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Quasi-Periodicity in Dissipative and Conservative Systems

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ABSTRACT. In the classical perturbation theory of conditionally periodic motions series occur that, due to resonances, diverge on a dense set. In the complement of the resonances, small divisors make convergence problematic. Nonetheless, convergence of the series can be established in a nowhere dense set of positive Hausdorff measure in a suitable dimension. In the product of phase space and parameter space this gives rise to quasi-periodic invariant tori with Diophantine frequency vectors. This kind of result belongs to KAM theory, as this developed from Kolmogorov's 1954 paper [77]. We sketch elements of this development, both in the dissipative and the conservative setting.

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1 Introduction

This paper describes the origin and certain developments of KAM theory, as the legacy of Kolmogorov's 1954 paper [76,77]. We partially follow [17,18], but at the end give more details on recent work. We roughly maintain the following scheme:

- Complex linearization (Poincaré-Siegel [5,110])
  Circle maps (Arnold) [4,5]
  Dissipative theory [31,32,92]
- Area preserving twist maps (Moser) [89,101]
  Conservative theory (Kolmogorov-Arnold-Moser) [1,76,77]
- Further developments:
  Global KAM theory, monodromy [20]
  Quasi-periodic bifurcations [23–27,30,32,33,45,46,64,65,118,119].

2 Complex linearization

We start explaining the problem of Poincaré-Siegel as described in [5].

2.1 Setting

Given is the germ of a holomorphic map

$$F : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \; F(z) = \lambda z + f(z)$$

with $f(0) = f'(0) = 0$. Here λ is a complex number. The search is for a linearizing biholomorphic diffeomorphism $\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, i.e., such that

$$\Phi \circ F = \lambda \cdot \Phi.$$

We start investigating the existence of a formal solution. Indeed, given $f(z) = \sum_{j\geq 2} f_j z^j$, we look for $\Phi(z) = z + \sum_{j\geq 2} \phi_j z^j$ and the claim is that such a formal solution in general only exists for complex $\lambda \neq 0$, which is no root of unity.

This claim easily follows by induction with respect to the degree j. Indeed, considering the equations for ϕ_j for $j \geq 2$, we see

$$\begin{aligned} \lambda(1-\lambda)\phi_2 &= f_2 \\ \lambda(1-\lambda^2)\phi_3 &= f_3 + 2\lambda f_2 \phi_2 \\ \cdots &= \cdots \\ \lambda(1-\lambda^{j-1})\phi_j &= f_j + \text{ known}, \end{aligned}$$

which gives a recursive process for the series Φ.

2.2 Convergent solutions

In the hyperbolic case where $0 < |\lambda| \neq 1$, this process converges to a biholomorphic solution Φ, as desired. This was shown by H. Poincaré using iteration, compare [5].

In the elliptic case where $\lambda \in \mathbb{T}^1$, even when λ is not a root of unity, the power series Φ suffers from small divisors, since the powers of λ accumulate at 1. This problem was solved by C.L. Siegel in 1942 [5,110] under the Diophantine (nonresonance) conditions that for certain constants $\gamma > 0$ and $\tau > 2$ and for all rationals p/q,

$$|\lambda - \mathrm{e}^{2\pi \mathrm{i} p/q}| \geq \gamma |q|^{-\tau}. \tag{1}$$

The corresponding subset of λ is easily seen to have full measure in the complex unit circle $\mathbb{T}^1$.

2.3 H. Cremer's 1928 counter example [47]

Cremer's example consists of a concrete nonlinear case $F(z) = \lambda z + z^2$, where $\lambda \in \mathbb{T}^1$ is not a root of unity, in which case there exists a formal linearization Φ. Note that the linear map $z \mapsto \lambda z$ is an irrational rotation of the plane. The interest is with values of λ for which F has periodic points in any neighbourhood of $z = 0$, which then proves that Φ must have zero radius of convergence.

To investigate periodic points of period q, observe that $F^q(z) = \lambda^q z + \cdots + z^{2^q}$. The point z is periodic of period q precisely if $F^q(z) = z$. Noticing that

$$F^q(z) - z = z(\lambda^q - 1 + \cdots + z^{2^q - 1})$$

and calling $N = 2^q - 1$, we conclude for the product of the non-zero q-periodic points $z_1, z_2, \ldots z_N$ that

$$z_1 \cdot z_2 \cdot \cdots \cdot z_N = \lambda^q - 1.$$

From this is follows that a non-zero q-periodic point exists within radius

$$|\lambda^q - 1|^{1/N}$$

of $z = 0$. Now we consider all values of $\lambda \in \mathbb{T}^1$, which are no root of unity, such that

$$\mathrm{liminf}_{q \to \infty} |\lambda^q - 1|^{1/N} = 0. \tag{2}$$

It can be shown that these constitute a residual set, also known as a dense G_δ or a set of 2nd Baire category. This set is comparable to that of the Liouville numbers. For this use of terminology compare with Oxtoby [99].

Clearly for λ satifying (2) periodic points exist in any neighbourhood of zero and the series for Φ necessarily has zero radius of convergence. This example was orally communicated by Alain Chenciner and Michel Herman, also see [14]. It is to be noted that by the end of the 20th J.-C. Yoccoz [121,122] completely solved the elliptic case with the so-called Brjuno-condition [40] on the continued fraction expansion of α, where $\lambda = \mathrm{e}^{2\pi i \alpha}$.

3 Dissipative KAM theory

Although KAM theory traditionally focuses on the conservative dynamics of classical, in particular celestial, mechanics [1,76,77,101], it is also known to apply to cases of reversible, volume preserving, or to generally dissipative systems [31,32,92]. To fix thoughts we first consider maps of the circle.

3.1 Setting

Given is the smooth (C^∞ or real analytic) family of circle maps

$$P_{\alpha,\varepsilon} : \mathbb{T}^1 \to \mathbb{T}^1,\ x \mapsto x + 2\pi\alpha + \varepsilon a(x,\alpha,\varepsilon),$$

where α and ε are real parameters, with $|\varepsilon| \ll 1$. The 'unperturbed' case $P_{\alpha,0}$ forms a family of rigid rotations and the general question is what happens to the corresponding 'parallel' dynamics for small non-zero ε.

