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# Infinite curves on closed surfaces

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Abstract. This work arose from studying flows (continuous 1-parameter transformation groups) on closed surfaces. Surfaces considered are those of nonpositive Euler characteristic $\chi$, since for those with $\chi > 0$ the well-known Poincaré-Bendixson theory provides a complete description of possible qualitative types of behavior of trajectories. A new feature in case $\chi \leq 0$ is that the universal covering surface is noncompact and can be endowed by such structure of Euclidean or hyperbolic (Lobachevsky) plane which is invariant with respect to the deck transformations; thus one can lift the trajectory to this plane and ask about behavior of the lifted trajectory, so to say, at infinity, using geometric notions appropriate to the above mentioned structure. Such approach in rather particular cases was suggested by A. Weil in the thirties, then forgotten and revived by N. Markley (USA) and myself in the sixties.

In the course of research this subject (originally belonging to the theory of ordinary differential equations and dynamical systems) was somewhat extended in a natural way and now my work concerns geometrical questions which can be asked not only about the (semi)trajectory, but about any (semi)infinite continuous curve having no self-intersections (again lifted to the covering plane). Special attention is paid to the "intermediate" case of leaves of foliations on surfaces with only finite number of singularities. So, the main content of this work is considering several classes of curves and comparing these classes with respect to several properties. The latter are such that they do not change if the lifted curve is replaced by another curve lying at a bounded Frechet distance from the first curve. (E.g., such is the property that the lifted curve is (or is not) bounded.) Such properties are, in a sense, more "elementary", "primitive" than properties of the limit behavior of the trajectories usually considered in the qualitative theory of ordinary differential equations.

First I shall dwell on some standing notations and recall or fix the terminology. (Some of this has already appeared in the abstract.)

By a "surface" $M$, we understand below a closed two-dimensional manifold; the closedness of this manifold implies that it is compact and has no boundary. It is not necessary to assume that the surface lies in $\mathbb{R}^3$, although one may think that it is so in the case when $M$ is orientable and this may seem to be descriptive. $\widetilde{M}$ denotes the universal covering surface of $M$. The Euler characteristic of $M$ is denoted by $\chi$. If $\chi = 0$, $M$ (which is either the torus or the Klein bottle) can be endowed by a Riemannian metric of zero curvature; when lifted to $\widetilde{M}$, this metric endowes the latter surface by such a structure of the Euclidean plane which is invariant with respect to the deck transformations. If $\chi < 0$, $M$ can be endowed by a Riemannian metric of a constant negative curvature; when lifted to $\widetilde{M}$, this metric endowes the latter surface by such a structure of the hyperbolic (Lobachevsky) plane which is invariant with respect to the deck transformations. Below $M$ and $\widetilde{M}$ are always endowed with such metrics. (There are many metrics of the type required, but our definitions and statements do not depend on the concrete choice of the metric used.)

As our subject is related to the study of flows on surfaces, I shall recall that (as it was already said in the abstract) a "flow" $\{\varphi_t\}$ is a continuous one-parameter group of transformations of a surface. Usually, a flow is defined in a certain way by a vector field, although this is unessential for our purposes. I would like to emphasize at once that besides the flows, we will also deal with one-dimensional foliations (which almost inevitably have singularities); if a foliation is orientable, its leaves represent the trajectories of a certain flow, and (unless the velocity along the trajectories is of special importance [1]), actually there is no difference between this flow and a foliation.

The theory I am going to speak about is a secondary but legitimate descendant of Poincaré's memoir "On the curves defined by differential equations". (Such combination — secondary and legitimate — would seem strange for the genealogy, but is rather common in many fields of human activity.) This memoir together with Poincaré's "New methods of celestail mechanics" (which appeared later) manifested the arising of the qualitative theory of ordinary differential equations (ODE) and the theory of dynamical systems (DS). Part of Poincaré's

[1] In certain cases, the velocity may be essential (this is the case in the ergodic theory); however, there are no such cases in the problems considered here.

