

Introduction to the physics of the Cosmic Microwave Background

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Résumé: FRW Cosmology

Friedmann equation

$$H^2 = \frac{8\pi G_N}{3} \rho - \frac{K}{a^2}$$

where

$$H = \frac{da}{adt} = \frac{\dot{a}}{a^2} \quad \text{Hubble parameter}$$

and

$$K = \frac{8\pi G_N}{3} \rho_0 - H_0^2 = H_0^2(\Omega_0 - 1) \quad \text{spatial curvature}$$

By definition, $a = 1$ today.

Critical density

$$\rho_c = \frac{3H^2}{8\pi G_N}$$

Density parameter

$$\Omega(\eta) = \frac{\rho}{\rho_c}$$

We denote the density parameter today simply as $\Omega(\eta_0) = \Omega$.

$$H_0 = 100h \text{ km sec}^{-1} \text{ Mpc}^{-1} = 2.133 \times 10^{-33} \text{ eV}$$

$$h = 0.72 \pm 0.08 \quad \text{HST}$$

$$\rho_c = 1.88h^2 \times 10^{-29} \text{ g cm}^{-3}$$

Matter Dominated Era (MD):

$$a \propto t^{2/3} \quad a \propto \eta^2 \quad \eta \propto t^{1/3}$$

where overdots refer to derivative wrt conformal time.

$$H = \frac{2}{3t} \quad H a = \frac{2}{\eta}$$

Radiation dominated era (RD)

$$a \propto t^{1/2} \quad a \propto \eta \quad \frac{\dot{a}}{a} = \frac{1}{\eta}$$

CMB temperature

$$T_0 = 2.725 \pm 0.00?$$

Radiation density

$$\rho_r = \rho_\gamma + \rho_\nu = \left[2 + 2 \times 3 \times \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} \right] \frac{\pi^2}{30} T^4$$

(massless or relativistic neutrinos, after electron-positron annihilation)

$$\Omega_\gamma = 2.47 \times 10^{-5} \times h^2$$

Matter-radiation equality

$$\frac{\rho_m}{\rho_r} = \frac{\Omega_m}{\Omega_r} a_{\text{EQ}} = 1$$

$$a_{\text{EQ}} = 4.15 \times 10^{-5} \times \Omega_m h^2$$

Flat matter dominated universe

$$a(\eta) = \left(\frac{\eta}{\eta_0} \right)^2 + \sqrt{2a_{\text{EQ}}} \frac{\eta}{\eta_0}$$

where

$$\eta_0 = \frac{2}{H_0 \Omega_m^{1/2}}$$

$$\eta_{\text{EQ}} \approx (\sqrt{3} - 1) \sqrt{\frac{a_{\text{EQ}}}{2}} \eta$$

Thermodynamics in FRW

Energy and number densities in relativistic regime (and negligible chemical potential):

$$\begin{aligned}\rho &= \begin{cases} g \frac{\pi^2}{30} T^4 & \text{Bose} \\ g \frac{7\pi^2}{8 \cdot 30} T^4 & \text{Fermi} \end{cases} \\ n &= \begin{cases} g \frac{\zeta(3)}{\pi^2} T^3 & \text{Bose} \\ g \frac{3\zeta(3)}{4\pi^2} T^3 & \text{Fermi} \end{cases} \\ p &= \frac{\rho}{3}\end{aligned}$$

where $\zeta(3) = 1.20206\dots$

Non-relativistic regime:

$$\begin{aligned}n &= g \left(\frac{mT}{2\pi} \right)^{3/2} e^{-(m-\mu)/T} \\ \rho &= mn \\ p &= nT \ll \rho\end{aligned}$$

Standard problems and inflation

Flatness problem

From

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{K}{a^2}$$

where K is a constant (the curvature today). let

$$\Omega_K = 1 - \Omega$$

with

$$\Omega_K = -\frac{K}{a^2 H^2}$$

We observe that the universe is close to flatness today. Hence $\Omega_K \lesssim 1$. However, in a universe dominated by matter or radiation, $a^2 H^2$ is a decreasing function of time since $aH = \dot{a}$ and

$$\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G_N}{3}(\rho + 3p)$$

(ordinary matter is attractive under gravity).

A flat universe is thus an unstable solution in standard cosmology. Conversely, if the universe is closed to flat today, it had to be very flat before. In a matter dominated universe for instance,

$$aH = \frac{H_0}{\sqrt{a}} \approx H_0 \sqrt{z}$$

which implies that if the universe is flat to within 10 percent today, it had to be flat to a precision better than 0.1 percent at the time of equality. More impressive statements can be made going further back in time.

A simple resolution is to assume that the universe underwent a period of accelerated expansion early on. This is possible for form of matter with an eq. of state such that

$$\frac{p}{\rho} = \omega \leq -\frac{1}{3}$$

Horizon problem

The size of the observable universe today is about $1/H_0 \sim 10^{28}$ cm. Since $a = T_0/T$, at the Planck scale, the universe was about 10^{-4} cm in size. Pretty small by still about 10^{29} larger than the Planck length ($L_P \approx 10^{-33}$ cm). That is, at the Planck scale, the universe was composed of about $(10^{29})^3 \approx 10^{87}$ causally disconnected regions. How comes it is so homogeneous and isotropic today? This is the essence of the horizon problem.

A more important implementation of the horizon problem rests on the isotropy of the cmbr. Recombination took place at conformal time $\eta_*/\eta_0 \sim 100$. The light we see thus comes from $\sim 10^6$ disconnected regions at the time of last scattering (the horizon at last scattering spans an angle of a degree scale).

Remarkably a period of accelerated expansion could also solve this issue. Consider a phase during which the Hubble parameter is approximately constant, corresponding to a form of energy such that $p \approx -\rho$. Then

$$a = \exp(Ht)$$

and the size of the particle horizon

$$d_{ph} = a(t) \int_0^t \frac{dt'}{a(t')} = \frac{1}{H} e^{Ht}$$

diverges exponentially with t . Supposing to simplify that inflation took place at the Planck scale, with Planck size causal regions, we have to ask that the universe grew by a factor of about $10^{29} \sim e^{60}$ during inflation to solve the causality problem. Note that implementing inflation leads to less severe requirements.

Implementing inflation

The simplest way to implement inflation in the early universe is to assume that the energy density of the universe was dominated by the potential energy of some scalar field ϕ called the inflaton. An homogeneous and isotropic scalar field has stress-energy tensor

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu - p g_{\mu\nu}$$

with

$$\rho = \frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 + V(\phi)$$

and

$$p = \frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 - V(\phi)$$

A static scalar field thus have

$$p = -\rho$$

like a cosmological constant. This is possible if the scalar field sits at a minimum of its potential $V(\phi) \neq 0$. This pose the probleme that it is not clear how inflation can stop (as it does, since eventually the universe becomes radiation and matter dominated). It is also possible to implement an effective period of acceleration if the kinetic energy is small compared to the potential energy, for

$$p \leq -\frac{1}{3}\rho$$

corresponds to

$$\left(\frac{d\phi}{dt} \right)^2 \leq V$$

The equation of motion of a scalar field in a FLRW universe are

$$H^2 = \frac{1}{3M_{\text{pl}}^2} \left[\frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 + V(\phi) \right]$$

and

$$\frac{d^2\phi}{dt^2} + 3H \frac{d\phi}{dt} + \frac{\delta V}{\delta \phi} = 0$$

where the friction term arises from expansion.

The standard approximation for analysing inflation is the so called slow-roll approximation. This consists in setting

$$H^2 \approx \frac{V}{3M_{\text{pl}}^2}$$

so as to get inflation and, for consistency, to set

$$3H \frac{d\phi}{dt} \approx -V'$$

so that the field evolves slowly. For this approximation to be valid, it is necessary that the following two parameters be small

$$\epsilon = \frac{M_{\text{pl}}^2}{2} \left(\frac{V'}{V} \right)^2 \ll 1$$

and

$$|\eta(\phi)| = M_{\text{pl}}^2 \left| \frac{V''}{V} \right| \ll 1$$

the latter not to be confused with conformal time. It is easy to verify these conditions by substitution. Note that these conditions are not sufficient, since it is still required that the scalar field evolves according to the approximate equation of motion. In general this corresponds to a condition on initial conditions on the field and its time derivative.

The simplest implementation is that of a field with potential

$$V = \frac{1}{2}m^2\phi^2$$

Slow roll conditions impose that

$$\phi^2 \geq M_{\text{pl}}^2$$

initially. This sort of potential leads to a scenario called chaotic inflation because the initial conditions are supposed to be set at random.

