

Henri Poincaré and XXth Century Topology

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ABSTRACT. About 110 years ago Poincaré published his famous work "Analysis Situs" starting Topology as a new area of mathematics. What was done in this and subsequent works of Poincaré exactly? How Topology was developed after that and became a central part of mathematics in the 1950s and 60s followed by a 10 year long period of decay in the 1970s? What is the status of Topology today? We present here our vision of these subjects.

* This lecture is dedicated to the 150th anniversary of Henri Poincaré and 100th anniversary of Henri Cartan.

1 Introduction

Exactly 110 years passed since Henri Poincaré wrote his famous memoir "Analysis Situs" (AS, 1894). Topology under the name Analysis Situs appeared as a new branch of Mathematics.

I published my first work in 1959. Since the 1970s I have worked in different areas of Mathematics and Mathematical Physics. However, I consider myself primarily as a topologist. During my lifetime I have heard a lot of romantic tales about the first works of Poincaré and his predecessors partly based on the oral traditions of topologists.

Without further quotations, I always refer to the Encyclopedia Volume [14] and recent Historical Article [15].

I am trying to answer the following questions I and II:

Question I. What exactly did Poincaré in his works?

Let me quote here the full list of his topological works [1–11].

The work [12] started what we call today "Symplectic Topology" (see [23]). Strictly speaking, it is not a topological work.

One more related work [13] on the non-selfintersecting closed geodesics on the boundary of the convex body in R^3 will be also discussed later: solving it, people started what we call now the Morse Theory (see [21]).

Question II. How were his ideas reflected in the works of XXth Century Topologists?

My opinion is expressed in the recent article [15] where more details can be found, including the names of the leading topologists after 1955.

2 Prehistory

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Ancient Greeks, knots and Alexander The Great:

Some topological differences between knots were observed by the Ancient Greeks. They offered a knot problem to Alexander the Great. He took sword and unknotted it in his own way. Is it not exactly what our Nature is doing with the linked spirals of DNA in living creatures? Otherwise, not one of us could be born.

Euler: Two topological observations

Probably, Euler was the first mathematician who started to do mathematics here. It was some sort of game for him, no motivation by any applications is known. Two topological observations are attributed to him:

1. Euler identity for the numbers of Vertices (V), Faces (F) and Edges (E) of the Convex Polytop in euclidean 3-space:

$$V - E + F = 2$$

According to Poincaré (see [3]), some french "admiral" extended this relation to the nonconvex case "with holes".

2. The Imbedding Problem of graphs in R^2 : Three houses on the plane cannot be connected with three wells by paths not crossing each other.

Gauss: Topological Quantities in the New Physics

He invented a Linking Number for any pair of closed curves $r_1(t), r_2(s)$ in the 3-space not crossing each other. Gauss wrote this quantity analytically:

$$\{r_1, r_2\} = \oint_{r_1} \oint_{r_2} \frac{(dr_1 \times dr_2, r_1 - r_2)}{|r_1 - r_2|^3}$$

He also found the famous global relation for the curvature K of the closed convex surface S :

$$\int \int_S K d\sigma = 4\pi$$

(see [18]). The idea of the Linking Number had a background in Electromagnetism. Gauss pose a problem to his pupil Listing to develop a mathematical theory of such properties. Listing did something in the 1840s and invented the word "Topology". Poincaré did not use it.

Maxwell: Critical points of functions.

For the isolated island there is a relation between minima M_0 , maxima M_2 and saddles M_1 of the height function:

$$M_0 + M_2 - M_1 = 1$$

Kelvin:

He found a 3D integral associated with vorticity:

$$I = \int \int \int_{R^3} (v, \text{curl } v) d^3x$$

About 1950 J.H.C. Whitehead expressed the so-called Hopf Invariant for the homotopy groups of spheres in the form of the Kelvin Integral (see in my Encyclopedia book [14]).

Kelvin intended to classify atoms through the topology of knots (turned out to be a completely wrong idea). His pupil Tait started to study knots in the late XIX Century. Some of his observations were proved only in the late 1980s using such new topological discoveries as the Jones Polynomial (see in the book [19]).

Cauchy and Riemann:

Complex Analysis and Theory of Riemann Surfaces were the immediate predecessors of Poincaré.

