtain:

$$t' = \frac{1}{g}(t - vx/c^2) . \quad (17)$$

Equations (4), (5), (6), (7) and (17) show clearly that Poincaré was right when, in 1900, he associated the Lorentz local time (5) to the Galileo transformation (4). And that he was right again when, in 1904, he associated the Lorentz local time (7) to the “contracted” Galileo transformation (6). But this analysis shows also that the contracting factor g is in fact totally arbitrary and totally independent of the synchronization. In this sense, the true contraction factor $g = (1 - v^2/c^2)^{1/2}$ remains indeed an independent hypothesis. In other words, the determination of g requires the introduction of some new element in the discussion.

6 The group of parallel translations

The next step of Poincaré will be to use the notion of “transformation group”, which is the real essence of the *Principle of Relativity*. The Lorentz relation that has just been established between a frame at rest in the ether and a frame moving with some velocity v with respect to the ether must also be true between two such moving frames! However, this group property will necessitate a revision of Galileo’s law of addition of velocities. Today we know that Poincaré reached his conclusions after a significant time of reflection, extending from September 1904 to June 1905. At some undefined epoch in between, Poincaré wrote a letter to Lorentz where he mentioned that the Lorentz transformations define such a mathematical group, and he made use of his discovery to solve the problem of finding the appropriate value of the factor g . It is not clear to me whether this first communication of the discovery concerns the restricted group of parallel translations (boosts) or the full Lorentz group (including boosts and spatial rotations). I guess that it only concerns the boosts, because the elimination of the “ g ambiguity” still makes use of some external arguments, coming from Lorentz theory of the electron. The content’s of this letter is essentially the following:

- Let us consider a first transformation (ε, k, f) from the ether frame (x, t) to a moving frame (x', t') :

$$\begin{aligned}x' &= fk(x - \varepsilon t) \\t' &= fk(t - \varepsilon x),\end{aligned}\tag{18}$$

($\varepsilon = v/c$ is the relative velocity in units of the velocity of light, $k = (1 - \varepsilon^2)^{-1/2}$ is the inverse of the Lorentz contraction factor, and f is some arbitrary parameter which in fact represents the ambiguity on our coefficient g).

- Let us also consider a second transformation (ε', k', f') from the same ether frame (x, t) to another moving frame (x'', t'') :

$$\begin{aligned}x'' &= f'k'(x - \varepsilon't) \\t'' &= f'k'(t - \varepsilon'x); \end{aligned}\tag{19}$$

- Then, a completely similar transformation $(\varepsilon'', k'', f'')$ exists between the two moving frames:

$$\begin{aligned}x'' &= f''k''(x' - \varepsilon''t') \\t'' &= f''k''(t' - \varepsilon''x'),\end{aligned}\tag{20}$$

with the following rules defining the corresponding parameters:

$$\varepsilon'' = \frac{\varepsilon' - \varepsilon}{1 - \varepsilon\varepsilon'},\tag{21}$$

$$k'' = (1 - \varepsilon''^2)^{-1/2},\tag{22}$$

$$f'' = ff'.\tag{23}$$

These results call for some commentaries:

- Formula (21) is the first appearance in the history of physics of a new rule for combining velocities ! Before, the composition rule was the old Galileo's relation: $\varepsilon'' = \varepsilon' - \varepsilon$.
- Formula (22) shows that the contraction is only "relative" and is in fact symmetrical: each of the two moving observers will notice a contraction in the frame of the other.
- Finally, the ambiguous factor f is submitted to a multiplication rule. In Lorentz theory of the electron, as well as in many models currently used at that epoch, one parameterized f by the formula:

$$f = f(\varepsilon) = (1 - \varepsilon^2)^m, \quad (24)$$

where the power m is model dependent. Formula () shows that according to the group property, the only possible value of m is zero, i.e. that f must be equal to one, a result already obtained by Lorentz by means of rather subtle dynamical considerations.