For our purposes we see this family as a 'vertical' map

$$P_\varepsilon : \mathbb{T}^1 \times [0,1] \to \mathbb{T}^1 \times [0,1],\ (x,\alpha) \mapsto (x + 2\pi\alpha + \varepsilon a(x,\alpha,\varepsilon), \alpha)$$

of the cylinder $\mathbb{T}^1 \times [0,1] = \{x,\alpha\}$. Note that here $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. In a naive way we start looking for a conjugacy Φ_ε :

$$\begin{array}{ccc} \mathbb{T}^1 \times [0,1] & \xrightarrow{P_0} & \mathbb{T}^1 \times [0,1] \\ \downarrow \Phi_\varepsilon & & \downarrow \Phi_\varepsilon \\ \mathbb{T}^1 \times [0,1] & \xrightarrow{P_\varepsilon} & \mathbb{T}^1 \times [0,1] \end{array}$$

i.e., with

$$P_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ P_0.$$

3.2 Formal solution, small divisors again

Let us assume that Φ_ε has the following near-identity format:

$$\Phi_\varepsilon(x,\alpha) = (x + \varepsilon U(x,\alpha,\varepsilon), \alpha + \varepsilon\sigma(\alpha,\varepsilon)),$$

then the conjugacy property leads to the following non-linear equation:

$$U(x+2\pi\alpha,\alpha,\varepsilon) - U(x,\alpha,\varepsilon) = 2\pi\sigma(\alpha,\varepsilon) + a\left(x + \varepsilon U(x,\alpha,\varepsilon), \alpha + \varepsilon\sigma(\alpha,\varepsilon), \varepsilon\right).$$

Expansion in powers of ε and comparison at lowest order gives a linear equation

$$U_0(x+2\pi\alpha,\alpha) - U_0(x,\alpha) = 2\pi\sigma_0(\alpha) + a_0(x,\alpha),$$

which can be directly solved in Fourier-series

$$\begin{aligned} a_0(x,\alpha) &= \sum_{k\in\mathbb{Z}} a_{0k}(\alpha)\mathrm{e}^{\mathrm{i}kx}, \\ U_0(x,\alpha) &= \sum_{k\in\mathbb{Z}} U_{0k}(\alpha)\mathrm{e}^{\mathrm{i}kx}. \end{aligned}$$

Indeed, we find as solutions $\sigma_0 = -\frac{1}{2\pi}a_{00}$, which amounts to a parameter shift, and

$$U_{0k}(\alpha) = \frac{a_{0k}(\alpha)}{\mathrm{e}^{2\pi\mathrm{i}k\alpha} - 1}.$$

We conclude that a formal solution (of the linear equation) generally only exists when α is irrational. As before, even for irrational α convergence is problematic by small divisors, since $\mathrm{e}^{2\pi\mathrm{i}k\alpha} - 1$ accumulates at 0.

3.3 Measure and Category

To overcome the small divisors we put Diophantine nonresonance conditions on α as follows. For given, fixed constants $\tau > 2$ and $\gamma > 0$, consider all $\alpha \in [0,1]$ such that for all p/q

$$|\alpha - p/q| \geq \gamma q^{-\tau}, \tag{3}$$

compare (1). This condition defines the Cantor set $[0,1]_{\tau,\gamma}$.[1] One easily shows that the measure of $[0,1] \setminus [0,1]_{\tau,\gamma}$ is of order γ as $\gamma \downarrow 0$. Notice that $[0,1]_{\tau,\gamma}$ as a Cantor set is topologically small (nowhere dense), but yet of positive measure, again compare with [99].

3.4 KAM on the circle

In the above circumstances we have the following result, which is the first KAM theorem of the present paper.

Theorem 1 *For $\tau > 2$, for $\gamma > 0$ sufficiently small and for $|\varepsilon| \ll 1$ there exists a smooth transformation $\Phi_\varepsilon : \mathbb{T}^1 \times [0,1] \to \mathbb{T}^1 \times [0,1]$, conjugating the restriction $P_0|_{[0,1]_{\tau,\gamma}}$ to a subsystem of P_ε*

We start taking $\gamma > 0$ sufficiently small such that $[0,1]_{\tau,\gamma}$ is a large measure Cantor set. Compare with [4, 5]. The present formulation has the structural stability format [31, 32]. Concerning the smoothness of Φ_ε compare with [101] and with [120, 124, 125].

3.5 Conclusion

In general a circle diffeomorphism smoothly conjugate to a rigid rotation $x \mapsto x + 2\pi\alpha$, with α irrational, is called *quasi-periodic*. It is known that each orbit of such a quasi-periodic map densely fills the circle [51]. Notice that for $\alpha \in [0,1]_{\tau,\gamma}$ the rigid rotation $P_{\alpha,0}$ surely is quasi-periodic.

A similar definition of quasi-periodicity holds for flows with invariant n-tori ($n \geq 2$), where the motion is conditionally periodic or parallel [6, 31, 32]. In these cases we require the existence of a smooth conjugacy with the constant vector field

$$\begin{aligned} \dot{x}_1 &= \omega_1 \\ \dot{x}_2 &= \omega_2 \\ \cdots &= \cdots \\ \dot{x}_n &= \omega_n \end{aligned}$$

on the standard n-torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$, so with the angles x_j counted mod 2π, $j = 1, 2, \ldots, n$.

Theorem 1 implies that the circle maps $P_{\alpha,\varepsilon}$, that are conjugate to one of these quasi-periodic rotations $P_{\alpha,0}$, are still quasi-periodic. Hence quasi-periodicity typically occurs with positive measure in the parameter space. It is to be noted that M.R. Herman [67] and J.-C. Yoccoz [121,122] have proven non-perturbative versions of this theorem. Also see [86].

[1] United with a countable set, ccording to the Cantor-Bendixson theorem, cf. [66].