Betti:

According to Poincaré, it was Betti (chemist) who invented homology.

We see that Physicists and Chemists always loved topological ideas if they are clear and non-abstract. Does not it remind the same phenomenon today?

3 Topological works of Poincaré. Their Contribution and immediate byproduct

As already mentioned, Poincaré published 11 works in pure Topology; the central one is the work Analysis Situs (AS)-see [3]. Two Comptes Rendus notes preceding AS were published in 1892 and 1893. He named Riemann and Betti as the Ideological Predecessors in this work: Riemann developed Analysis Situs for Riemann Surfaces; Betti invented Cycles and Homology. According to my information, Betti was a Chemist. It explains such terminology as Homology.

Now we start to discuss the contents of the work *Analysis Situs*:

No motivation by the applications was given. Poincaré mentioned only that these kind of ideas already were used by him in order to construct the qualitative theory of dynamical systems. He expected that these ideas would be enormously useful in future mathematics.

The following subjects were discussed in detail in AS:

1. Definition of Manifolds;
2. Cycles and Homology;
3. Intersection Index and Duality;
4. Differential Forms and Cycles;
5. Extension of the Euler Characteristics for Polyhedra;
6. Fundamental Group;
7. Manifolds and Discrete Groups;
8. Another approach to Manifolds, Polytopes.

His definition of manifolds exactly describes the class C^1 -manifolds with nondegenerate imbedding in the euclidean space. Poincaré understood the idea of orientation and used it.

As we know, these ideas could not be taken as an initial base for the construction of Topology at that time. Only after the discovery of transversality and other tools for dealing with differentiable manifolds by Whitney in the 1930s did such a program became realizable (see in [14]). Poincaré dropped this approach in the later works.

The definition of cycles and homology classes is taken from Betti. Cycles are the linear combinations of closed orientable submanifolds with integral or rational coefficients. The homological equivalence of cycles is given by the submanifolds with boundary. **This definition is also wrong as we know:** it is based on the nonlocal objects. After the proper corrections it leads to the "bordism" and "cobordism" groups instead of homology. It was M. Atiyah who invented this extraordinary homology theory in the early 1960s. This nonlocal theory is much more complicated and surprisingly rich (see in [14]). Details can be found in my article [17] where the methods of Algebraic Topology were reconsidered and refined using the Cobordism Theory. Let me point out that it could be constructed only after the Great Revolution in Topology (1935-1955). It was R. Thom who clarified relationship between the cycles and submanifolds in the early 1950s; his works used all of the new machinery of Algebraic Topology. No way is known to prove these results elementary for multidimensional manifolds $n \geq 5$.

The right local combinatorial approach to the construction of homology theory was found by Poincaré in his later works. So one might say that it was Poincaré who invented what we now call homology (see [18], volume 3).

Poincaré formulated the Duality Law for the complementary Betti numbers of the oriented closed manifolds $b_i = b_{n-i}$. He understood it as a byproduct of the nondegeneracy property of the intersection index form between the cycles of complementary dimensions and formulated corollaries of that for the middle-dimensional homology. Within this approach one cannot prove these facts. Only later works of Poincaré contain a basis for that. S. Lefschetz developed these ideas very far in the 1920s and 1930s. The proofs of his results for algebraic manifolds were found only in the 1950s in some cases (see [18], vol 3).

Poincaré described differential forms (without formalism of external product). He formulated what we call "De Rham Theorem". He never returned to

differential forms in the later topological works. This approach was developed in the 1920s: E.Cartan invented convenient formalism of exterior product and constructed the (co)homology ring based on differential forms. How to prove that they are isomorphic to the ordinary homology? It was formulated by Poincaré but no ideas were presented by him. This problem was solved by E.Cartan himself in the special cases like symmetric manifolds. De Rham completely solved this problem in the 1930s, but convenient simple proof was found later through the theory of sheafs (see [18,20]).

The universal extension of Euler Characteristics for all Polyhedra and invention of the Fundamental Group were the impressive achievements of Poincaré in AS. There was no rigorous way for him at that time to calculate fundamental group except comparison with discrete groups but he presented ideas of how to write generators and relations. Combinatorial group theory did not exist at that time. One might say that Poincaré started it. It was developed by M.Dehn in the first decade of the XXth Century who actively applied it in the theory of knots. It is interesting that no traces of knots can be found in the works of Poincaré (I failed to find any).