To see the connection between the slow roll conditions and inflation consider

$$\frac{1}{a} \frac{d^2 a}{dt^2} = \frac{dH}{dt} + H^2$$

Acceleration requires

$$-\frac{dH}{dt} \frac{1}{H^2} < 1$$

(note that normally H is a decreasing function of time). Substituting the slow roll conditions gives

$$-\frac{dH}{dt} \frac{1}{H^2} \approx \frac{M_{\text{pl}}^2}{2} \left(\frac{V'}{V} \right)^2 = \epsilon$$

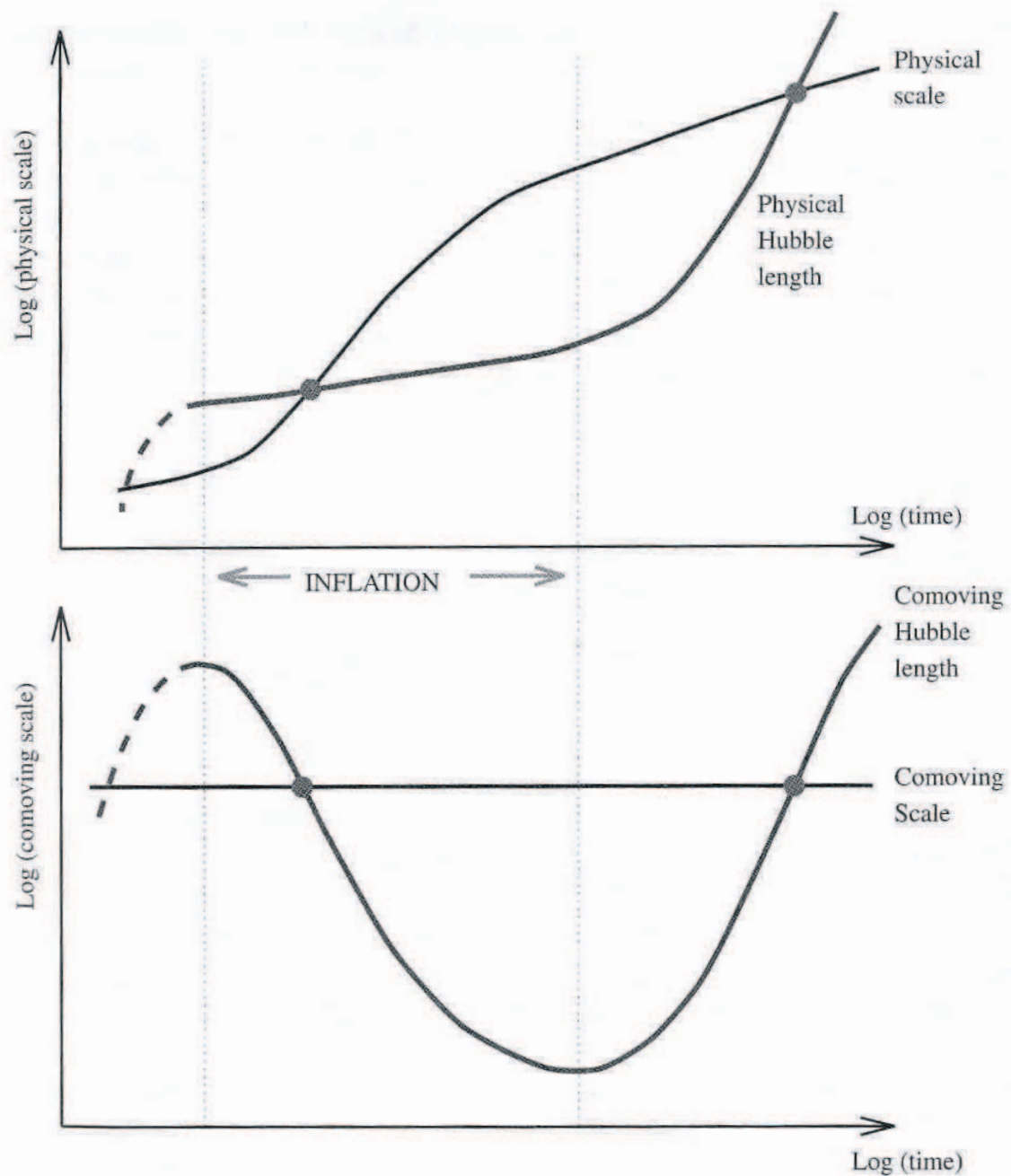
The amount of inflation is given by the number of e-folds

$$N(t) = \ln \frac{a(t_{\text{end}})}{a(t)} = \int_t^{t_{\text{end}}} H dt \approx \frac{1}{M_{\text{pl}}^2} \int_{\phi_{\text{end}}}^{\phi} \frac{V}{V'} d\phi$$

Note that the Hubble length is approximately constant during inflation. This scale sets the physical distance within which objects can influence each other, i.e. the causal horizon is

$$d_H = a(t) \int_t^{\infty} \frac{dt'}{a(t')} \approx \frac{1}{H}$$

Incidentally, the comoving horizon is **shrinking** during inflation. This provides an alternative description of inflation (see figure).



II. Boltzmann Equations

$$\frac{df}{d\lambda} = C[f]$$

*

$f(x, p, t)$ = distribution function (DF)



for photons, baryons, electrons, neutrinos and cold dark matter in curved space-time...

Fortunately, only small fluctuations early on and/or on large scales



Linear perturbations around homogeneous & isotropic expanding background

*We are in a relativistic context: λ parametrizes the worldline of the particles of the fluid = proper time for timelike path.

Free Boltzmann equation for photons

The DF depends on $x^\mu = (t, \vec{x})$ and

$$p^\mu \equiv \frac{dx^\mu}{d\lambda} \quad p^2 = P_\mu P^\mu = 0$$

In Newtonian Conformal Gauge (NCG)

$$P^2 = -(1 + 2\Psi)(P^0)^2 + p^2 = 0 \quad p^2 \equiv g_{ij}P^iP^j$$

or

$$\boxed{P^0 = p(1 - \Psi)} \quad (\text{ie to first order!}) \quad (1)$$

☞ Redshift as photons moves out of potential $\Psi < 0$

Introduce unit spatial vector

$$\hat{p}^i = \hat{p}_i \quad \delta_{ij}\hat{p}^i\hat{p}^j = 0$$

Then (exercise)

$$\boxed{p^i = p \hat{p}^i \frac{1 - \Phi}{a}} \quad (2)$$

$$p^0 \frac{df}{d\lambda} \equiv \frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{\frac{dx^i}{dt} \frac{\partial f}{\partial x^i}}_B + \underbrace{\frac{dp}{dt} \frac{\partial f}{\partial p}}_C + \underbrace{\frac{d\hat{p}^i}{dt} \frac{\partial f}{\partial \hat{p}^i}}_A \quad (3)$$

A) This term is second order (why?) hence $\equiv 0$.

B) From (1) and (2)

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} \equiv \frac{p^i}{p^0} = \frac{\hat{p}^i}{a} (1 + \Psi - \Phi)$$

☞ Photons “slow down” in overdense region $\Psi < 0$ & $\Phi > 0$

Since $\partial f / \partial x^i$ is first order, keep only

$$\frac{dx^i}{dt} = \frac{\hat{p}^i}{a}$$

C) Time component of geodesic equation

$$\frac{dP^0}{d\lambda} + \Gamma_{\alpha\beta}^0 P^{\alpha\beta}$$

First use (1)

$$\frac{d}{dt} [p(1 - \Psi)] = -\Gamma_{\alpha\beta}^0 \frac{p^\alpha p^\beta}{p} (1 + \Psi)$$

or

$$\frac{dp}{dt} = p \left\{ \frac{\partial \Psi}{\partial p} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right\} - \Gamma_{\alpha\beta}^0 \frac{p^\alpha p^\beta}{p} (1 + 2\Psi)$$

Second use the metric to express the Christoffel symbols (keeping first order terms).

Get

$$\frac{1}{p} \frac{dp}{dt} = -H - \frac{\partial \Phi}{\partial t} - \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i}$$

1) First two terms is redshift due to (local) expansion.



2) Last term is *e.g.* the energy loss of a photon leaving a potential well

$$\hat{p}^i \frac{\partial \Psi}{\partial x^i} > 0$$

Altogether

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial f}{\partial x^i} - p \frac{\partial f}{\partial p} \left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right]} \quad (4)$$

Remark:

The a factors are there because $a\vec{x}$ is the physical position vector.

Finally expand the distribution function around the equilibrium Bose-Einstein distribution,

$$f^{(0)}(p) = \frac{1}{\exp(p/T) - 1}$$

$$\begin{aligned} f(\vec{x}, p, \hat{p}, t) &= f^{(0)}(p) + \delta f(\vec{x}, p, \hat{p}, t) \\ &\hat{=} f^{(0)}(p) - p \frac{df^{(0)}(p)}{dp} \Theta(\vec{x}, p, \hat{p}, t) \end{aligned}$$

where

$$\Theta(\vec{x}, \hat{p}, p, t) \equiv \frac{\delta T}{T} = \text{Brightness Function}$$

This definition amounts to replacing

$$T = T(t) = \text{average temperature}$$

by

$$T(1 + \Theta)$$

in the equilibrium distribution

$$f^{(0)} = \frac{1}{\exp(p/T) - 1}$$

Important remark:

Θ will turn out to depend on the direction \hat{p} but not on p .