As a concrete example consider the Arnold family of circle maps given by

$$P_{\alpha,\varepsilon}(x) = x + 2\pi\alpha + \varepsilon \sin x,$$

compare with [4,5,51]. In the (α, ε)-parameter plane resonance tongues emanate from the points $(\alpha, \varepsilon) = (p/q, 0)$ into the half planes $\varepsilon \neq 0$, in such a way that for small $|\varepsilon| \neq 0$ an open dense subset is covered. In the tongue labeled by p/q, the dynamics is periodic with rotation number p/q. For definitions see [51]. Theorem 1 implies that in the complement of the union of tongues there exists and uncountable union of smooth curves filling out positive measure and corresponding to quasi-periodic dynamics.

Remark. A more precise formulation of theorem 1 asserts that for $|\varepsilon| \ll 1$ and γ of the form $\gamma = c\varepsilon$ the conclusions hold. This implies that, inside the α-axis, the parameter region of Diophantine invariant circles tends to a full measure set as $\varepsilon \downarrow 0$.

As a more general class of examples consider weakly coupled Van der Pol type oscillators

$$\begin{aligned} \ddot{y}_1 + c_1\dot{y}_1 + a_1 y_1 + f_1(y_1, \dot{y}_1) &= \varepsilon g_1(y_1, y_2, \dot{y}_1, \dot{y}_2) \\ \ddot{y}_2 + c_2\dot{y}_2 + a_2 y_2 + f_2(y_2, \dot{y}_2) &= \varepsilon g_2(y_1, y_2, \dot{y}_1, \dot{y}_2), \end{aligned}$$

i.e., where the free oscillators have a hyperbolic periodic attractor, yielding a vector field in the phase space $\mathbb{R}^2 \times \mathbb{R}^2 = \{(y_1, z_1), (y_2, z_2)\}$ with $\dot{y}_j = z_j, j = 1, 2$. In these cases, in the 4-dimensional phase space a 2-torus attractor occurs which for small $|\varepsilon|$ has a Poincaré map P as above. Theorem 1 then implies similar results regarding quasi-periodic 2-tori with Diophantine frequency ratio's. Inside the resonance tongues the dynamics on the 2-tori is phase locked.

This example directly generalizes to more weakly coupled oscillators in higher dimension, where next to periodicity and quasi-periodicity also chaotic dynamics coexist. This setting, as well as the transitions or bifurcations between the various kinds of dynamics, fits in turbulence scenario's according to Landau-Hopf-Lipschitz [71,81,82] and Ruelle-Takens, et al. [97,107]. In the last section we shall return to this subject. In all cases the dynamics contain *families* of quasi-periodic attractors [106] that constitute a certain order amidst chaos, filling out positive measure in parameter space.

4 Conservative KAM theory

Next we turn to the conservative setting [1, 76, 77, 101], starting with annulus maps [89].

4.1 The Twist Mapping Theorem

An 2-dimensional annulus Δ has coordinates $(\varphi, I) \in \mathbb{T}^1 \times \mathbf{K}$, with $\mathbf{K}$ compact. Let Δ be endowed with the area form $\sigma = d\varphi \wedge dI$.

A σ-preserving smooth map $P_\varepsilon : \Delta \to \Delta$ is given by

$$P_\varepsilon(\varphi, I) = (\varphi + 2\pi\alpha(I), I) + O(\varepsilon),$$

where we assume that the map $I \mapsto \alpha(I)$ is a (local) diffeomorphism. This is the so-called *twist* condition. The dynamics of P_0, restricted to invariant circles $I = \text{cst.}$, is parallel and – as before – the general question is: what is the fate of this dynamics for $|\varepsilon| \ll 1$?

The Diophantine conditions are as before in (3): for given constants $\tau > 2$ and $\gamma > 0$ let

$$|\alpha(I) - p/q| \geq \gamma q^{-\tau},$$

for all rationals p/q. The pull back of the corresponding set to Δ we denote by $\Delta_{\tau,\gamma}$, a set which is nowhere dense and yet of positive measure. Indeed, the complement $\Delta \setminus \Delta_{\tau,\gamma}$ has small measure as $\gamma \downarrow 0$. In the same spirit as theorem 1, we now have the following.

Theorem 2 *Assume the twist condition to be valid. Then, for $\gamma > 0$ sufficiently small and for $|\varepsilon| \ll 1$ there exists a smooth transformation $\Phi_\varepsilon : \Delta \to \Delta$, conjugating the restriction $P_0|_{\Delta_{\tau,\gamma}}$ to a subsystem of P_ε*

Theorem 2 generally is referred to as the Twistmapping Theorem, and it goes back to J.K. Moser [89], although the present formulation is closer to [101], also compare with [31, 32]. Regarding the smoothness the same remarks apply as those following theorem 1.

4.2 Conclusion

Comparing theorems 1 and 2, we observe that the role of the parameter α in theorem 1 has been taken by the action coordinate I in theorem 2. A difference between the two theorems is that now the setting is area preserving. It is to be noted, however, that generally the conjugacy P_ε does not preserve the area form σ. As a consequence of theorem 2 we conclude that quasi-periodicity typically occurs with positive measure in the phase space.

In the complement of the quasi-periodic regime, in the 2D phase space we can have periodicity and chaos. As in the case of theorem 1, we can consider coupled oscillators

$$\begin{aligned}
\ddot{y}_1 &= -\omega_1^2 \sin y_1 - \varepsilon \frac{\partial U}{\partial y_1}(y_1, y_2) \\
\ddot{y}_2 &= -\omega_2^2 \sin y_2 - \varepsilon \frac{\partial U}{\partial y_2}(y_1, y_2),
\end{aligned}$$

leading to a 4-dimensional Hamiltonian vector field

$$\begin{aligned}
\dot{y}_j &= z_j \\
\dot{z}_j &= -\omega_j^2 \sin y_1 - \varepsilon \frac{\partial U}{\partial y_j}(y_1, y_2),
\end{aligned}$$

$j = 1, 2$, with Hamilton function $H_\varepsilon(y_1, z_1, y_2, z_2) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 - \omega_1 \cos y_1 - \omega_2 \cos y_2 + \varepsilon U(y_1, y_2)$. In this setting the iso-energetic Poincaré map, for small $|\varepsilon|$, obtains the form P_ε. In the unperturbed case, where $\varepsilon = 0$, the energy hypersurfaces are foliated by invariant 2-tori (and two 'singular' circles, carrying conditionally periodic or parallel flow [6, 31, 32]. This leads to the conclusion that quasi-periodicity occurs with positive measure in energy hypersurfaces of H_ε.