It was finally clarified in the 1950s and 60s by Papakiriakopulos, Haken and Waldhausen that the fundamental group provides a complete set of invariants for knots (see my recent history article [17]). The theoretical algorithm was found to test unknotness. In the 1930s Nielsen and Magnus made an important contribution to the combinatorial group theory, later developed by the famous people in Algebra and Theory of Algorithms. It is interesting to mention, however, that even in the late XXth Century some important new group theoretical ideas and results heavily used its association with topology.

Poincaré had already began to work with polyhedra in AS but only in the next works did this approach start to dominate. **In the series of works after AS Poincaré found the right local construction of the homology theory based on the chains in polyhedra.** It leads to the exact treatment of Poincaré duality, to the calculations of homology. The Fundamental Group also was effectively developed through the combinatorial approach. The Torsion numbers appeared in his work.

Poincaré finally established that the full nonabelian fundamental group is needed for the characterization of the 3-sphere (The famous **Poincaré Conjecture**—see [11]).

Let us mention that the fundamental group combined with all homological quantities cannot characterize 3-manifolds completely: A new remarkable invariant was discovered by Reidemeister in the 1930s leading to the classification of lens spaces. It is homotopy non-invariant quantity. Later its extension by J.H.C. Whitehead led to the theory of noncommutative determinant about 1940 (see [14]).

The development of Poincaré duality by Lefschetz and Alexander led to the duality in the group theory (Pontryagin), to the idea of cohomology and its ring structure in the 1930s (Kolmogorov and Alexander).

The following Conjecture was formulated by Poincaré in the work [13] published in 1905: Prove that the number of nonselfintersecting closed geodesics on the convex 2-surfaces is 3 or more?

A new variational approach was found by Birkhoff in 1913 and developed by Morse in the 1920s. Finally this conjecture was proved by Liusternik and Schnirelman in the early 1930s for all Riemannian manifolds homeomorphic to

the 2-sphere; Complete verification of this proof was only finished 60 years later.

Poincaré also started topology of symplectic maps in his last mathematical work [12] published in 1912. Later this discovery led to Symplectic Topology (see [23]).

Let me point out that in the philosophical article published in 1912 Poincaré wrote that the most important topological problem for him is how to prove that our space is 3-dimensional. Is this quantity a topological invariant?

Our Conclusion from that is the following: **Poincaré considered Continuous Homeomorphism as a most fundamental equivalence relation for manifolds.** For the C^1 -diffeomorphic manifolds their dimensions are obviously equal. This problem was solved by Brauer in 1913 who invented the degree of map. In 1915 Alexander proved homotopy invariance of homology and invented homotopy type. His results were made rigorous in the 1940s by Eilenberg and others who formalized the singular homology theory. Cell complexes appeared in the 1940s. About 1950 they were used by Thom and others in order to make rigorous the so-called Morse Theory with the help of transversality technic.

4 Topology after Poincaré

I see XXth Century Topology as a series of the following periods:

I. The Post-Poincaré period

Several outstanding people developed the ideas discussed by Poincaré. Most of their names have already appeared in the previous part. Let me add the name of H.Hopf, who really discovered in the 1930s the new deep homotopy topology associated with the homotopy groups of spheres. Hopf started many fundamental directions between 1920 and 1950.

II. The Great Revolution in Topology (1935-1955)

1.The Theory of Smooth Manifolds, including the idea of Transversality and Analysis on Manifolds was created; The Fibre Bundles, Connections and Characteristic Classes were discovered,

2.Various Homotopy Obstruction Theories were constructed. The Categorical Homology Theories of Spaces, Sheafs and Fibre Bundles were developed leading to the great Homological Methods like Exact and Spectral Sequences, Cohomological Operations and other tools; Huge machinery for calculation of the Homotopy Groups of Spheres and other spaces was constructed; the Cobor-disms were essentially calculated; The Homological Algebra and Hopf Algebras were invented.

Let me give the list of the most important names associated with that period:

H.Whitney, H.Hopf, L.Pontryagin, S.Chern, Hodge, N.Steenrod, J.H.C.Whitehead, S.Eilenberg-S.McLane, J.Leray, J-P.Serre, H.Cartan, R.Thom, A.Borel.