Applying Boltzmann (4) to zero-order gives

$$\frac{df^{(0)}}{dt} = \frac{\partial f^{(0)}}{\partial t} - Hp \frac{\partial f^{(0)}}{p} \equiv 0$$

which implies (exercice) that

$$T \propto \frac{1}{a}$$

The time derivative of $f^{(0)}$ vanishes by very definition of equilibrium.

To first order, one gets the time derivative of the brightness function is

$$\boxed{\left. \frac{df}{dt} \right|_{f.o.} = -p \frac{\partial f^{(0)}}{\partial p} \left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right]} \quad (5)$$

(Rem: we used

$$T \frac{\partial f^{(0)}}{\partial T} = -p \frac{\partial f^{(0)}}{\partial p}$$

)

Collision term

The leading interaction which can affect the photon distribution is Compton scattering off electrons

$$\gamma(\vec{p}) + e(\vec{q}) \leftrightarrow \gamma(\vec{p}') + e(\vec{q}')$$

The collision term is

$$C[f(\vec{p})] = \int \prod_{\vec{k}=\vec{q},\vec{q}',\vec{p}} \frac{d^3\vec{k}}{(2\pi)^3 2E_k} (2\pi)^4 \delta^4(\vec{p}+\vec{q}-\vec{p}'-\vec{q}') |\mathcal{M}|^2 \{ \underbrace{f_e(\vec{q}')f(\vec{p}')}_{\text{inward flow}} - \underbrace{f_e(\vec{q})f(\vec{p})}_{\text{outward flow}} \}$$

We can always neglect Fermi blocking, because the occupation numbers f_e are small at the epoch of interest.

We also can neglect stimulated emission to first order.

This part is a bit technical but it is worth doing things in details (the devil is in the details)

Steps:

I. The electron is non-relativistic at recombination. Replace E_e by m_e in numerator of collision term. Do the \vec{q}' integral using momentum conservation.

$$C[f(\vec{p})] = \frac{\pi}{4m_e^2 p} \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \delta(p + q^2/2m_e - p' - (\vec{q} + \vec{p} - \vec{p}')^2/2m_e) \\ \times |\mathcal{M}|^2 \{f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\}$$

II. Little energy is transferred in non-relativistic Compton scattering.

$$E_e(q) - E_e(\vec{q} + \vec{p} - \vec{p}') \approx \frac{(\vec{p}' - \vec{p}) \cdot \vec{q}}{m_e} \lesssim \frac{Tq}{m_e} \sim T\nu_b$$

for $p, p' \ll q \sim m_e$, and where ν_b is small. Since the initial energy is order T , the change in energy of the electron is small. We thus expand the delta function around the incoming electron energy $E = q^2/2m_e$

$$\delta(p + q^2/2m_e - p' - (\vec{q} + \vec{p} - \vec{p}')^2/2m_e) \approx \delta(p - p') - \frac{\vec{q} \cdot (\vec{p} - \vec{p}')}{m_e} \frac{\partial \delta(p - p')}{\partial p}$$

III. Use $f_e(\vec{q} + \vec{p} - \vec{p}') \approx f_e(\vec{q})$

IV. Take the low energy limit of the unpolarized Compton scattering amplitude to be constant (elastic photon scattering in electron rest frame):

$$|\mathcal{M}|^2 = 8\pi m_e^2 \sigma_T$$

where

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} \approx 0.67 \times 10^{-24} \text{cm}^2$$

is the Thomson cross-section.

This neglects the spin and angular dependence of the amplitude. The latter is a small effect $\sim 1\%$. The former is larger $\sim 10\%$ and will interest us when we will discuss polarization effects. See later. Perhaps.

V. Use

$$f(\vec{p}) = f^{(0)}(p) - p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p})$$

VI. Define the **temperature monopole**

$$\Theta_0(\vec{x}, t) \equiv \frac{1}{4\pi} \int d\Omega' \Theta(\hat{p}', \vec{x}, t)$$

This represent the photon temperature perturbation averaged over all direction and measured at time t at position \vec{x} . It is unobservable here and today but makes sense anywhere else.

All of these steps give the first order approximation to the collision term

$$C[f(\vec{p})] = -n_e \sigma_T p \frac{\partial f^{(0)}}{\partial p} [\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b] \quad (6)$$

In absence of velocity, the effect of Compton scattering is to drive Θ to its monopole,

$$\Theta \approx \Theta_0$$

ie photons from all incoming directions have roughly the same temperature.

If the electrons have a velocity, there is also a dipole moment, fixed by the amplitude and direction of the electron velocity,

$$\Theta \approx \Theta_0 + \hat{p} \cdot \vec{v}_b$$

Efficient Compton scattering makes the description of the photon perturbation easy. It is described solely by its monopole and dipole moment of the distribution function, *ie* like a fluid.

Boltzmann equation for photons

Eqs. (5) and (6) taken together give

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i \partial \Theta}{a \partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i \partial \Psi}{a \partial x^i} = n_e \sigma_T [\Theta_0 - \Theta + \hat{p} \cdot \vec{v}_b]$$

Since these equations are linear, it is convenient to work in Fourier space. Our convention is

$$\Theta(\vec{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot \text{vec} x} \Theta(\vec{k})$$

where \vec{k} are comoving wavenumber.

The baryon velocity field, like any vector field, can be uniquely decomposed into an irrotational and a rotational part. We assume that only the former is relevant. In Fourier space, this means that

$$\vec{v}_b(\mathbf{k}) = \hat{k} v_b(\mathbf{k})$$

Using conformal time, the photon Boltzmann equation becomes

$$\boxed{\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Theta = -\dot{\tau} [\Theta_0 - \Theta + \mu v_b]} \quad (7)$$

where

$$\mu = \hat{k} \cdot \hat{p} = \cos \theta$$

and

$$\dot{\tau} = \frac{d\tau}{d\eta} = -n_e \sigma_T a$$

is the derivative (wrt conformal time) of the **optical depth**.

The optical depth is defined by

$$\tau(\eta) = \int_{\eta}^{\eta_0} d\eta n_e \sigma_T a$$

The quantity $\sigma_T n_e a$ is the rate of photon collision per unit of conformal time. Hence the optical depth is simply the number of photon collisions between conformal time η and today $\eta = \eta_0$.

Since the rate of collision is $\sigma_T n_e a = -\dot{\tau}$, the probability that a photon, now observed, has travelled freely since time η is $P = e^{-\tau}$. (Note that $\dot{P} = -\dot{\tau}P$ and $P(\eta_0) = 1$.) It will be also useful to introduce the **visibility function**

$$g(\eta) = -\dot{\tau}(\eta)e^{-\tau(\eta)},$$

which is the probability density that a photon last scattered at conformal time η .

In the absence of reionization, the optical depth is practically zero until we reach the decoupling epoch, where it rises sharply. Correspondingly, the visibility function is sharply peaked around the epoch of last scattering.

Boltzmann equation for cold dark matter

Basic assumptions:

I. Dark matter, by definition, does not interact with any other form of matter or radiation.

II. Dark matter is non-relativistic.

Let

$$g_{\mu\nu}P^\mu P^\nu = -m^2$$

and

$$E = \sqrt{p^2 + m^2}$$

where

$$p^2 = g_{ij}P^i P^j$$

These identities imply

$$P^\mu = \left(E(1 - \Psi), p\hat{p}^i \frac{1 - \Phi}{a} \right)$$

The time derivative of the dark matter distribution function becomes

$$\frac{df_{dm}}{dt} = \frac{\partial f_{dm}}{\partial t} + \frac{\partial f_{dm}}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f_{dm}}{\partial E} \frac{dE}{dt} + \frac{\partial f_{dm}}{\partial \hat{p}^i} \frac{d\hat{p}^i}{dt}$$

Working the algebra of Christoffel symbols,

$$\frac{dx^i}{dt} = \frac{p\hat{p}^i}{aE}$$

to zeroth order and

$$\frac{dE}{dt} = -\frac{p^2}{E} \left(H + \frac{\partial \Phi}{\partial t} \right) - \frac{p\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i}$$

The last term is negligible as for photons. It is the product of two first order effect, hence next to next to leading order.

Altogether

$$\frac{\partial f_{dm}}{\partial t} + \frac{\hat{p}^i p}{aE} \frac{\partial f_{dm}}{\partial x^i} - \frac{\partial f_{dm}}{\partial E} \left[H \frac{p^2}{E} + \frac{p^2}{E} \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a} \frac{\partial \Psi}{\partial x^i} \right] = 0 \quad (8)$$

The assumption of non-relativistic dark matter will allow us to neglect effects which are second order in the velocity $p/E = v$. That is we consider the density of CDM and its velocity field, but assume the higher moments are negligible. This will drastically simplify the discussion of cdm.