The latter example again generalizes to more weakly coupled oscillators, in a higher dimensional phase space. Given appropriate nondegeneracy conditions, we can apply the so-called iso-energetic KAM theorem [6, 29, 93, 94], ensuring positive measure of quasi-periodic tori in each energy hypersurface. These tori are Lagrangian, having the same dimension as the number of degrees of freedom.

4.3 Classical KAM theory

This leads us to KAM theory [5, 76, 77, 101], which takes the setting of smooth nearly-integrable Hamiltonian systems defined on the phase space $\mathbb{T}^n \times A \subseteq \mathbb{T}^n \times \mathbb{R}^n = \{\varphi_1, \varphi_2, \ldots, \varphi_n, I_1, I_2, \ldots, I_n\} = \{\varphi, I\}$, with $A \subseteq \mathbb{R}^n$ open and bounded, with the symplectic form $\sigma = dI \wedge d\varphi = \sum_{j=1}^n dI_j \wedge d\varphi_j$. In the case where a given Hamiltonian system is Liouville integrable [6,48], up to a canonical change of variables, the Hamiltonian function $H = H(I)$ is independent of the angles φ and takes the (local) form X_H :

$$\dot{\varphi} = \frac{\partial H}{\partial I}, \quad \dot{I} = 0. \tag{4}$$

If the energy hypersurface $H(I) = \text{cst.}$ is compact, the nearby phase space is foliated by invariant Lagrangian n-tori of the form $I = \text{cst.}$ The tori carry conditionally periodic (or parallel) motion with frequency vector

$$\omega(I) = \frac{\partial H}{\partial I}(I),$$

and the system is called *Kolmogorov nondegenerate* if the frequency mapping $I \mapsto \omega(I)$ is a (local) diffeomorphism. In the nearly integrable case one considers perturbations $\tilde{H}(I, \varphi) = H(I) + \varepsilon H_1(I, \varphi)$, where H_1 is bounded in a suitable topology on the function space. Let $X_{\tilde{H}}$ denote the corresponding Hamiltonian vector field. The present interest is with the persistence of quasi-periodic X_H-invariant, Lagrangian n-tori where now the following Diophantine conditions are assumed. For given constants $\tau > n - 1$ and $\gamma > 0$,

$$|\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}, \tag{5}$$

for all integer vectors $k \in \mathbb{Z}^n \setminus \{0\}$. The set of all such $\omega \in \mathbb{R}^n$ is denoted by $\mathbb{R}^n_{\tau,\gamma}$, which is another nowhere dense set, the intersection of which with any bounded set has a complement with measure of order $O(\gamma)$ as $\gamma \downarrow 0$. For details on its geometry and the relationship with the previous Diophantine conditions (1), (3) see, e.g., [20, 46, 101]. Let $A_{\tau,\gamma} \subseteq A$ be the pull back of $\mathbb{R}^n_{\tau,\gamma}$ under the frequency map.

Theorem 3 *Under the Kolmogorov nondegeneracy condition, for $\gamma > 0$ sufficiently small and $|\varepsilon| \ll 1$ there exists a smooth diffeomorphism $\Phi_\varepsilon : \mathbb{T}^n \times A \to \mathbb{T}^n \times A$, such that, the restriction $\Phi|_{\mathbb{T}^n \times A_{\tau,\gamma}}$ conjugates the integrable subsystem $X_H|_{\mathbb{T}^n \times A_{\tau,\gamma}}$ to a subsystem of $X_{\tilde{H}}$.*

This is the classical KAM theorem, as it goes back to [5, 76], in the version of [31, 32, 101]. Notice that we took the (structural) stability formulation of the KAM theorem, in correspondence with the theorems 1 and 2. In [31, 32] this stability is coined *quasi-periodic* stability, in reminiscence of 'omega-stability' when restricting to the non-wandering set.

Many remarks as following theorem 1 and 2 also apply here. We conclude that, as a consequence of theorem 3, for Hamiltonian systems typically quasi-periodicity on Lagrangian invariant tori occurs with positive measure in phase space. The iso-energetic KAM theorem, as referred to above, is a variation on theorem 3 where one restricts to the energy hypersurfaces, assuming an appropriate nondegeneracy condition. Both theorems are closely related [29].

4.4 Conclusion

Regarding the physical and philosophical consequences of KAM theorem 3 much has been said, e.g., compare [2, 52, 93, 94]. Historically the KAM theory is closely related to perpetual stability problems, in particular related to the 3-body problem or the solar system. One of the consequences indeed is, that with 'positive probability' the evolution is very orderly and confined to an invariant torus. Whether this kind of argument applies to anything close to the actual solar system, has to be doubted [83]. For tutorials in the direction of KAM theory see [46, 84, 103].

In this respect the difference between the cases $n = 2$ and $n \geq 3$ has to be mentioned, where n is the number of degrees of freedom. Indeed, in the former case motions can be captured in between Cantor families of KAM tori, since these are of codimension one in the 3-dimensional energy hypersurface. In the latter case, however, the notion 'in between' no longer makes sense and evolutions can escape, a phenomenon referred to as Arnold diffusion [3]. On the other hand, quasi-periodic motions are known to be 'sticky' in the sense that evolutions can be captured for long time intervals in their neighbourhood [95], which makes the Arnold diffusion quite ineffective.

Another issue is the fact that KAM theory shows that typically Hamiltonian systems, even when restricting to energy hypersurfaces, are not ergodic. This follows from the fact, that the Cantor families of invariant tori provide nontrivial invariant sets [8, 76, 77]. A question is what is the value of this 'obstruction' to ergodicity as $n \to \infty$. This question is of importance for the dynamics of infinite systems as described in statistical mechanics. It turns out that there is a lot of evidence that the density of the KAM tori decays rapidly as $n \to \infty$ [3, 73]. For approaches to the KAM theory of ∞-dimensional systems we refer to [75, 78].