memoir is devoted to local questions and part — to global ones; it is the second part which is relevant for us. In more modern terms, here he considered the global behavior of trajectories of flows (i.e. of continuous 1-parameter transformation groups) on surfaces. Flows were defined by the phase velocity vector fields which he usually assumed to be real analytic, but actually this was not important for the most part of his considerations. Poincaré considered almost exclusively 2-sphere, projective plane and 2-torus; together with the Klein bottle (which he did not consider specially) this is the complete set of closed surfaces with nonnegative Euler characteristic $\chi$. As regards to the case $\chi > 0$, his memoir essentially contains what is called now the Poincaré – Bendixson theory, although it is not exposed there in the general systematic form which was given to this theory later by Bendixson. The theory provides a complete description of possible qualitative types of behavior of trajectories. As regards to the toral case, Poincaré studied only one type of flows existing on this surface — those which in terms of cyclic coordinates $(x, y)$ measured modulo 1 can be described by an ODE

$$\dot{x} = f,\ \dot{y} = g \tag{1}$$

where $f$ and $g$ are periodic in $x$ and $y$ with period 1, and $f > 0$ everywhere. This type attracted his attention because it is one of those cases when the flow has the properties different comparatively to the properties inherent for the case $\chi = 2$ and because it seems to be the most important of such cases. I shall recall that Poincaré reduces the study of (1) to the study of the first return map $\Phi$ of the topological circle $C$ — the toral meridian which has equation $x \equiv 0 \bmod 1$. (For any point $z \in C$ point $\varphi_t(z)$ moving along the positive semitrajectory of (1) returns from time to time to the circle $C$. Let $t(z)$ be the time when the first return happens, i.e. $\varphi_{t(z)}(z) \in C$ and $\varphi_t(z) \notin C$ for all $t \in (0, t(z))$; then $\Phi(z) = f_{t(z)}(z)$.) Poincaré studies iterations $\Phi^n(z)$ and introduces the "rotation number" $\alpha_\Phi$ which is the number $\alpha$ such that the rotation of the circle $C$ (identified with $\mathbb{R}/\mathbb{Z}$) on the angle $\alpha$, i.e. the map

$$C \to C \quad z \mapsto z + \alpha,$$

"resembles" the map $\Phi : C \to C$ better than the rotation on any other angle. In the formal definition of $\alpha$ the lift $\widetilde{\Phi}$ of $\Phi$ to $\mathbb{R}$ is used — namely,

$$\alpha_\Phi = \lim_{n\to\infty} \frac{1}{n}\widetilde{\Phi}^n(z).$$

(One proves that this limit exists for all $z$ and does not depend on $z$.)

As regards to other surfaces, Poincaré only proved for them that if a vector field tangent to a surface has only finitely many zeroes, then the sum of their indices equals to the Euler characteristic. Essentially this is a topological theorem, not a theorem on DS.

After the Poincaré – Bendixson theory was developed, it was natural to pay special attention to flows on the surfaces $M$ with $\chi \le 0$. One approach to the study of flows on such surfaces $M$ is based on lifting a trajectory to the universal covering plane $\widetilde{M}$ and asking about the behavior of the lifted trajectory — mainly about its behavior, so to say, at infinity — using geometric notions appropriate to the above mentioned structure on $\widetilde{M}$. Such approach in rather particular cases was suggested by A. Weil in the thirties who first obtained new

Poincaré's results on toral flows using this approach and later payed some attention to somewhat different flows on other surfaces. He restricted the analysis to a certain particular class of flows and foliations, apparently keeping in mind that these flows and foliations represent the most natural analogues of the flows Poincaré considered on a torus. However, the general character of his approach is obvious. Besides this, as a passing remark he paid some attention to infinite non-self-intersecting curves which need not be trajectories of any flow.

(Note that Poincaré also uses some universal covering — the universal covering $\mathbb{R}$ of the meridian $C$. This is more "economical" than the covering of the whole surface M, but involves ambiguity in the construction — clearly $C$ can be replaced by any other cross-section of the flow, i.e. by any other closed curve $C$ such that locally trajectories intersect it passing from one side of $C$ to another.)