Defining the density

$$n_{\text{dm}} = \int \frac{d^3p}{(2\pi)^3} f_{\text{dm}}$$

and velocity fields

$$v^i = \frac{1}{n_{\text{dm}}} \int \frac{d^3p}{(2\pi)^3} f_{\text{dm}} \frac{p \hat{p}^i}{E}$$

and integrating over momentum gives

$$\frac{\partial n_{\text{dm}}}{\partial t} + \frac{1}{a} \frac{\partial (n_{\text{dm}} v^i)}{\partial x^i} + 3 \left[H + \frac{\partial \Phi}{\partial t} \right] n_{\text{dm}} = 0$$

Define

$$n_{\text{dm}} = n_{\text{dm}}^{(0)}(t) (1 + \delta(\vec{x}, t))$$

The zeroth order equation gives

$$\frac{dn_{\text{dm}}^{(0)}}{dt} = -3Hn_{\text{dm}}^{(0)}$$

or $n_{\text{dm}}^{(0)} \propto a^{-3}$ To first order,

$$\boxed{\frac{\partial \delta}{\partial t} + \frac{1}{a} \frac{\partial v^i}{\partial x^i} + 3 \frac{\partial \Phi}{\partial t} = 0} \quad (9)$$

So far we have one equation for two unknown, the density perturbation δ and velocity field \vec{v} , which is basically the continuity equation in a curved spacetime. (Note that $\partial \Phi / \partial t$ is like a local correction to the Hubble parameter.) We still need an Euler equation for cdm. The strategy is to multiply the Boltzmann equation (8) by velocity P^i/P^0 and integrate over momentum, to get an equation for the first moment of the distribution

$$0 = \frac{\partial}{\partial t} \int \frac{d^3p}{(2\pi)^3} f_{\text{dm}} \frac{p \hat{p}^i}{E} + \frac{1}{a} \frac{\partial}{\partial x^i} \int \frac{d^3p}{(2\pi)^3} f_{\text{dm}} \frac{p^2 \hat{p}^i \hat{p}^j}{E^2} - \left[H + \frac{\partial \Phi}{\partial t} \right] \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{\text{dm}}}{\partial E} \frac{p^3 \hat{p}^i}{E^2} - \frac{1}{a} \frac{\partial \Psi}{\partial x^i} \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^i \hat{p}^j p^2}{E}$$

The first term is the time derivative of $n_{\text{dm}}v^i$. The second is second order in the velocity and can be dropped. The other terms require more care.

The term

$$\int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{\text{dm}}}{\partial E} \frac{p^2 \hat{p}^j}{E} = \int \frac{d\Omega}{(2\pi)^3} p^j \int_0^\infty dp \frac{p^4}{E} \frac{\partial f_{\text{dm}}}{\partial p} = - \int \frac{d\Omega p^j}{(2\pi)^3} \int_0^\infty dp f_{\text{dm}} \left(\frac{4p^3}{E} - \frac{p^5}{E^3} \right)$$

The last term is negligible. The first term is simply equal to $-4n_{\text{dm}}v^j$. The same strategy applied to the last term, using

$$\int d\Omega \hat{p}^i \hat{p}^j = \delta^{ij} \frac{4\pi}{3}$$

gives altogether

$$\frac{\partial(n_{\text{dm}}v^i)}{\partial t} + 4Hn_{\text{dm}}v^i + \frac{n_{\text{dm}}}{a} \frac{\partial \Psi}{\partial x^i} = 0$$

We can replace n_{dm} by $n_{\text{dm}}^{(0)}$ everywhere since this equation has no zeroth order counterpart,

$$\boxed{\frac{\partial v^i}{\partial t} + H v^i + \frac{1}{a} \frac{\partial \Psi}{\partial x^i} = 0} \quad (10)$$

The hierarchy of equations stops here because we neglected terms which were second order in the velocity. Hence Cold Dark Matter can be described as a fluid.

Using conformal time and switching to momentum space the continuity and Euler equation for cold dark matter take the following form

$$\dot{\delta} + ikv + 3\dot{\Phi} = 0$$

and

$$\dot{v} + \frac{\dot{a}}{a} v + ik\Psi = 0$$

where overdot refer to derivative wrt to conformal time.

Boltzmann equation for baryons

NB: by baryons cosmologists mean protons and leptons.

Electrons and protons are coupled by Coulomb scattering, with a rate of interaction which is larger than the expansion rate. This enforces local neutrality

$$\frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} = \delta_b$$

Similarly

$$c v_e = \vec{v}_p = \vec{v}_b$$

We start from the Boltzmann equations for protons and electrons

$$\frac{df_e(\vec{x}, \vec{q}, t)}{dt} = \langle c_{ep} \rangle_{QQ'q'} + \langle c_{e\gamma} \rangle_{pp'q'}$$

and

$$\frac{df_p(\vec{x}, \vec{Q}, t)}{dt} = \langle c_{ep} \rangle_{qq'Q'}$$

Conventions are p, p' are the initial and final photon momenta, q, q' the electron ones and Q, Q' for the protons.

Consider the Compton collision term.

$$\begin{aligned} \langle c_{e\gamma} \rangle_{pp'q'} &= \int \prod_i \frac{d^3 p^i}{(2\pi)^3 E_{p^i}} (2\pi)^4 \delta^4(p + q - p' - q') \\ &\quad |\mathcal{M}|^2 [f_e(q') f_\gamma(p') - f_e(q) f_\gamma(p)] \end{aligned}$$

The Coulomb term is analogous. There is no Compton term for the protons since it is suppressed by a factor of m_e^2/m_p^2 wrt to the electron one. Also, to simplify, we don't include capture and ionization term.

Then we proceed as for CDM and integrate over electron phase space to get an equation for the first moment. The LHS is like that of CDM,

$$\frac{\partial n_e}{\partial t} + \frac{1}{a} \frac{\partial(n_e v_b^i)}{\partial x^i} + 3 \left[H + \frac{\partial \Phi}{\partial t} \right] n_e = \langle c_{ep} \rangle_{QQ'qq'} + \langle c_{e\gamma} \rangle_{pp'qq'}$$

Both term on the RHS vanish. Technically this is because the integrand is asymmetric. Physically this is because what goes out must comes in: *ie* electron number is conserved in Coulomb and Compton.

Obviously we obtain the same equation if we do the integral for protons. Altogether then, using conformal time and going to momentum space

$$\boxed{\dot{\delta}_b + ikv_b + 3\dot{\Phi} = 0} \quad (11)$$

To get a second equation, we consider the first moment of both proton and electron equation. Unlike for CDM, we multiply by the momentum q or Q , not the velocity. The LHS is than the same as for CDM taking into account a factor of m_e or m_p . Adding the two equations gives

$$m_p \frac{\partial(n_b v_b^j)}{\partial t} + 4Hm_p n_b v_b^j + m_p \frac{n_b}{a} \frac{\partial \Psi}{\partial x^j} = \langle c_{ep}(q^j + Q^j) \rangle_{QQ'qq'} + \langle c_{e\gamma} q^j \rangle_{pp'qq'}$$

where the LHS is dominated by the protons, since $m_p \gg m_e$.

We can use conservation of momentum to set the first term on the RHS to zero (exercise).

The final step is to evaluate the average momentum \vec{q} in Compton scattering.

Momentum conservation gives

$$\langle c_{e\gamma} \vec{q} \rangle_{pp'qq'} = -\langle c_{e\gamma} \vec{p} \rangle_{pp'qq'}$$

II. Switching to momentum space, multiply by \hat{k} with $\hat{k} \cdot \vec{p} = \mu$. We have already computed $\langle c_{e\gamma} \rangle_{pp'qq'}$. Now

$$\begin{aligned} -\langle c_{e\gamma} p \mu \rangle_{pp'qq'} &= n_e \sigma_T \int \frac{d^3 p}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p} \mu [\Theta_0 - \Theta(\mu) + v_b \mu] \\ &= n_e \sigma_T \int_0^\infty \frac{dp}{2\pi^2} p^4 \frac{\partial f^{(0)}}{\partial p} \int_{-1}^1 \frac{d\mu}{2} \mu [\Theta_0 - \Theta(\mu) + v_b \mu] \end{aligned}$$

Introducing the first moment (dipole) of the photon distribution

$$\Theta_1 = i \int_{-1}^1 \frac{d\mu}{2} \mu \Theta(\mu)$$

and integrating over p finally gives, using conformal time and Fourier space

$$\boxed{\dot{v}_b + \frac{\dot{a}}{a} v_b + ik\Psi = \dot{\tau} \frac{4\rho_\gamma}{3\rho_b} [3i\Theta_1 + v_b]} \quad (12)$$

were we used the equation for $n_b^{(0)}$.

The factor of $\rho_b = m_b n_b$ arise because it is difficult for electrons to move since they are tightly bound to protons.

III. Einstein equations

Leaving aside the question of gauge freedom (see end of section), this chapter is completely straightforward. I give details only when strictly necessary (relative concept).