5 Global Hamiltonian KAM theory

In many Liouville integrable Hamiltonian systems the union of Lagrangian tori forms a nontrivial bundle over the range of the action variables. In certain cases this can be measured by *monodromy*. This nontriviality is of importance for semi-classical quantizations, where it explains so-called spectral defects [49, 50, 88, 98].

As an example consider the spherical pendulum [48, 54] with configuration space $\mathbb{S}^2 = \{q \in \mathbb{R}^3 \mid \langle q, q \rangle = 1\}$ and phase space $T^*\mathbb{S}^2 \cong \{(q, p) \in \mathbb{R}^6 \mid \langle q, q \rangle = 1 \,\&\, \langle q, p \rangle = 0\}$, As gravitation points vertically downward the system is axially symmetric and by the Noether theorem [6], the momentum M with respect to the vertical axis is an extra integral. If the energy is called E, we get

$$\mathcal{EM} : T^*\mathbb{S}^2 \to \mathbb{R}^2, \; (q, p) \mapsto (M, E) = \left(q_1 p_2 - q_2 p_1, \frac{1}{2}\langle p, p \rangle + q_3 \right)$$

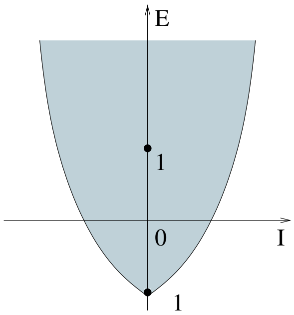

Figure 1: Range of the energy–momentum map of the spherical pendulum.

as the energy-momentum mapping. See figure 1. On an open piece of the range of $\mathcal{EM}$ the fibers of the map are Lagrangian 2-tori: here the system is Liouville integrable. The upward equilibrium point has coordinates $(q, p) = ((0, 0, 1), (0, 0, 0))$ with $(M, E) = (0, 1)$, and is a singularity of the fibration. A circle in the range of $\mathcal{EM}$ with winding number 1 around $(M, E) = (0, 1)$, collects the following nontrivial monodromy [54]:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{Z}),$$

where we use a suitable basis of the corresponding period lattice. This shows that the 2-torus bundle of Lagrangian tori is nontrivial, which prevents the existence of globally defined action angle variables.

Many further examples have been found of integrable systems with nontrivial Lagrangian torus bundles, e.g., in normal form truncations of the Hamiltonian Hopf bifurcation [57].

A natural question is: what is the effect on the bundle geometry when perturbing the system? Within the class of integrable systems, e.g., think of perturbing the spherical pendulm while maintaining the axial symmetry, the monodromy is constant under sufficiently small perturbation. For a treatment in 2 degrees of freedom see [87, 126], for the general case see [20]. For non-integrable perturbations, by KAM theory certain Diophantine Lagrangian tori can be proven to persist. It turns out that the corresponding union forms a Whitney smooth bundle, which allows for smooth interpolations that are isomorphic to the integrable bundle [20]; for a treatment in 2 degrees of freedom see [105]. The monodromy therefore persists for sufficiently small perturbations. In [23, 24] this idea is pursued for non-integrable perturbations of the Lagrange top near gyroscopic stabilization; here the Hamiltonian Hopf bifurcation and its quasi-periodic analogue play a role.

5.1 Conclusion

The monodromy of integrable Hamiltonian torus bundles has been well understood in the case of 2 degrees of freedom [87,126]. Indeed, if the 2-torus fibration has a singularity, which is a k-pinched 2-torus, $k \geq 1$, then the monodromy matrix is given by

$$\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{Z});$$

in the case of the spherical pendulum $k = 1$, where the 1-pinched 2-torus exactly is the intersection of stable and unstable manifold of the unstable equilibrium.

In the semi-classical quantizations of integrable systems [49, 50, 88] it turns out that the asymptotics of the energy-momentum spectrum, in the limit for Planck's constant $h \downarrow 0$, is related to the monodromy operator [98]. In the applications regarding the Hamiltonian Hopf bifurcation [57], an integrable normal form truncation has to be taken, which is somewhat artificial. We expect that the results of [20] may be useful when studying the quantized family in dependence of both parameters h and ε.

6 Quasi-periodic bifurcations

Loss of normal hyperbolicity is a notorious reason for invariant manifolds to bifurcate. In the case of quasi-periodic tori a 'generic' bifurcation theory has been developed in various contexts, like the general (dissipative) setting, the Hamiltonian and the reversible settings. The germ 'generic' refers to persistence under small perturbations. This *quasi-periodic bifurcation theory* is based on integrable approximations, where integrability generally is defined by equivariance with respect to a suitable (locally) free action of the torus group [31, 32]; this definition generalizes the notion of Liouville integrability in Hamiltonian systems [6, 48]. In the integrable case the theory reduces to that of relative equilibria or of relative periodic solutions that undergo more classical bifurcations as these go back on Whitney, Pontrjagin, Thom, Arnold [7, 116] and where the bifurcation sets are given by semi-algebraic stratifications as well as differentiable immersions of these. The nearly integrable analogues all give rise to, perturbed and 'Cantorized' stratifications, due to Diophantine nonresonance conditions [21, 23–26, 28, 32, 46, 64, 65, 118, 119].

6.1 Dissipative cases

In the dissipative case, the theory for bifurcations of equilibria and periodic solutions at the codimension one level contains saddle-node and Hopf bifurcations, as well as period doubling and Hopf-Neĭmark-Sacker bifurcations [63, 80]. Their analogues are the quasi-periodic saddle-node, period doubling and Hopf bifurcation [31, 32].

6.1.1 The quasi-periodic saddle-node bifurcation

In the case of the quasi-periodic saddle-node bifurcation the following is going on. As described earlier, in the integrable approximation two branches of n-tori looses their normal hyperbolicity and get annihilated at the bifurcation curve.