Before going further in the analysis of Poincaré's work, I would like to make the following remark. It is somewhat surprising that Poincaré didn't try to go a bit further in the analysis of the multiplication rule (6). Once the group structure is identified, one can just as well consider the following set of transformations: firstly from (x, t) to (x', t') , secondly from (x', t') to (x'', t'') , and then apply the group relations (21)-(6) to the transformation from (x, t) to (x'', t'') . If the second transformation is an infinitesimal transformation $\delta\varepsilon$, equation () becomes:

$$f(\varepsilon) \cdot f(\delta\varepsilon) = f\left(\frac{\varepsilon + \delta\varepsilon}{1 + \varepsilon \cdot \delta\varepsilon}\right), \quad (25)$$

Expanding (25) to first order in $\delta\varepsilon$ and taking the evident condition $f(0) = 1$ into account, equation (25) becomes the differential equation:

$$\frac{df}{d\varepsilon} = \frac{f'(0)}{1 - \varepsilon^2} f(\varepsilon), \quad (26)$$

which is easily integrated. The solution [with $f(0) = 1$] is

$$f(\varepsilon) = \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^{f'(0)/2}. \quad (27)$$

It shows (a posteriori) that, even after the discovery of the group of boosts, some external argument remains necessary in order to eliminate the ambiguity on the contracting factor g . In order to obtain $f(\varepsilon) = 1$, the minimal condition is to assume that $f'(0) = 0$. This condition is of course fulfilled in all old electron theories where, for reasons of symmetry, $f(\varepsilon)$ is assumed to be an even function of the velocity.

7 The Lorentz group

Poincaré went rapidly much farther and considered the full group of transformations, i.e. parallel transformations and spatial rotations. He mentioned briefly his results in a short paper to the *Comptes Rendus*, on June 5 1905 [11]:

“Le point essentiel, établi par Lorentz, c’est que les équations du champ électromagnétique ne sont pas altérées par une certaine transformation (que j’appellerai du nom de Lorentz) et qui est de la forme suivante:

$$(1) \quad x' = fk(x + \varepsilon t), \quad y' = fy, \quad z' = fz, \quad t' = fk(t + \varepsilon x),$$

x, y, z sont les coordonnées et t le temps avant la transformation, x', y', z' et t' après la transformation. D’ailleurs ε est une constante qui définit la transformation

$$k = \frac{1}{\sqrt{1 - \varepsilon^2}}$$

et f est une fonction quelconque de ε . On voit que dans cette transformation l’axe des x joue un rôle particulier, mais on peut évidemment construire une transformation où ce rôle serait joué par une droite quelconque passant par l’origine. L’ensemble de toutes ces transformations, joint à l’ensemble de toutes les rotations d’espace, doit former un groupe; mais pour qu’il en soit ainsi, il faut que $f = 1$; on est donc conduit à supposer $f = 1$ et c’est là une conséquence que Lorentz avait obtenue par une autre voie.”

One month later, he sent a much more detailed paper to the Rendiconti del Circolo matematico di Palermo [12]. Concerning the invariance of electromagnetism, it contains the first complete demonstration of this property with ε and f considered as independent parameters. In this paper, Poincaré makes a detailed analysis of the group of transformations that he generously terms “the Lorentz group”. This analysis is so complete that it remains practically unchanged in today’s teaching. The results are briefly mentioned as follows:

“Nous sommes donc amenés à envisager un groupe continu que nous appellerons le groupe de Lorentz et qui admettra comme transformations infinitésimales:

- $\frac{1}{4}$ La transformation T_0 qui sera permutable à toutes les autres;
- $\frac{2}{4}$ Les trois transformations T_1, T_2, T_3 ;
- $\frac{3}{4}$ Les trois rotations $[T_1, T_2], [T_2, T_3], [T_3, T_1]$.

Une transformation quelconque de ce groupe pourra toujours se décomposer en une transformation de la forme:

$$x' = fx, \quad y' = fy, \quad z' = fz, \quad t' = ft \quad (28)$$

et une transformation linéaire qui n’altère pas la forme quadratique

$$x^2 + y^2 + z^2 - t^2. \quad (29)$$

Nous pouvons encore engendrer notre groupe d’une autre manière. Toute transformation du groupe pourra être regardée comme une transformation de la forme:

$$(1) \quad x' = fk(x + \varepsilon t), \quad y' = fy, \quad z' = fz, \quad t' = fk(t + \varepsilon x), \quad (30)$$

précédée et suivie d’une rotation convenable. Mais pour notre objet, nous ne devons considérer qu’une partie des transformations de ce groupe; nous devons supposer que f est une fonction de ε , et il s’agit de choisir cette fonction, de façon que cette partie du groupe, que j’appellerai P , forme encore un groupe.”