I shall describe briefly some of Weil's work. Lifting the Poincaré-type flow from the torus to $\mathbb{R}^2$ means considering (1) on $\mathbb{R}^2$. In this case it is clear that as the time $t \to \infty$, any trajectory goes to infinity, as already $\lim_{t\to\infty} x(t) = \infty$. Weil proved that it has an "asymptotic direction in infinity", i.e. that when $t \to \infty$, there exists a limit of the straightlinear ray joining a fixed initial point (which will be sometimes called "an observation point") with $(x(t), y(t))$. The inclination of this limit ray (i.e. the tangent of the angle between the positive $x$-semiaxis and this ray) turns out to coincide with the Poincaré's rotation number.

Poincaré flows on the torus have no equilibria (rest) points, but there are other flows and 1-dimensional foliations on the torus sharing the same property: those with the "Kneser annulus" [2]. In terms of the behavior of the trajectories at infinity, the latter leads to the phenomenon called "a saddle in infinity".

If $\chi < 0$, any flow (foliation) on the closed surface $M$ must have rest (singular) points. Weil's flows and foliations are, so to say, the most "economic" ones in this respect: all their rest (singular) points are saddles. Besides this, Weil demanded the absence of saddle connections and saddles in infinity for lifted flow (foliation). He proved that in this case any lifted semitrajectory (semileaf) tends to infinity and has an asymptotic direction there, except for the separatrices (which tend to saddle points).

As regards to general simple curves on the torus, Weil formulated the following statement: if the lifted curve tends to infinity, it has an asymptotic direction there. As far as I know, he never published the proof.

This point of view was forgotten for many years. It was revived by N. Markley (USA) and myself in the sixties. One of the first steps was to prove Weil's statement and its analogous for $\chi < 0$ (conjectured by myself). The first step was done by Markley [3], second — by V. Pupko. Whatever strange, the latter case was more easy (usually the hyperbolic case is more difficult).

In the course of research the subject of the theory of ODE and DS was somewhat extended in a natural way, and now my work concerns geometrical questions which can be asked not only about the (semi)trajectory, but about any (semi)infinite continuous curve having no self-intersections (simple curves); we again lift them to the covering plane. I consider several classes of curves and

[2]It somehow resembles the well-known "Reeb component" of some 2-dimensional foliations on 3-dimensional manifolds, e.g. on the 3-dimensional sphere. One can find in the literature Reeb's name associated with the Kneser annulus, although Kneser's paper was published much earlier amd although Reeb's contribution concerned 3-dimensional case.

[3]Later I was told that he formulated and proved the corresponding statement also for $\chi < 0$, but it remained unpublished.

compare these classes with respect to several properties. The classes of curves considered are:

$\mathfrak{K}_1$: class of all semi-infinite simple curves on $M$;

$\mathfrak{K}_2$: class of all semi-trajectories af all flows on $M$;

$\mathfrak{K}_3$: class of all semi-trajectories of all $C^\infty$ flows on $M$;

$\mathfrak{K}_4$: class of all semi-trajectories of all $C^\infty$ flows on $M$ having a prescribed invariant measure with a "good" (say, $C^\infty$) density;

$\mathfrak{K}_5$: class of all semitrajectories of all real-analytic flows on $M$;

$\mathfrak{K}_6$: class of all semi-leaves of all 1-dimensional foliations on $M$ having only finite number of singularities;

$\mathfrak{K}_7$: class of all semi-trajectories af all flows on $M$ having only finite number of equlibria points (singularities).