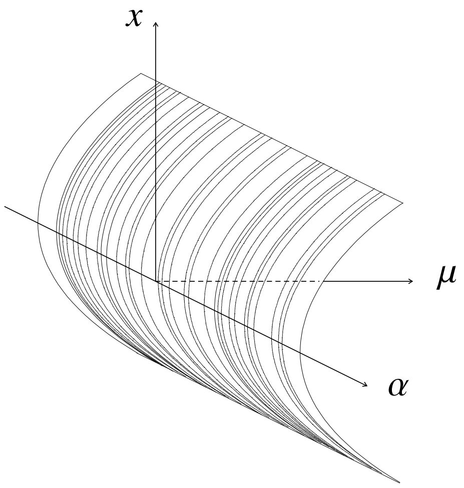

Figure 2: Cantorized bifurcation diagram of the quasi-periodic saddle-node bifurcation.

Apart from a bifurcation parameter, we need sufficiently many parameters to obtain a situation which is persistent under small perturbations. For this it is needed that the parameters can 'keep track' of the frequencies of the n-tori.

To fix thought we consider a concrete example of the quasi-periodic saddle node bifurcation of invariant circles, consisting of a skew family of circle maps:

$$P_{\alpha,\mu}(\varphi, x) = (\varphi + 2\pi\alpha, x + (\mu - x^2)) + \varepsilon f(\varphi, x, \alpha, \mu).$$

Probably a simple model given by the choice $f(\varphi, x, \alpha, \mu) = (x + \sin\varphi, a\cos\varphi)$ already contains much of the possible dynamical richness, that we describe now. Note that the integrable case $\varepsilon = 0$ is the straight product of a circle map and a saddle-node bifurcation of a fixed point. The bifurcation set in the (α, μ)-plane is just a fold line: on one side ($\mu > 0$) two invariant circles exist that annihilate each other at the fold line in an integrable saddle-node bifurcation. On the other side ($\mu < 0$) of the fold line no circles persist. For $0 \neq |\varepsilon| \ll 1$ the picture changes, where now the resonances $(\alpha, \mu) = (p/q, 0)$, $p/q \in \mathbb{Q}$, come into play, since here the normal hyperbolicity of the invariant circles generically vanishes. A stability result like theorems 1, 2 and 3 can be formulated for this situation [31, 32] where in the α-direction Diophantine conditions (3) are imposed.

As said earlier, in the present setting integrable means that there is a free $\mathbb{T}^1$-symmetry, which implies independence of the angular variable φ. The integrable situation can be largely described in terms of relative equilibria, leading to the fold

$$\mathcal{F} = \{(x, \alpha, \mu) \in \mathbb{R}^3 \mid \mu = x^2\}.$$

The subset of $\mathcal{F}_{\tau,\gamma} \subseteq \mathcal{F}$ where the $\alpha \in [0,1]_{\tau,\gamma}$ are Diophantine we loosely call 'a Cantorized version' of $\mathcal{F}$, see figure 2. In the nearly integrable perturbation we obtain a near-identity diffeomorphic image of this, where the integrable and nearly-integrable dynamics are conjugate; again compare with the theorems 1, 2 and 3. As announced before, for a proper description of this phenomenon we need to include the parameter α, which is in the same spirit as earlier in the paper.

The Cantorized fold surface, compare with figure 2, governs the quasi-periodic (even Diophantine) invariant circles that pairwise annihilate at the fold line. In the gaps of the Cantor set several things may happen. First, away from the fold line, the quasi-periodic invariant circles are normally hyperbolic. By [69] the parameter values of persistence of normally hyperbolic invariant manifolds form and open and hence the region of normally hyperbolic invariant circles can be 'fattened' from the Cantorized fold surface. Indeed, it can be shown that the complement of this domain consists of a countable union of small resonance 'bubbles', centered around the resonances $(\alpha, \mu) = (p/q, 0)$, $p/q \in \mathbb{Q}$. The dynamics in the domain of hyperbolic circles generically is either periodic or quasi-periodic, compare with the Arnold family of section 3.5, but inside the 'bubbles' more interesting dynamics may occur, like cantori, strange attractors, etc. For details connected to case studies, see [9, 41–43].

6.1.2 The quasi-periodic Hopf bifurcation

The Hopf bifurcation (or Poincaré-Andronov-Hopf bifurcation) and its periodic and quasi-periodic versions are a bit more involved First, for equilibria in planar vector fields, under generic conditions and with an appropriate choice of signs, it has the following topological normal form [16, 96, 100]

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \mu & -\alpha \\ \alpha & \mu \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{6}$$

where we use μ as a parameter and α can be kept constant (and even scaled to 1). The variables y_1, y_1 and μ vary near 0. In this case, for $\mu \geq 0$ an attracting periodic solution (limit cycle) branches off, of amplitude $\sqrt{\mu}$.

Next we consider the analogue of (6) for fixed points of planar diffeomorphisms. The lower order part now is equivalent to

$$P(y) = \mathrm{e}^{2\pi(\mu + i\alpha)} y + O(|y|)^2, \tag{7}$$

where $y \in \mathbb{C} \cong \mathbb{R}^2$ varies near 0. To begin with α is a constant, which is not rational with a denominator less than 5, compare with [5, 113]. If the $O(|y|^2)$-terms are generic, for a proper choice of signs, the map P has an attracting invariant circle, branching off for $\mu \geq 0$ with an amplitude that is again of order $\sqrt{\mu}$ [69]. Here, due to the invariance of the rotation numbers of the invariant circles, there is no question of topological stability [96]. Still, the Hopf bifurcation for fixed points can be characterized by many persistent properties. To describe this well we need to include α as a parameter as well. In fact, the Arnold family $x \mapsto x + 2\pi\alpha + \varepsilon \sin x$ of circle maps [4, 5, 51], as mentioned before in section 3.5, is quite a good model. Instead of the (α, ε)-plane for the Arnold family, we now consider the (α, μ)-plane for the Hopf bifucation. Here we have a similar parametrization of periodicity, namely by a countable union of resonance tongues emanating from the points $(\alpha, 0) = (p/q, 0)$, where p/q is the rational rotation number of the invariant circle. For small values of μ, the union of tongues is residual [35]. In their complement, quasi-periodicity is given by a nowhere dense union of smooth curves. Again compare with theorem 1.