J.Milnor and A.Grothendick began their activity at the end of that period inventing new great ideas. They started the next period in which I worked. The names of the main topological players are dropped here. They can be found in my articles [14, 15].

III. Byproduct of the Great Revolution (1955-1970); Topology as a Centrum of Interaction. (see [14, 15, 17–22, 24–27]).

During that period many fundamental problems of Topology were solved; Several applications of topological ideas in other areas of mathematics were

found. (The subjects where I was involved myself are given in italics here). The problems whose solution really required the use of the new algebraic methods are marked at the beginning by the sign (!); The results obtained in the late 1960s and 1970s, whose full proof is not written yet in the literature, are marked at the beginning by the sign (?).

1. Manifolds:

(!) Nonstandard Differentiable Structures on the 7-Sphere were discovered and classified for $n > 4$; Non-smoothable manifolds were found; The torsion of Pontryagin Classes was found to be topologically noninvariant; Poincaré Conjecture and h-cobordism theorem were proved for $n > 4$; (!) The classification of immersions and imbeddings of manifolds were obtained; (!) *The Classification Theory of multidimensional smooth manifolds was constructed*; (!) Relationship between smooth and PL manifolds was understood; (!) *Topological Invariance of the Periods of Pontryagin Classes along the cycles was proved*; (!) The so-called Annulus Conjecture was proved; (!)(?) Hauptvermutung was proved for manifolds without 2-torsion in the 3-homology; (!) Counterexamples to Hauptvermutung were found for the polyhedra (non-manifolds) first, and later for manifolds; (!)(?) Some sort of classification of Topological Manifolds was obtained for $n \neq 3$.

Several fundamental problems of the 3D Topology and knot theory were also solved in the 1960s. **Let us mention that in the 1980s the same people realized the great computational program leading to the proof of the famous 4-color problem.**

The existence and uniqueness of differentiable structure on 3-manifold was established in the early 1950s by the elementary methods. New achievements of the 3D Topology of the 1960s made possible the development of the Hyperbolic Topology of 3-manifolds in the 1970s. The technic of differential topology was extended in the 1970s to the 4-manifolds. It led to the construction of the purely continuous homeomorphisms only: it was proved that homotopy equivalent simply connected 4-manifolds are homeomorphic.

The discovery in the 1980s of nonstandard differential structures on the 4-manifolds belongs to the new era in Topology: it is a byproduct of interaction with the activity of quantum field physicists and achievements of the qualitative theory of nonlinear PDEs.

2. Calculations:

The stable homotopy groups of the classical Lie Groups were found through the calculus of variations; (!) Hopf Invariant Problem was solved, the nonexistence of the Divisible Algebras in higher dimensions was proved; (!) The Extraordinary Homology Theories were invented: the K -Theory brought new dimensions to the homological methods; (!) *The Cobordism Theory was developed; it improved methods of studying stable homotopy groups of spheres and fixpoints of the compact groups acting on manifolds*; (!) The Categorical "Localization Idea" was invented helping to solve some deep problems of the Homotopy Topology; Nontrivial finite-dimensional H -spaces were discovered.

3. Interaction with other areas of Mathematics:

(!) Riemann-Roch Theorem in Algebraic Geometry was proved in the 1950s as an application of the Cobordism Theory. A completely new approach was discovered leading to the so-called K -Theory; (!) The index problem for the Elliptic PD Operators was solved on the basis of Cobordisms and K -Theory; The revolutionary new understanding of Topology of Multidimensional Dynamical

Systems was invented by topologists; *The main problems were solved for the codimension one foliations, including the proof of existence of compact leaves on the 3-sphere; New areas of Algebra were created, such as the Algebraic K-Theory, Theory of Hopf Algebras, The Homological Algebra.*

IV. Dispersion of Multidimensional Topology in 1970s, The Hyperbolic 3D Topology; Discovery of various Topological Phenomena in the observable events of real Physics.

V. The Revival of Topology; Physicists in Topology (1980-2000).

VI. New ideas:

Is it possible to solve main problems in the theory of 3-manifolds analytically? At the moment we don't know the answer.

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