There are evident set-theoretic relations between these classes:

$$\mathfrak{K}_1 \supset \text{all } \mathfrak{K}_i, \quad \mathfrak{K}_2 \cap \mathfrak{K}_6 \supset \mathfrak{K}_7; \quad \mathfrak{K}_5 \subset \mathfrak{K}_4 \subset \mathfrak{K}_3 \subset \mathfrak{K}_2;$$

of course, all the inclusions here are strict. But I compare these classes with respect to several properties, all formulated in terms of lifts of these curves to the universal covering plane, as are the properties in the Weil-Markley-Pupko theorem above. Let $L = \{z(t); t \in \mathbb{R}_+\}$ be the semi-infinite curve on $M$ considered and $\widetilde{M} = \{\widetilde{z}(t)\}$ be one of its lifts to the universal covering plane $\widetilde{M}$ endowed by the Euclidean or Lobachevsky structure (in dependence of $\chi$) with metric $\rho$; here $\widetilde{z(t)}$ covers $z(t)$. I deal with the following properties of $\widetilde{L}$:

(a) unboundedness;

(b) tending to infinity ($\rho(\widetilde{z(t)}, a) \to \infty$ for a certain (and, hence, any) fixed point $a \in \widetilde{M}$ as $t \to \infty$);

(c) oscillation ($\liminf \rho(\widetilde{z(t)}, a) < \infty, \ \limsup \rho(\widetilde{z(t)}, a) = \infty$);

(d) tending to infinity in the asymptotic direction $l$ characterized by a straight ray originating at $a$ (this means that $\widetilde{L}$ tends to infinity and that the segments $\overline{a\widetilde{z(t)}}$ tend to $l$ as $t \to \infty$). One can associate with $\widetilde{M}$ points at infinity $p_l$ that are in a bijective relation with the rays $l$ beginning at $a$ and introduce an appropriate topology in the completed plane (in the hyperbolic case, this means the addition of an absolute[4]; however, in the flat case, this is not equivalent to the addition of the line at infinity as it is done when passing to the projective plane); then, this property is reformulated as $\lim \widetilde{z(t)} = p_l$;

(e) a bounded or unbounded deviation of $\widetilde{L}$ from the asymptotic ray $l$, provided that the preceding property holds; the unbounded deviation may occur either only to one side of $l$ or to both sides;

(f) in the presence of oscillation, a set $\omega(\widetilde{L})$ of rays is defined that represent the limits of the segments $\overline{a\widetilde{z(t_n)}}$ taken along the sequences $t_n$ such that $\rho(a, \widetilde{z(t_n)}) \to \infty$ (in other terms, the set of points $p_l$ from property (d) that represent the limit points for $\widetilde{L}$); if $\omega(\widetilde{L})$ coincides with a certain preassigned set or belongs to a certain system of sets, then this is a certain property of the curve $\widetilde{L}$;

[4]In the Poincaré model of a hyperbolic plane in a disk $D$, an absolute is a circle that bounds $D$.

(g) quasimonotonicity. This property was only introduced for the case when $M$ is a torus in $\mathbb{R}^2/\mathbb{Z}^2$, while $\widetilde{z(t)} = (x(t), y(t))$ and $\sup y(t) = \infty$; the formulation of this property is distinct form the previous ones. We say that there exist time intervals with a quasimonotonic arbitrarily large growth of $y(t)$ if there exists an $a > 0$ such that, for any $A > 0$, there exists a time interval $[t_1, t_2]$ on which the increment of $y(t)$ is greater than $A$ (i.e., $y(t_2) - y(t_1) > A$), and this growth is quasimonotonic in the sense that, if $t_3, t_4 \in [t_1, t_2]$ and $t_3 < t_4$, then $y(t_4) - y(t_3) > -a$.

It is obvious that, if a certain of the curves $\widetilde{L}$ covering $L$ possesses one of the properties (a)–(g), then any lift of $L$ to $\widetilde{M}$ also possesses the same property; the only exception is the final part of property (f): another lift $\widetilde{L}'$ is obtained from $\widetilde{L}$ under a certain deck transformation, and the same deck transformation translates $\omega(\widetilde{L})$ to $\omega(\widetilde{L}')$. (There are no exceptions when $M$ is a torus.) Therefore, these properties can be considered as properties of $L$.