The Hopf bifurcation for fixed points applies for the theory of ordinary differential equations (or, more generally, of vector fields) in cases when the diffeomorphism is the Poinaré return map of a periodic solution. Here the invariant circles correspond to invariant 2-tori. The dynamics given by the resonance tongues is called phase locked, compare with section 3.5. In this formalism one usually speaks of the Neĭmark-Sacker bifurcation [63, 80].

Remark. There exists a normal form (or averaging) theory that, at least up to order $O(|y|^4)$, reduces the Hopf case for fixed points to that of equilibria for vector fields [5,16,63,113,114]. In that sense the case for diffeomorphisms can be viewed as a perturbation of the vector field case. The vector field then serves as an integrable (i.e., rotationally symmetric) approximation of the original system. The invariant circle in this setting is a perturbation of a limit cycle. We mention that in the case of resonance, the circle generically is only finitely differentiable, where the degree of differentiability depends on the resonance [112].

The quasi-periodic Hopf bifurcation is the present KAM analogue of the Hopf bifurcation for equilibria and of the Hopf-Neĭmark-Sacker bifurcation for periodic solutions. In an integrable approximation, a family of parallel n-tori in the phase space $\mathbb{T}^n \times \mathbb{R}^2$ with coordinates x mod 2π and y has normal linear part

$$\begin{aligned} \dot{x} &= \omega \\ \dot{y} &= \begin{pmatrix} \mu & -\alpha \\ \alpha & \mu \end{pmatrix} y, \end{aligned} \tag{8}$$

where we now use $(\omega, (\alpha, \mu)) \in \mathbb{R}^n \times \mathbb{R}^2$ as parameters. Here μ and y are local variables, while the pair (ω, α) may vary over a bounded set in $\mathbb{R}^n \times \mathbb{R}$. The integrable approximation again has a free $\mathbb{T}^n$-symmetry, which here implies that the higher order terms are independent of the angular variables x. Now the invariant n-tori correspond to relative equilibria and the $(n+1)$-tori to relative limit cycles, where we assume similar generic conditions as before. Given the right choices of sign, the asymptotics on the amplitude of the relative limit-cycle again is of order $\sqrt{\mu}$ as $\mu \downarrow 0$. The question now is to what extent this n- and $(n+1)$-tori are persistent for small, but arbitrary perturbations.

Also we need a variation on the Diophantine condition (3) in the sense that for constants $\tau > n - 1$ and $\gamma > 0$ it is required that

$$|\langle \omega, k \rangle + \alpha \ell| \geq \gamma |k|^{-\tau} \tag{9}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ and for all $\ell \in \mathbb{Z}$ with $|\ell| \leq 2$. As a subset of $\mathbb{R}^{n+1} = \{(\omega, \alpha)\}$, this set has similar properties to the subset of $\mathbb{R}^n = \{\omega\}$ defined by the Diophantine condition (3). On the one hand it is nowhere dense, but on the other hand the intersection with any bounded set has a complement that has measure of order $O(\gamma)$ as $\gamma \downarrow 0$. As mentioned in the remark in section 3.1 also here we can take γ small with the perturbation. The general conclusion is that, for sufficiently small perturbations, and restricted to the Diophantine set (9) a Whitney smooth family of invariant n-tori survives. Moreover, when restricting to a similar large measure set, a Whitney smooth family of invariant $(n+1)$-tori branches off, as indicated by the integrable approximation. For an appropriate choice of signs, the $(n+1)$-tori are all attracting.

As in the previous saddle-node case, the integrable approximation is largely determined by relative equilibria and relative limit cycles, organized in a smooth stratification[2] in the product of phase space and parameter space. The Diophantine conditions (9) Cantorize this stratification and in the nearly integrable case we obtain a near-identity smooth image of this. Here we also speak of Cantor stratification. Another analogy with the saddle-node case is that we can fatten the domains of both the invariant n- and $(n+1)$-tori by normal hyperbolicity [69]. The extra tori do not have to be quasi-periodic and in the case $n+1 \geq 3$, apart from periodic attractors, can also have strange attractors [96,107]. In this way, near the set $(y, \mu) = (0, 0)$ a zone of resonance bubbles is left over. Inside a bubble we are close to a resonance $\langle \omega, k \rangle + \alpha \ell = 0$. Note that for $\ell = 0$ this is an internal resonance of the n-torus, while for $\ell \neq 0$ the resonance is normal-internal. We mention [9, 42, 43, 115] for results in this direction.

6.1.3 Conclusion

For a full treatment of quasi-periodic versions of the saddle-node and the Hopf bifurcation, as well as the *quasi-periodic period doubling*, we refer to [32], for certain further explanations also see [31, 46, 72]. A generalization of the quasi-periodic saddle-node to quasi-periodic cuspsoids is given in [118].

One general aspect of this approach is the way in which parameters are dealt with. In the above treatment, for simplicity, we include the 'frequency vector' $\omega \in \mathbb{R}^n$ as an n-dimensional parameter; similarly for other constants like 'the rotation number' $\alpha \in \mathbb{R}$. A more systematic approach introduces frequency maps with ω, or (ω, α) as its component functions which are assumed to be submersive, compare with the Kolmogorov nondegeneracy assumption in the classical Hamiltonian KAM theory of Lagrangian tori, see section 4.3.

A second general aspect is that we consider simple losses of normal hyperbolicity of quasi-periodic tori. Therefore we can easily embed all systems in higher dimensional phase spaces, where the present phenomena take place in a center manifold [69].

A third remark which generally applies refers back to the term 'generic' used at the beginning of this section. Indeed, the theory covers open classes of dynamical systems, say in the C^∞-topology [68]. This notwithstanding the fact that all integrable systems have infinite codimension. A related aspect is the fact that presently all normal linear parts are reducible to Floquet form. For related KAM theory on a skew Hopf bifurcations we refer to [36, 38, 39, 115]. Here there are topological obstructions against reducibility.