Scalar perturbations

Perturbed Ricci tensor and scalar

The Einstein equations have 10 independent components ($D = 4$ symmetric two indices tensor equation). We will only be concerned with linear perturbations to a FRW universe. These can be separated into tensor, vector and scalar perturbations. Fortunately, to leading order, these perturbations don't mix. We can thus separate the problem into studying either one of these different sort of perturbations. Furthermore we will mostly consider inflation as the source of perturbations and vector perturbations don't arise in this context.

We begin with scalar perturbations. In many respects, tensor perturbations (gravity waves) are easier. Those will be treated separately. Scalar perturbations can be parameterized by two gravitational potentials Φ and Ψ with

$$ds^2 = -(1 + 2\Psi(\vec{x}, t))dt^2 + a^2(1 + 2\Phi(\vec{x}, t))\vec{d}x^2 \quad (13)$$

where

$$\vec{d}x^2 = \delta_{ij}dx^i dx^j$$

The choice of space-time coordinates such that the metric of scalar perturbation to FRW takes the form of Eq.(13) is called the Newtonian conformal gauge. Given a system of coordinates, it is trivial to compute the Christoffel symbols, the Riemann tensor, the Ricci tensor and scalar and finally the Einstein tensor.

Christoffel Symbols

From

$$\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2}g^{\alpha\beta} [g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}]$$

we get

$$\Gamma_{00}^0 = \Psi_{,0} \quad (14)$$

$$\Gamma_{0i}^0 = \Psi_{,i} \equiv ik_i\Psi \quad (15)$$

$$\Gamma_{ij}^0 = \delta_{ij}a^2 [H + 2H(\Phi - \Psi) + \Phi_{,0}] \quad (16)$$

$$\Gamma_{00}^i = \frac{ik^i}{a^2}\Psi \quad (17)$$

$$\Gamma_{j0}^i = \delta_{ij}(H + \Phi_{,0}) \quad (18)$$

$$\Gamma_{jk}^i = i\Phi[\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i] \quad (19)$$

where

$$H = \frac{a_{,0}}{a}$$

As before, we hope that the distinction between configuration space and momentum space is clear from the context.

Ricci tensor

From

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^{\alpha} - \Gamma_{\mu\alpha,\nu}^{\alpha} + \Gamma_{\beta\alpha}^{\alpha}\Gamma_{\mu\nu}^{\beta} - \Gamma_{\beta\nu}^{\alpha}\Gamma_{\mu\alpha}^{\beta}$$

we get

$$R_{00} = -3\frac{1}{a}a_{,00} - \frac{k^2}{a^2}\Psi - 3\Phi_{00} + 3H(\Psi_{,0} - 2\Phi_{,0})$$

and

$$R_{ij} = \delta_{ij} \left[(2a^2H^2 + a^2a_{,00}) (1 + 2\Phi - 2\Psi) + a^2H(6\Phi_{,0} - \Psi_{,0}) + a^2\Phi_{,00} + k^2\Phi \right] + k_ik_j(\Phi + \Psi)$$

Ricci scalar Let

$$R = \bar{R} + \delta R = g^{\mu\nu} R_{\mu\nu}$$

Then

$$\bar{R} = 6H^2 + 6\frac{a_{,00}}{a}$$

is the unperturbed Ricci scalar while

$$\delta R = -12\Psi\bar{R} + \frac{2k^2}{a^2}\Psi + 6\Phi_{,00} - 6H(\Psi_{,0} - 4\Phi_{,0}) + 4\frac{k^2\Phi}{a^2}$$

Einstein Equations

The Einstein equations give the evolution equations for Ψ and Φ ,

$$G^\mu{}_\nu = 8\pi G_N T^\mu{}_\nu$$

where G_N is Newton's constant.

Consider first $G^0{}_0 = \bar{G}^0{}_0 + \delta G^0{}_0$. To leading order

$$\bar{G}^0{}_0 = -3H^2$$

while to first order,

$$\delta G^0{}_0 = -6H\Phi_{,0} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2}$$

To complete the equation, we need an expression for the stress-energy tensor to first order. The contribution of each particle species to the energy density T^0_0 is given by an integral over the distribution function

$$T^0_0 = - \sum_{\text{species}} g_i \int \frac{d^3p}{(2\pi)^3} E_i(p) f_i(\vec{p}, \vec{x}, t)$$

For non-relativistic species, this is simply

$$T^0_0|_i = -\rho_i(1 + \delta_i)$$

For photons

$$T^0_0 = -2 \int \frac{d^3p}{(2\pi)^3} p \left(f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta \right) = -\rho_\gamma(1 + 4\Theta_0)$$

after integrating by part and using the approximation that the temperature perturbation depends only on the direction, not the magnitude of the momentum. The factor of 4 comes of course from the fact that $\rho \propto T^4$. The subscript refers to the fact that only the monopole component of the temperature fluctuation survives the integration of momentum.

Similarly for neutrinos

$$T^0_0 = \rho_\nu(1 + 4\mathcal{N}_0)$$

Altogether

$$\boxed{\frac{k^2}{a^2} \left[\Phi - 3 \frac{H^2 a^2}{k^2} \Psi + 3 \frac{H a^2}{k^2} \Phi_{,0} \right] = 4\pi G_N [\rho_{\text{dm}} \delta + \rho_b \delta_b + 4\rho_\gamma \Theta_0 + 4\rho_\nu \mathcal{N}_0]} \quad (20)$$

Note that this equation reduces to the Poisson equation for the Newtonian potential $\Phi = -\Psi$ on scales within the horizon $k^2 \gg a^2 H^2$, where k^2 is the comoving momentum squared.

We need another equation for Φ and Ψ . This is *a priori* given by the space-space component of the Einstein equation G^i_j . However, even to linear order, there is a plethora of terms in this equation. A simple way out is to project it out and consider only the longitudinal, traceless part of G^i_j ,

$$(\hat{k}_i \hat{k}^j - 1/3 \delta_i^j) G^i_j = \frac{2}{3a^2} k^2 (\Phi + \Psi)$$

Also

$$(\hat{k}_i \hat{k}^j - 1/3 \delta_i^j) T^i_j = \sum_{\text{species}} g_i \int \frac{d^3p}{(2\pi)^3} \frac{\mu^2 p^2 - 1/3 p^2}{E_i(p)} f_i(p)$$

Since

$$\mathcal{P}_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$$

the projection picks the quadrupole part of the perturbed distribution function. The integral for photons gives

$$-2 \int \frac{dp p^2}{2\pi^2} p^2 \frac{\partial f^{(0)}}{\partial p} \int_{-1}^1 \frac{d\mu}{2} \frac{2\mathcal{P}_2(\mu)}{3} \Theta(\mu) = -\frac{\rho_\gamma \Theta_2}{3}$$

This component is called the anisotropic stress. Non-relativistic species (baryons, DM) do not contribute.

Altogether, we get our second equation for the gravitational potentials

$$\boxed{k^2(\Phi + \Psi) = -32\pi G_N a^2 [\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2]} \quad (21)$$

The gravitational potentials are equal and opposite unless there is appreciable quadrupole moment. In practice the photon quadrupole is negligible. Only neutrinos contribute during the radiation dominated era.

Tensor perturbations

From previous section: scalar perturbations are sourced by density perturbations.

Many theories of structure formation predict the existence of tensor perturbations as well.

Metric ansatz. We work in the so-called TT-gauge (transverse, traceless gauge; See MTW). There is an implicit choice of axes (propagation along z – axis).

$$g_{00} = -1 \quad ; \quad g_{ij} = a^2(\delta_{ij} + \mathcal{H}_{ij}) = a^2 \begin{pmatrix} 1 + h_+ & h_\times & 0 \\ h_\times & 1 - h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$k^i \mathcal{H}_{ij} = k^j \mathcal{H}_{ij} = 0 \quad ; \quad \mathcal{H}^i{}_i = 0$$

Christoffel symbols

$$\begin{aligned} \Gamma_{00}^0 &= \Gamma_{i0}^0 = 0 \\ \Gamma_{ij}^0 &= -\frac{g^{00}}{2} g_{ij,0} = \frac{1}{2} g_{ij,0} \\ \Gamma_{ij}^0 &= H g_{ij} + \frac{a^2 \mathcal{H}_{ij,0}}{2} \\ \Gamma_{0j}^i &= H \delta_{ij} + \frac{1}{2} \mathcal{H}_{ij,0} \\ \Gamma_{ij}^k &= \frac{i}{2} [k_k \mathcal{H}_{ij} + k_j \mathcal{H}_{ik} - k_i \mathcal{H}_{jk}] \end{aligned}$$

Ricci tensor

$$R_{00} = -3\frac{\dot{a}_{,00}}{a}$$

as in unperturbed FRW spacetime.

$$R_{ij} = g_{ij} \left(\frac{\dot{a}_{,00}}{a} + 2H^2 \right) + \frac{3}{2}a^2 H \mathcal{H}_{ij,0} + a^2 \frac{\mathcal{H}_{ij,00}}{2} + \frac{k^2}{2} \mathcal{H}_{ij}$$

You can check that

$$\delta R = 0$$

to first order because \mathcal{H} is traceless.