The properties considered here are such that they are invariant under the replacement of $\widetilde{L}$ by another curve $\widetilde{L}' = \{\widetilde{z}'\}$ lying at a finite Frechet distance from $\widetilde{L}$. The Frechet distance $\rho(\widetilde{L}, \widetilde{L}')$ is defined as $\inf \sup_t \rho(\widetilde{z(t)}, \widetilde{z(t)}')$, where the infimum is taken over all possible parameterizations of our curves. This definition does not imply in itself that $\widetilde{L}$ and $\widetilde{L}'$ are obtained by lifting certain non-self-intersecting curves $L$ and $L'$ lying on $M$ to $\widetilde{M}$. However, the Frechet distance in this general case does not possess the properties that are usually required of the distance: it may be zero for $\widetilde{L} \neq \widetilde{L}'$ or may be infinite. The first "trouble" is impossible in our case (it is only required that $\widetilde{L}$ and $\widetilde{L}'$ do not have self-intersections, irrespective of whether or not they are obtained by lifting certain curves from $M$). The second trouble remains; however, it is not so essential.

In fact, we only need the property $\rho(\widetilde{L}, \widetilde{L}') < \infty$ rather than the Frechet distance itself. This property is equivalent to the fact that $\sup_t \rho(\widetilde{z(t)}, \widetilde{z'(t)}) < \infty$ under certain parameterization of these curves. Finally, since our $\widetilde{L}$ and $\widetilde{L}'$ are the lifts of certain $L$ and $L'$ from $M$ to $\widetilde{M}$, we will say that the curves (that are, as usually, semi-infinite and simple) $L$ and $L'$ on $M$ are Frechet-equivalent ($F$-equivalent) if certain lifts of these curves on $\widetilde{M}$ lie at a finite Frechet distance from each other. (This is indeed an equivalence relation.)

Properties (a)-(g) (as well as any F-invariant properties) are, in a sense, more "elementary", "primitive" than properties of the limit behavior of the trajectories usually considered in the qualitative theory of ODE (which of course may change when replacing a curve by a F-equivalent curve), and in this sense the theory presented here is "underlying" the usual qualitative theory.

Up to the $F$-equivalence, there are the following relations between these classes. Each curve from $\mathfrak{K}_1$ is $F$-equivalent to a curve from $\mathfrak{K}_4$; however, there are curves in these classes that are $F$-equivalent to no curves from $\mathfrak{K}_6$; there are curves in $\mathfrak{K}_6$ that are $F$-equivalent to no curves from $\mathfrak{K}_5 \cup \mathfrak{K}_7$. So from our point of view there is no difference between classes $\mathfrak{K}_1 - -\mathfrak{K}_4$, but $\mathfrak{K}_5$ is "smaller" and $\mathfrak{K}_5, \mathfrak{K}_7$ are even smaller. (The last two classes are "the same".)

In all these cases $F$-equivalence is proved by an "approximation" of a curve from a broader class by a curve for a smaller class. The difference of "smaller" classes from "bigger" ones follows from theorems claiming that for curves from a smaller class some of the properties (a) – (g), or their negations, or their combinations are impossible, while there are examples of the curves belonging to a “bigger” class possessing these properties (or their negations, or their combinations). Study of all cases, except the comparison of $\mathfrak{K}_1$ and $\mathfrak{K}_6$, is based on the properties (a) – (f) which can be considered as concerning the behavior of the lifted curves at infinity. (Of course this is an intuitively convincing, but non-formal statement, since the term ”behavior at infinity,” just as the term ”behavior”, is an informal descriptive expression.) When comparing $\mathfrak{K}_1$ and $\mathfrak{K}_6$, I use the property (g) which can hardly be considered as having to do with the specific features of the behavior at infinity. Although the problem ”compare $\mathfrak{K}_1$ and $\mathfrak{K}_6$” is solved in the form as it stands, there remain unsolved questions “around it”. The situation still remains unclear for a relatively simple concrete curve I had in mind in 1995 — for the projection on $\mathbb{R}^2/\mathbb{Z}^2$ of the graph of the function $\sin\frac{1}{x}$ considered on $(0, \frac{1}{2})$. While it is clear that properties (a) - (f) are important, this is by no means clear for the new property (g). And as for the behavior at infinity, I said that several more or less exotic variants of such behavior have been realized by means of foliations with a finite number of singular points, but it is unknown whether other (more exotic?) variants (that are possible for arbitrary simple $\widetilde{L}$) are realized by such foliations.