In the latter sentence, Poincaré clearly states that in a relativistic approach of physics, the parameters of the transformation can only depend on the relative

velocity. He then proceeds to determine the value of the function $f(\varepsilon)$, making use only of group properties:

“Faisons tourner le système de $180\frac{1}{4}$ autour de l’axe des y , nous devons retrouver une transformation qui devra encore appartenir à P . Or cela revient à changer le signe de x, x', z et z' ; on trouve ainsi:

$$(2) \quad x' = fk(x - \varepsilon t), \quad y' = fy, \quad z' = fz, \quad t' = fk(t - \varepsilon x), \quad (31)$$

Donc f ne change pas quand on change ε en $-\varepsilon$. D’autre part, si P est un groupe, la substitution inverse de (1), qui s’écrit:

$$(3) \quad x' = (k/f)(x - \varepsilon t), \quad y' = y/f, \quad z' = z/f, \quad t' = (k/f)(t - \varepsilon x), \quad (32)$$

devra également appartenir à P ; elle devra donc être identique à (2), c’est-à-dire que

$$f = 1/f.$$

On devra donc avoir $f = 1$.”

It is not my task to recall here the numerous original results that are contained in this paper of July 1905. But I cannot leave the subject without mentioning at least those results that are directly related to his group analysis, i.e. results that nobody but he has found:

- He shows that the set of transformations that he terms “Lorentz transformations” defines, together with the set of spatial rotations, a mathematical group that he terms “Lorentz group”, a name that is currently accepted today. He constructs the infinitesimal operators of the group and he discovers the “invariant quadratic form” $x^2 + y^2 + z^2 - t^2$. He shows that the Lorentz group is in fact the group of rotations around the origin in some four dimensional space with coordinates: x, y, z, it .
- He shows that several individual physical quantities, either mechanical or electromagnetical, should be taken together as “four partners” which transform under the group transformations as the coordinates do. This is the discovery of quadrivectors, but the term is not used.
- He discovers the electromagnetic invariants:

$$\mathbf{E} \cdot \mathbf{H} \quad \text{and} \quad \mathbf{E}^2 - \mathbf{H}^2,$$

and he uses them in a clever way to study the radiation emitted by a moving electric charge. He also discovers the invariance of the electromagnetic action-integral.

- In his study of the electron, he introduces a new element that is today termed “Poincaré pressure” ; however, its meaning is not yet fully elucidated. He shows that the dynamics of the electron can then be formulated in a complete relativistic way. He writes explicitly the correct relativistic equation of motion of a particle. Unfortunately, this relativistic equation of motion of Poincaré is written in units where $m = 1$ and $c = 1$, and that circumstance makes that the famous mass energy relation is not clearly exhibited.

- He formulates the constructive hypothesis that all forces in nature should be constrained by the Lorentz covariance law (alike the electromagnetic force).
- Finally, he uses his full understanding of the Lorentz group and his remarkable mathematical skill to try to construct a relativistic theory of gravitation. He formulates the hypothesis of the existence of gravitational waves propagating with the same velocity as light. And he adds that if this was true, this remarkable property should be considered as a property of the ether.

8 About some critics on Poincaré's work

A recurrent reproach to the work of Poincaré on Relativity is that he didn't deduce the Lorentz transformations from "*first principles*". On this point, I would like to make the following general remarks:

- Firstly, it is extremely rare in science that true "creators" make their discoveries when starting from "first principles". Small and successive creation steps, if not the method of trials and errors, generally makes real innovations. Scientists that succeed to create "something new" when starting from first principles are more often followers who already know the results from the work of other scientists. That they do or do not mention their original sources is something that should be examined. Something also that is certainly related to their own responsibility as scientists. and to their honesty.
- Secondly, Poincaré was a mathematician which didn't much like "axiomatic" theories. In his mathematical work, he often reserved an important part of the creation to intuition. Physics is essentially a matter of experience. Mathematical physics is a branch of Physics and of Mathematics where "Fundamental Principles" are to be considered as an elegant way to formulate the results of many experiences. He explicitly recalled the point in his talk at the St-Louis Conference which was precisely titled: "The Principles of Mathematical Physics" [10].

Another recurrent reproach to the work of Poincaré on Relativity is that he still used the notion of "ether", even after 1905. This is of course a rather bad trial based only on the affirmation that ether was definitively eliminated from physics after this date. But if one carefully reads Poincaré's works, one realizes that his position concerning the notion of ether was much more ambiguous than for the large majority of the physicists of this epoch. Poincaré uses ether as a way of thinking Physics and of presenting it to other people. For instance, we read in "*La Science et l'Hypothèse*" Ch. XII (1902) :

"Peu nous importe que l'éther existe réellement, c'est l'affaire des métaphysiciens; l'essentiel pour nous c'est que tout se passe comme s'il existait et que cette hypothèse est commode pour l'explication des phénomènes. Après tout, avons nous d'autre raison de croire à l'existence des objets matériels? Ce n'est là aussi qu'une hypothèse commode; seulement elle ne cessera jamais de l'être, tandis qu'un jour viendra sans doute où l'éther sera rejeté comme inutile. Mais ce jour-là même, les lois de l'optique et les équations qui les traduisent

analytiquement resteront vraies, au moins comme première approximation. Il sera donc toujours utile d'étudier une doctrine qui relie entre elles toutes ces équations.”