Einstein equations for tensor perturbations

These reduce to

$$\delta G^i_j = \delta R^i_j = \delta^{ik} \left[\frac{3}{2} H \mathcal{H}_{kj,0} + \frac{\mathcal{H}_{kj,00}}{2} + \frac{k^2}{2a^2} \mathcal{H}_{kj} \right]$$

To derive an equation for tensor perturbations, we consider

$$\delta G^1_1 - \delta G^2_2 = 3H \dot{h}_{+,0} + \ddot{h}_{+,00} + \frac{k^2 h_+}{a^2}$$

The source term for this combination vanishes if the temperature perturbation only depend on $\mu = \hat{p} \cdot \hat{z}$, (ie not on ϕ),

$$\delta T^1_1 - \delta T^2_2 \propto \int \frac{d^3 p}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p} \Theta(\mu) (\hat{p}_1^2 - \hat{p}_2^2) \equiv 0$$

A similar conclusion of course holds for h_\times . Using conformal time, the equations of motion take the form

$$\boxed{\ddot{h}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + k^2 h_\alpha = 0} \quad (22)$$

where $\alpha = +, \times$.

This is the equation of motion of a damped harmonic oscillator. For a purely matter or radiation dominated expansion, explicit solutions are given in term of spherical Bessel functions.

For a radiation dominated universe,

$$a = \frac{\eta}{\eta_0} \quad \text{and} \quad \frac{\dot{a}}{a} = \frac{1}{\eta}$$

The solution is a Bessel function of the first kind

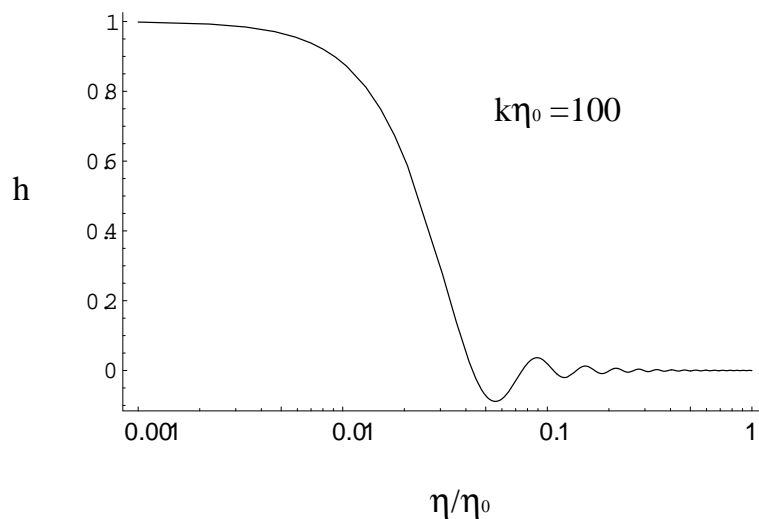
$$h(k, \eta) = C \frac{J_{1/2}(k\eta)}{k\eta}$$

These solutions are constant for $k\eta \lesssim 1$, until the mode enters the horizon and start oscillating, but with a decaying amplitude. See figure. Higher k modes enter the horizon earlier.

For a matter dominated universe, the solution is

$$h(k, \eta) = C(k\eta)^{-3/2} J_{3/2}(k\eta)$$

There is no analytical solution for a universe which goes from radiation dominated to matter dominated. However the generic picture is the same. The figure below shows the evolution of a tensor perturbation in a MD universe.



The issue of gauge freedom

L & L; E & B; D

Consider scalar perturbations. We have chosen to parametrize them as

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)d^2\vec{x}$$

However, the metric would look different in another coordinate system, through which

$$x^\mu \rightarrow \tilde{x}^\mu$$

and

$$g_{\mu\nu}dx^\mu d^\nu = \tilde{g}_{\alpha\beta}d\tilde{x}^\alpha d\tilde{x}^\beta$$

The choice of a specific system of coordinates on spacetime is referred to as a choice of gauge. This issue is rather subtle and will only be sketched here. For further details on the meaning of gauge choices, see Ellis (George, not John) & Bruni (Carlo, not Carla).

The Newtonian view of spacetime is adequate on scales inside the horizon. We want to follow perturbations on scales beyond the horizon. There is no problem for tensor perturbations (gravity waves) but will somewhat complicate the discussion of scalar perturbations.

Let us use (t, \vec{x}) to denote the coordinates of spacetime. A choice of coordinates defines a threading of spacetime into timelike lines of constant \vec{x} and a slicing into spacelike hypersurfaces of constant coordinate time t .

For perturbations to be meaningful, we have to define what is unperturbed spacetime. A natural choice is a spatially homogeneous and isotropic expanding FLRW spacetime. In this case, there are preferred coordinates. Threading corresponds to the worldlines of observers who see zero momentum density at their position. To them, expansion appears **isotropic** and their worldlines are **geodesics**. Slicing is orthogonal to the threading, and on each slice, space is **homogeneous**.

In presence of perturbations, no coordinates exist in which all these properties are verified. The minimal requirement is that they reduce to those of RWFL spacetime when perturbations are absent. There are many reasonable choices which satisfy these properties. Each choice **defines** the perturbations and perturbations can look very different between two coordinate choices, very much like the gauge potential does in electromagnetism. Hence the nomenclature. The gauge adopted here is called the Conformal Newtonian Gauge, although for the sake of illustration we also consider so-called comoving gauges.

It is important, both as a matter of principle and for practical application to understand the effect of gauge transformations on perturbations. One remark before doing so. Since we only consider linear perturbations (thanks God), once defined, perturbations leave in unperturbed spacetime, in the sense that a perturbation

$$g(t, \vec{x})$$

depends on the coordinates (t, \vec{x}) of unperturbed (FLRW) spacetime.

The meaning of changing gauge

Consider a given slicing. We can then define for instance the energy density of slices,

$$\rho(t, \vec{x}) = \rho(t) + \delta\rho(t, \vec{x})$$

where t and \vec{x} in $\delta\rho$ refer to the coordinates of unperturbed spacetime to first order.

Now change the slicing. This means introducing a new time coordinate

$$\tilde{t}(t, \vec{x}) = t + \delta t(t, \vec{x})$$

On the two slicings, perturbations are different

$$\begin{aligned}\rho(t, \vec{x}) &= \rho(t) + \delta\rho(t, \vec{x}) \\ \tilde{\rho}(\tilde{t}, \vec{x}) &= \tilde{\rho}(\tilde{t}) + \delta\tilde{\rho}(\tilde{t}, \vec{x})\end{aligned}$$

where, according to our definition of perturbations

$$\rho(t) = \tilde{\rho}(\tilde{t}) \quad \text{if} \quad t = \tilde{t}$$

To first order

$$\tilde{t}(\vec{x}, t - \delta t) = \tilde{t} - \frac{\partial \tilde{t}}{\partial t} \delta t = \tilde{t} - \delta t = t$$

ie to first order, δt is the time displacement in going from a slice of fixed \tilde{t} to a slice of fixed t such that $\tilde{t} = t$. (See figure.)

If we compare the densities ρ and $\tilde{\rho}$ (at the same spacetime point) we get

$$\begin{aligned}\rho(t) + \delta\rho(t, \vec{x}) &= \tilde{\rho}(\tilde{t}) + \tilde{\rho}(t, \vec{x}) \\ &= \tilde{\rho}(t) + \frac{d\tilde{\rho}}{dt} \delta t + \tilde{\rho}(t, \vec{x}) \\ \rightarrow \quad \delta\tilde{\rho}(t, \vec{x}) &= \delta\rho(t, \vec{x}) - \frac{d\rho}{dt} \delta t(t, \vec{x})\end{aligned}$$

where we used the equality of unperturbed densities when their arguments have the same numerical value.

We have not specified the motion of the observers who are supposed to measure $\rho(t, \vec{x})$. In the unperturbed case, these are comoving observers. In presence of perturbations, ρ can be measured by any observer, regardless of the threading (corrections are $\mathcal{O}(v^2)$ as we shall see shortly). This is important and general so we emphasize: scalar perturbations are independent of threading.