The following two restrictions for the curves $L$ from the classes $\mathfrak{K}_7$ and $\mathfrak{K}_5$ are known (for $\chi = 0$ they are due to myself, for $\chi < 0$ — to S. Kh. Aranson and V. Z. Grines). Their coverings $\widetilde{L}$ cannot oscillate. If $\widetilde{L}$ tends to infinity, it has only a finite deviation from its asymptotic ray $l$. None of these restrictions is obligatory for the curves of other classes. Concerning the deviation, there is a general theorem which I shall formulate first for the torus. In this case the asymptotic direction is characterized by the inclination of any corresponding ray (all they have the same inclination). It turns out that if the inclination is a rational number, the deviation can be unbounded in one side from this ray (so to say, above or below the corresponding straight line), but not in both sides. If the inclination is irrational, then one can construct examples where the deviation is unbounded in both sides or in one side.

If $M$ is the Klein bottle, it turns out that there are only 4 points at infinity $p_l$ which can be asymptotic directions of $\widetilde{L}$ for $L \in \mathfrak{K}_1$. Realize the Klein bottle as $\mathbb{R}^2$ factorized by the action of the group generated by $\mathbb{Z}^2$ (acting by shifts) and also by the transformation $(x, y) \mapsto \left(x + \frac{1}{2}, -y\right)$. Four $p_l$ mentioned correspond to two directions (positive and negative) of the two coordinate axes. It turns out that if $\widetilde{L}$ tends to infinity in the direction of the $x$-axis, its deviation from this axis is bounded. If $\widetilde{L}$ tends to infinity in the direction of the $y$-axis, its deviation from this axis can be unbounded in one side (so to say, “to the right side” or “to the left side”), but not in both sides.

Now I shall formulate an analogous statement for the hyperbolic case (case $\chi < 0$). First about the terminology. For both $\chi = 0$ and $\chi < 0$, choosing various observation points $a$ on the covering plane $\widetilde{M}$, we get a two-parameter family of the limit rays laying on some straight lines. The family of these lines depends on one parameter and is a so-called “pencil” of straight lines (to be more precise, a parabolic pencil, at least it is so in the hyperbolic case); I have already said that it consists of lines passing through a fixed point $p_l$ at infinity. Call the pencil and this point “rational”, if some of the straight lines belonging to the pencil projects onto a closed line on $M$. The name “rational” is explained by the fact that in the toral case projections of the asymptotic rays are closed just when the inclination of the asymptotic ray(s) is rational. (In this case either all of these projections are closed or none is closed.) Otherwise call the pencil and the point at infinity "irrational" or "arational" (the letter word was used by J. Nielsen). It turns out that if $p_l$ is rational, the deviation of $\widetilde{L}$ from any of its asymptotic rays can be unbounded in only one side, while there are examples when $p_l$ is arational and deviation can be made unbounded in both sides as well as in one side. However, at the moment examples demonstrating various possibilities are constructed only for some asymptotic directions — those which, so to say, arise from the asymptotic directions on the torus when we glue to it a handle or a Moebius strip.

The first of the above mentioned restrictions for curves from $\mathfrak{K}_5 \cup \mathfrak{K}_7$ (impossibility of oscillation) takes also place for semi-trajectories of somewhat more general types of flows than in $\mathfrak{K}_7$, namely, of flows such that the set $R$ of their rest points can be contracted to one point on $M$ (i.e. there exists a continuous map $\psi : R \times [0,1] \to M$ such that $\psi(z,0) = z$ for all $z \in R$ and $\psi(R,1) = \{\text{one point}\}$). Presumable the second restriction (bounded deviation from the asymptotic ray for unbounded $\widetilde{L}$) is also valid (this is known to be the case for $\chi = 0$).