It is clear that Poincaré's discovery of the Lorentz group reduces *de facto* the kinematical role of the ether to nothing at all. But he probably considered that it was useful to keep on speaking of it, not necessarily as an existing being, but simply as a useful pedagogical tool.

Addendum

During the discussion that followed this communication, a distinguished physicist of the audience argued that Poincaré's understanding of the relativistic time transformation was not correct, even as late as 1908. His argumentation concerns a rather strange formula that can be found in two different papers by Poincaré, although not exactly written in the same way (cf. below Refs. [14] and [15]). This formula looks as follows [15]:

$$\tau = t - AB' \frac{e}{v\sqrt{1-e^2}} .$$

AB' is the difference of the abscissae of two stations (observers) A and B , at rest in a frame moving in the ether along the x -axis, stations that do not necessary lie on this axis. In [14], the formula contains a factor 2 in front of AB' . With modern usual notations, one would write:

$$\tau = t - \Delta x \frac{\beta}{v\sqrt{1-\beta^2}} .$$

The formula is certainly *not* correct if one interprets the symbols τ and t as the time coordinates of an event in two different systems in relative motion. But this is *not* the case: t is the time delay for a light signal going from A to B , and τ is the time delay for the return signal from B to A ! Looking back at the context, one realises that Poincaré was investigating two different problems:

- firstly, he wanted to show (without using group theoretical arguments) that the Lorentz-Fitzgerald contraction factor [i.e. $g = (1 - v^2/c^2)^{1/2}$] is the only possible choice that can give a consistent synchronization between arbitrary positioned stations (i.e. stations that do not necessary lie on the x -axis).
- secondly, he wanted to show that the difference of the time delays for a signal going from A to B and a signal coming back from B to A is rigorously proportional to the difference of the abscissae. This circumstance is a necessary condition to explain Michelson's results.

I agree that Poincaré's argumentation can be somewhat confusing for readers who are not familiar with his rather subtle use of “elongated light ellipsoids”. However, his reasoning is easily converted into an ordinary synchronization calculation for stations positioned as above. Let us consider two stations at rest in the moving system: station A being as usual located at the origin of the coordinate system, station B being located at some point $x' = L, y' = H, z' = 0$. Let us make the same conventions as before: (i) at true time $t = 0$, the two

coordinate systems coincide and the local time of A is also $t' = 0$; (ii) for the observer at rest in the ether, there exists a contraction factor g along the x -axis so that the difference of the abscissae of A and B in the ether system is gL . A simple adaptation of the calculation presented in Sec. 5 above gives then the following results (in ether time) :

- the (true) time delay of a light signal going from A to B is:

$$t_1 = \frac{1}{c^2 - v^2} \left(gLV + \sqrt{(c^2 - v^2)H^2 + (cgL)^2} \right)$$

- the (true) time delay of the light signal coming back from B to A is:

$$t_2 = \frac{1}{c^2 - v^2} \left(-gLV + \sqrt{(c^2 - v^2)H^2 + (cgL)^2} \right)$$

Therefore the true time total duration of this physical go and back process is:

$$T = \frac{2}{c^2 - v^2} \sqrt{(c^2 - v^2)H^2 + (cgL)^2}$$

Since A and B are in relative rest, this go and back process registered at A has a “local A time” duration:

$$T' = \frac{2}{c} \sqrt{L^2 + H^2} .$$

However, A being located at the origin of the moving frame, the relation between “local” and “true” times at A is simply:

$$T = g^{-1}T' .$$

Therefore, we see that the Lorentz-Fitzgerald choice $g = (1 - v^2/c^2)^{1/2}$ is the only possibility! On the other hand, the difference $t_1 - t_2$ of the duration’s is clearly proportional to L , i.e. to the difference of abscissae of the stations. Once the value of the contraction factor has been settled, it becomes evident that this difference is the one given by Poincaré [14]:

$$t_1 - t_2 = 2 \frac{gv}{c^2 - v^2} L = 2 \frac{v}{c\sqrt{1 - v^2/c^2}} AB' .$$

This answers the question.

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