A possible choice of slicing is the **comoving slicing**, orthogonal to the worldline of comoving observers. Those move with the cosmic fluid and see no momentum density. In particular, given a perturbation in a scalar quantity, say the density ρ

$$\delta\rho(t, \vec{x}) = 0$$

This is related to the scalar perturbation on a different slice by

$$\delta\rho = -\frac{d\rho}{dt}\delta t$$

Note that making ρ spatially homogeneous doesn't mean that we could reduce the expansion to that of a FLRW universe. This is only so if there is no pressure perturbation. The Euler equation in the comoving gauge takes the form

$$\alpha_i(t, \vec{x}) = -\frac{1}{\rho + P} \frac{\partial \delta P}{\partial x^i}$$

where α_i is the acceleration of the fluid as measured by a freely falling (inertial) observer who is instantaneously at rest wrt to the fluid (ie $v = 0$ at the position of the observer). The presence of acceleration implies that comoving worldlines are not geodesics. In particular the coordinate time t separating slices cannot be identified with proper time as in the case of a FLRW universe. The two are related by

$$\vec{\nabla} \left(\frac{dt}{d\tau} \right) = -\vec{a} \left(\frac{dt}{d\tau} \right)$$

where τ is the proper time of the observer. (Hence the peculiar sign. You can derive this relation starting from the standard time dilation formula

$$\delta\tau = \sqrt{1 - v^2} dt \quad)$$

Using the Euler equation to first order gives

$$\frac{dt}{d\tau} = 1 + \frac{\delta P}{\rho + P}$$

Density is the same everywhere on a comoving slice (gauge) but, to a comoving observer instantaneously at rest with the fluid at some spacetime point, time flows differently in adjacent threads.

From gauge to gauge

Consider the most general first-order scalar perturbation to a FRW metric

$$ds^2 = -(1 + 2A)d^2t - aB_i dt dx^i + ((1 + 2\psi)\delta_{ij} - 2E_{,ij}) dx^i dx^j$$

The term B_i is called the shift function. It specifies the relative velocity between the threading and the worldlines orthogonal to slicing. The A term is the lapse function, which specifies the relation between proper time t_{pr} and coordinate time t , $dt_{pr} = (1 + A)dt$.

Consider then a gauge transformation

$$\tilde{x}^\alpha = x^\alpha + \delta x^\alpha(x)$$

with

$$\delta x^0 = \xi^0$$

and

$$\delta x^i = \xi_i = ik_i \xi$$

Using the transformation law of the metric

$$\begin{aligned} \tilde{g}_{\mu\nu}(x) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x - \delta x) \\ &\approx g_{\mu\nu}(x) - g_{\alpha\nu} \partial_\mu \delta x^\alpha - g_{\mu\alpha} \partial_\nu \delta x^\alpha - \delta x^\lambda \partial_\lambda g_{\mu\nu} \end{aligned}$$

Remark: we could have considered a more general form of the metric, including tensor and vector perturbations (in the sense that coordinate transformation derive from a curl.) The former are gauge invariant. The latter are not generated during inflation. For these reasons, and to avoid cluttering of symbols, we only consider scalar perturbations, which, a priori, are parameterized by four functions. An important subremark is that scalar, tensor and vector perturbations do not mix under coordinate transformation (to first order). This is one facet of the decomposition theorem.

Under coordinate transformations, you can check that, to first order,

$$\begin{aligned}\tilde{A}(x) &= A(x) - \frac{1}{a}\dot{\xi}^0 \\ \tilde{B} &= B - \frac{1}{a}\xi^0 + \dot{\xi} \\ \tilde{\psi} &= \psi + \frac{k}{3}\xi - H\xi^0 \\ \tilde{E} &= E + \xi\end{aligned}$$

There are four functions, but we have to gauge transformations at our disposal. So we can get rid of two functions, for instance E and B as in the Newtonian conformal gauge used in these notes, where we define $A = \Psi$ and $\Phi = \psi$ (I know, this is confusing).

More generally, Bardeen has introduced two gauge invariant objects:

$$\boxed{\Phi_H = -\psi + aH(B - \dot{E})} \quad (23)$$

where the overdot refer to derivation wrt to conformal time, and

$$\Phi_A = A + \frac{1}{a} \frac{\partial}{\partial \eta} [a(\dot{E} - B)] \quad (24)$$

Similarly, the components of the stress-energy tensor change under coordinate transformations.

Bardeen constructed

$$\boxed{v = ikB + \frac{\hat{k}^i T^0_i}{(\rho + p)a}} \quad (25)$$

which remains invariant under gauge transformations. In conformal Newtonian gauge, for matter, v is just the velocity of matter defined in the chapter on the Boltzmann equations. For radiation

$$v = -3i\Theta_{r,1}$$

Another invariant is related to the energy density

$$\epsilon_m = -1 - \frac{T^0_0}{\rho} + \frac{3H}{k^2\rho} k^i T^0_i$$

On small scales, for radiation $\epsilon_m = 4\Theta_{r,0}$ and for matter $\epsilon_m = \delta$. They reduce to their newtonian counterparts, as they should.

We will make use of gauge invariant variables in the next chapter, where we discuss generation of perturbations during inflation.

Inflation and the Initial Conditions

Where we learn how inflation, besides solving some otherwise puzzling cosmological problems, could have created the seeds for formation of large scale structures.

Initial Conditions

That is

$$k\eta \ll 1$$

Consider the Boltzmann equation for Θ . The first term is $\dot{\Theta}$. The second is $ik\mu\Theta$. The ration of these two terms is

$$\frac{\dot{\Theta}}{ik\mu\Theta} \sim \frac{1}{k\eta} \gg 1$$

Hence we can neglect the second term wrt the first one. More generally, we can neglect all terms $\propto k$.

The equation for the monopole then reduces to

$$\dot{\Theta}_0 + \dot{\Phi} = 0$$

and

$$\dot{\mathcal{N}}_0 + \dot{\Phi} = 0$$

while the Euler equations for the velocity of baryons and dark matter reduce to

$$\dot{\delta} = -3\dot{\phi}$$

and

$$\dot{\delta}_b = -3\dot{\phi}$$

The assumption of strong coupling implies

$$v = -3i\Theta_1$$

Now let us look at the Einstein equations.

The $0 - 0$ component of Einstein reduces to

$$3H(\dot{\Phi} - H\Psi) = 16\pi G a^2 (\rho_\gamma \Theta_0 + \rho_\nu \mathcal{N}_0)$$

neglecting matter density (radiation era).

Since $a \propto \eta$, $H = 1/\eta$. Multiplying by η^2 and defining

$$f_\nu = \frac{\rho_\nu}{\rho_\gamma + \rho_\nu}$$

gives

$$\dot{\Phi}\eta - \Psi = 2((1 - f_\nu)\Theta_0 + f_\nu \mathcal{N}_0)$$

Deriving wrt to conformal time and using $\dot{\Theta}_0 = \dot{\mathcal{N}}_0 = -\dot{\phi}$ finally gives

$$\ddot{\Phi}\eta + \dot{\Phi} - \dot{\Psi} = -2\dot{\Phi}$$

The space-space component of the Einstein equation implies that

$$\Phi \approx -\Psi$$

Altogether, these equations imply that the gravitational potential obeys

$$\ddot{\Phi}\eta + 4\dot{\Phi} = 0$$

which has two solutions

$$\Phi = \text{const} \quad \text{and} \quad \Phi \propto \eta^{-3}$$

The decaying mode is not of much interest. The constant mode, if excited, may be responsible for the formation of structure in the universe.

The first Einstein equation implies that

$$\Phi = 2((1 - f_\nu)\Theta_0 + f_\nu\mathcal{N}_0)$$

The photon and neutrino overdensities are constant.

Most models of structure formation don't differentiate between photons and neutrinos, hence

$$\Theta_0 \equiv \mathcal{N}_0$$

We will assume this to be the case from now on.

This means that

$$\boxed{\Phi(k, \eta_i) = 2\Theta_0(k, \eta_i)} \quad (26)$$

The Boltzmann equations for matter imply that

$$\delta = 3\Theta_0 + \text{const} \quad \text{and} \quad \delta_b = 3\Theta_0 + \text{const}'$$

Primordial perturbations such that the constants of integration are vanishing are called *adiabatic perturbations*.^{*}This terminology stems from the fact that matter traces radiation

$$\delta = \delta_b = 3\frac{\delta T}{T} \equiv \frac{3}{4}\delta_\gamma$$

^{*}Taking adiabatic perturbations, the constant solution for ϕ gives immediately $\Theta = 1/2\Phi$.

The initial condition (26) is an important relation so it is worth understanding its meaning at a qualitative level.

In a FRW universe,

$$\rho_\gamma \propto a^{-4}$$

Hence

$$\frac{\delta\rho_\gamma}{\rho_\gamma} = -4\frac{\delta a}{a}$$

On the other hand

$$a \propto t^{1/2}$$

which implies

$$\frac{\delta\rho_\gamma}{\rho_\gamma} = 4\Theta_0 = -2\frac{\delta t}{t}$$

where δt is the time delay in the expansion due to overdensity in radiation. ($\delta < 0$ means earlier, hence denser universe.) The latter can be related to the Newtonian potential through the standard gravitational redshift

$$\frac{\delta t}{t} = \Psi$$

Finally, use $\Psi = -\Phi$ to get

$$\Phi = 2\Theta_0$$

Exercise: what is the initial condition in the matter dominated era? That is, repeat the previous steps. Verify that Φ is still constant. Verify that

$$\Phi = \frac{3}{2}\Theta_0$$

Derive this result using the heuristic argument.