Starting speaking about the asymptotic directions at infinity, I shall note an important difference between the cases of the torus, the Klein bottle and the hyperbolic surfaces. In the first case any point at infinity $p_l$ can be a point of asymptotic direction of some curve $\widetilde{L}$ (covering some simple curve $L$ on the torus). In the second case only four $p_l$ can be such points. In the third case the set of those $p_l$ which are the asymptotic directions of various $\widetilde{L}$ — let us call these points "admissible" — is infinite, but it is less than the whole absolute. It seems that the points of the absolute can be described using some infinite symbolic sequences and that admissible points are characterized by some conditions imposed on the corresponding sequences, but unfortunately this is not done, at least in completely "convincing" way. In some papers dealing with the geodesics on the closed surfaces one can find some steps towards this goal, but in different papers (even written by the same author) descriptions can be different. Thus at the moment there is no "universally accepted" description. May be, this is unavoidable and there can be several "equally good" descriptions (perhaps one must use different descriptions for different purposes), but then it is desirable to elaborate a general point of view on this subject explaining the connections between different descriptions and allowing, so to say, to "grasp" them simultaneously. Possibly the lack of a "good" description of "admissible" $p_l$ can be a reason of the above-mentioned uncompleteness of results about the boundedness or unboundedness of the deviation of $\widetilde{L}$ from the asymptotic ray $l$ (see two paragraphs above).

What can be said about the set $\omega(\widetilde{L})$ for $L \in \mathfrak{K}_1$ (what for our purposes is practically equivalent to $f \in \mathfrak{K}_4$)? I constructed an example where this set coincides with the whole absolute. So an oscillating curve can manage to approach such points at infinity which are "prohibited" for curves "honestly" tending to infinity. In the toral case it is easy to construct $L$ with $\omega(\widetilde{L})$ consisting of two points which in the evident sense are "mutually diametrally opposite". In the same case A. A. Glutsuk was able to describe all sets which can be $\omega(\widetilde{L})$: besides those which we already know, they can be closed arcs $\Delta$ of the absolute which do not exceed its half, i.e. the rays going from a fixed point of the plane to all points of $\Delta$ completely fill an angle $\leq 180°$ (including its sides). For other $M$ the answer is unknown.

The following interesting result due to D. A. Panov lies slightly outside of the framework of this surway (because Panov deals with a property which is not invariant with respect to the F-equivalence), but is close to it and deserves mentioning. Panov constructed a (nonorientable) foliation on the torus with 3 singularitries having a leaf $L$ such that its lift $\widetilde{L}$ is dense on the covering plane. Also, any leaf of this foliation is dense on the torus. The foliation is $C^{\infty}$ outside of the singular points and has a "good" transversal measure given locally by a closed pfaffian form.

The construction of some examples is involved. To be more precise, usually the corresponding picture seems to be quite "understandable" after it is drawn, but it is the description of the construction and the formal proof that it actually provides such and such properties which are complicated. This can be due to a lack of technique — all my examles were constructed more or less by the naked hands. Contrary to this, Panov uses some technique which is not very complicated but looks "advanced" when compared to simple drawing. He starts from some special "flat" structure (so-called "half translation structure") with 3 singular points on the torus and takes an expanding foliation of some automorphism of this structure (which acts trivially on the first homology group, but is hyperbolic in the local affine coordinates — a peculiar combination!) Perhaps in other cases special "flat" structures (with singularities where their curvature is "concentrated") and some geometry related to them (say, geodesics) also can provide a good tool and convenient language for our purposes, as well as well as the branched coverings (already used in related situation by Aranson and Grines).

Instead of giving a long list of references, I refer to a recent survey which I wrote together with E. V. Zuzhoma and which contains many references (including references to surveys where other references are given) [1]. It appeared in a series which should be translated into English, so soon it will become available to English-speaking readers. (We — Zhuzhoma and I — wrote also a book which will appear as v.259 in the same series, but it will become available later.) Besides our survey, I give the reference to the article by Panov which was not cited in our survey [2].

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