In applications the quasi-periodic bifurcations often occur subordinate to a singularity of higher codimension. They show up after a normalization or averaging process when in the truncation the corresponding bifurcations for (relative) equilibria are present. An example of this is the Hopf-Hopf bifurcation where two conjugate pairs of simple eigenvalues cross the imaginary axis [10, 63, 127]. This is a codimension 2 bifurcation and in a generic 2-parameter unfolding quasi-periodic Hopf bifurcations occur in a persistent way.

Also the occurence of n-quasi-periodicity for many, maybe even infinitely many, values of n in certain infinite dimensional dynamical systems has been

[2] This is the smooth image of a semi-algebraic stratification.

observed [15, 58]. This brings us to the natural philosophy on turbulence and its onset, as given by Landau-Hopf-Lipschitz and by Ruelle-Takens [71, 81, 82, 97, 107], for a discussion also see [31]. We emphasize that no controversy arises between 'genericity' and 'large measure': we showed that for an open class of systems, i.e., for generic families, the quasi-periodic attractors occur with positive measure in the parameter space.

As a final area of applications we mention the quasi-periodic response problem as raised by Stoker and Moser [90, 91, 111]. Here a nonlinear oscillator is forced quasi-periodically and the question is whether quasi-periodic response solutions exist of the same frenquencies. This problem can be rephrased in the above setting and the answer is affirmative. It turns out that quasi-periodic Hopf bifurcations can occur in this setting [11].

Figure 3: Cantorized bifurcation set of the quasi-periodic center-saddle bifurcation for $n = 2$. The lower part corresponds to hyperbolic tori and the upper part to elliptic ones.

6.2 Hamiltonian and reversible cases

Already from the early years of KAM theory [90, 92] it was clear that a general approach is possible, within certain Lie algebras of vector fields. Apart from the classical, Hamiltonian case, this also includes the settings of volume preserving or dissipative systems. See above for examples in the latter setting. The approach also covers systems that are reversible with respect to a given involution, also see [30, 109]. Compare with [31] and references given there.

Regarding quasi-periodic bifurcations in the Hamiltonian and reversible setting we can report the following. Generally speaking, the program sketched before, directly carries through. However we note that the fattening of torus domains by normal hyperbolicity does not have to apply. What is left is the Cantorization of semi-algebraic stratifications or their differentiable images by Diophantine non-resonance conditions.

A first example in the Hamiltonian context is provided by Hanßmann [64] who considers the quasi-periodic center-saddle bifurcation. Starting point is the saddle-center bifurcation as this occurs for equilibria in 1 degree of freedom. A singularity theory model for this is given by

$$H_\mu(p, q) = \frac{1}{2}p^2 + V_\mu(q), \text{ with } V_\mu(q) = \frac{1}{3}q^3 - \mu q, \tag{10}$$

where for $\mu \geq 0$ a center and a saddle-point branch off from $(q, p) = (0, 0)$, with a mutual distance of order $\sqrt{\mu}$, in a fold catastrophe [116]. In the quasi-periodic analogue we consider an extension of (10) to $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2 = \{\varphi, I, (q, p)\}$ of the form

$$\tilde{H}_{\mu,\omega}(I, \varphi, p, q) = \langle \omega, I \rangle + \frac{1}{2}p^2 + V_\mu(q) + \text{ HOT}, \tag{11}$$

where HOT stands for higher order terms that can depend on all variables - including the angles φ – and parameters. Again, the analysis of the integrable approximation, i.e., (11) with the HOT deleted, reduces to the above case by considering relative equilibria. The latter correspond to normally elliptic and hyperbolic invariant tori with parallel dynamics. For nearly integrable perturbations we have to take Diophantine conditions (3) and (5) into account, which Cantorize the parabola like geometry as depicted in figure 3.

This theory can be extended in the direction of more general normally parabolic tori [25], so where the potential V depends on more parameters, like

$$V_{\mu_1,\mu_2}(q) = \frac{1}{4}q^4 + \frac{1}{2}\mu_1 q^2 + \mu_2 q,$$

which corresponds to the cusp catatrophe [116]. In fact, all cuspoid singularities are covered at once, including the hierarchies in their Cantor stratifications. In the same spirit quasi-periodic normally umbilic bifurcations [26] as well as the quasi-periodic Hamiltonian Hopf bifurcation [23, 24, 28] could be treated. For an overview see [65].

6.3 Conclusion

Quasi-periodic bifurcations of invariant tori, as treated above, are comparable to certain bifurcations of periodic solutions, in the sense that they go back on the case of relative equilibria. A difference with the periodic case is the Cantorization process invoked by the Diophantine nonresonance conditions. This difference is stressed in a Hamiltonian case study, that involves quasi-periodic center-saddle and period doubling bifurcations [21]. Regarding ongoing and future work in this direction we mention [19,22,45] for the quasi-periodic reversible Hopf bifurcation.

In the dissipative setting, in [33] we apply [118] for diffeomorphisms of the plane, where a general approach is given to subordinate quasi-periodic bifurcations of invariant circles. This approach merries KAM theory to singularity theory (with contact equivalence), to give a better description of the Cantor stratifications. It covers the KAM aspects of the general codimension k Hopf bifurcation, but also reveals a rich bifurcation structure of invariant circles near homoclinic tangencies.

Regarding the Hamiltonian setting, research is ongoing regarding to bundles of invariant tori in nearly integrable systems, where the tori can be more generally isotropic (and not only Lagrangian). In particular the interest is with nontriviality of the bundles and the role this plays and yet may play in semi-classical quantum mechanics, where nontrivial monodromy has been proven to connect to certain spectral defects [49, 98]. It is worth mentioning that for the generic Hamiltonian Hopf bifurcation nontrivial monodromy has been shown to exist [57]. Observe that since the work of Eliasson and Puig [34, 55] the ideas of Cantorized spectra, i.e., of spectral measures supported Cantor sets of positive measure, and devil's staircases of rotation numbers are no longer strange to all physicists.

From the general point of view we mention ongoing work [119], which reconsiders the general approach, separating more clearly the KAM theory aspects from those of singularity theory. Finally, in [27] an overview on quasi-periodic bifurcation theory is presented, where a lot of attention is given to applications and examples in the various settings and the role of Cantorized bifurcation diagrams.

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