Velocity initial conditions, although small, will also be of interest.

First we use the vanishing of the collision term in Eq.(82):

$$v_b = -3i\Theta_1$$

The Euler equation (82) then reduces to

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi$$

Since

$$\dot{v}_b = -3i\dot{\Theta}_1$$

we need the equation for the dipole.

In the regime of interest,

$$\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = \frac{k}{3}\Psi$$

which implies

$$\dot{v}_b = -ik(\Theta_0 + \Psi)$$

and thus

$$\frac{\dot{a}}{a}v_b = ik\Theta_0$$

Using the initial condition on Θ_0 , we finally get that

$$\boxed{\Theta_1 = \frac{iv}{3} = \frac{iv_b}{3} = -\frac{k}{3H\alpha}\Theta_0 \equiv -\frac{k}{6H\alpha}\Phi} \quad (27)$$

Tensor Perturbations

Tensor perturbations = gravity waves \rightarrow not coupled to density perturbations.

Induce fluctuations in the CMB: almost unique signature of inflation.

Technically simpler than scalar perturbations (do not couple to other perturbations).

We work in the so-called TT-gauge (transverse, traceless gauge; See MTW). There is an implicit choice of axes (propagation along z – axis).

$$g_{00} = -1 \quad ; \quad g_{ij} = a^2(\delta_{ij} + \mathcal{H}_{ij}) = a^2 \begin{pmatrix} 1 + h_+ & h_\times & 0 \\ h_\times & 1 - h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$k^i \mathcal{H}_{ij} = k^j \mathcal{H}_{ij} = 0 \quad ; \quad \mathcal{H}^i_j = 0$$

Equations of motion:

$$\ddot{h}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + k^2 h_\alpha = 0$$

with $\alpha = +, \times$. Overdots refer to derivation wrt to conformal time.

When considering quantum fluctuations, we would like tensor perturbations to be canonically normalized. The metric is dimensionless and so is the perturbation h . To define a gravity perturbation with mass dimension we define

$$\tilde{h} = \frac{h}{\sqrt{16\pi G_N}}$$

That this is the correct factor can be seen from the Hilbert action.

We don't want to enter in the details of quantizing a field in a curved space-time. Our approach is thus heuristic. We try to bring the equation in the form of that of an harmonic oscillator and then mimic quantization in flat spacetime. Furthermore, we will ask that modes reduce to that of flat spacetime at early times. This amount to a choice of vacuum (Bunch-Davies vacuum). Needless to say, there is an extensive litterature on these questions.

From the definition of \tilde{h} and the equation of motion for h we get

$$\ddot{\tilde{h}} + \left(k^2 - \frac{\ddot{a}}{a} \right) \tilde{h} = 0$$

This is, as promised, the equation of an harmonic oscillator with a (conformal) time dependent mass $-\ddot{a}/a$. This form makes it clear that something special is going to happen if the expansion is accelerated. During normal expansion

$$\ddot{a} < 0$$

and the effective mass is positive. During inflation however, the effective mass is negative and the harmonic oscillator is then top-down.

To estimate the effect of expansion, we approximate the inflationary stage by apure de Sitter expansion (slow-roll phase)

$$H \approx \text{const}$$

Then

$$d\eta = \frac{dt}{a} = e^{-Ht} dt$$

leads to

$$\eta = -\frac{1}{aH}$$

and

$$\frac{\ddot{a}}{a} = -\frac{2}{\eta^2}$$

To quantize tensor perturbations, we write

$$\tilde{h}(\vec{k}, \eta) = v(\vec{k}, \eta) a(\vec{k}) + v(\vec{k}, \eta)^* a(\vec{k})^\dagger$$

where the function $v(\vec{k}, \eta)$ also satisfies

$$\ddot{v} + \left(k^2 - \frac{2}{\eta^2} \right) v = 0$$

and interpret a and a^\dagger as creation/destruction operators. That is, we impose canonical commutation relations

$$[a(\vec{k}), a(\vec{k}')^\dagger] = \delta^3(\vec{k} - \vec{k}')$$

As before, the mode equation can be solve in term of Bessel functions. The solution is simple and neat

$$\boxed{v = \frac{e^{-ik\eta}}{\sqrt{2k}} \left[1 - \frac{i}{k\eta} \right]} \quad (28)$$

For modes which, at a given time are well within the horizon, the solution reduces to

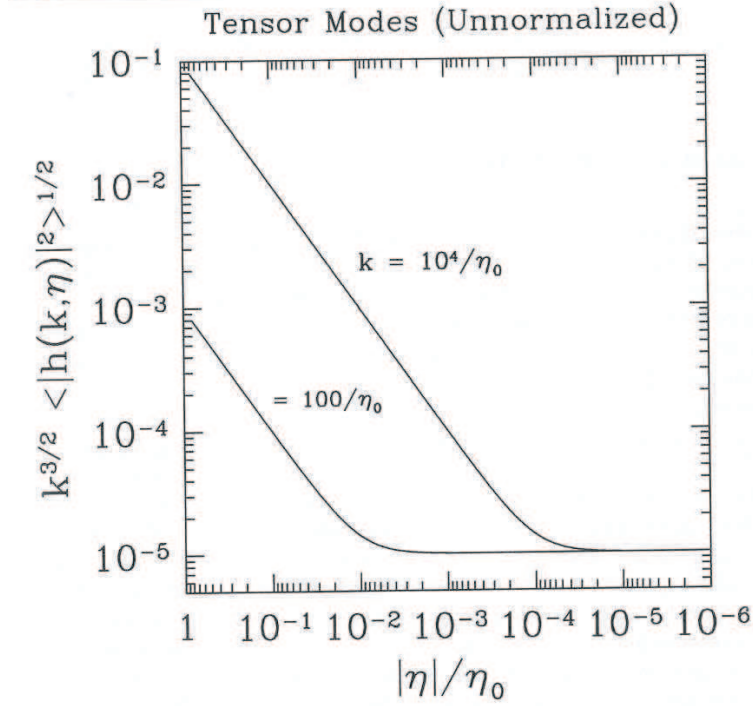
$$v(k) \sim \frac{1}{\sqrt{2k}} e^{-ik\eta}$$

which is indeed that of a free, massless plane wave, *ie* a vacuum solution of Minkowski space. After the mode is way beyond the horizon however, $k\eta \ll 1$, the solution behaves like

$$v(k) \approx \frac{1}{\sqrt{2k}} \frac{1}{2ik\eta}$$

ie it is essentially time-independent.

Moreover, $h \propto \tilde{h}/a$ and $a \propto -1/\eta$. The conclusion is that the mode amplitude decreases with expansion as long as it is within the horizon. Beyond the horizon on the other hand, its amplitude gets frozen. (See figure.)



The variance of the field \tilde{h} is

$$\langle \tilde{h}^\dagger(\vec{k}, \eta) \tilde{h}(\vec{k}', \eta) \rangle = |v(\vec{k}, \eta)|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

Since $\tilde{h} = ah/\sqrt{16\pi G_N}$,

$$\begin{aligned} \langle \tilde{h}^\dagger(\vec{k}, \eta) \tilde{h}(\vec{k}', \eta) \rangle &= \frac{16\pi G_N}{a^2} |v(\vec{k}, \eta)|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \\ &= (2\pi)^3 P_h(k) \delta^3(\vec{k} - \vec{k}') \end{aligned}$$

where P_h is the **power spectrum** of the tensor perturbations. Notice that sometimes the power spectrum is defined with an extra factor of k^3 . This is because

$$\langle h^2(\vec{x}) \rangle = \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k} k^3 P_h(k)$$

so that $k^3 P_h(k)$ is the power spectrum per $d \ln(k)$. If $k^3 P_h(k)$ is independent of k , the power spectrum is said to be a scale invariant or Harrison-Zel'dovich-Peebles spectrum.

On scales beyond the horizon, the power spectrum during the slow-roll part of inflation universe is

$$\boxed{\frac{1}{2\pi^2}k^3P_h(k) = \frac{16\pi G_N}{a^2} \frac{k^3}{2\pi^2} \frac{1}{2k^3\eta^2} = 16\pi G_N \left(\frac{H}{2\pi}\right)^2} \quad (29)$$

This is a powerful prediction of inflation: the spectrum of tensor perturbations should be approximately scale invariant.

Another important feature of tensor perturbations is that a observations of gravity waves would lead to a direct measurement of the Hubble parameter during inflation. Since the energy density is thought to have been dominated by the potential energy of a scalar field, this would yield a direct measurement of the potential. Remember that we are talking of something that happened at a scale which is presumably, but not with certainty, at a scale close to the Planck scale.

Yet another important feature is that the spectrum tensor perturbations is Gaussian. For particles physicists, this just means that the the metric perturbation field h is almost free, at least if inflation took place below the Planck scale. Technically speaking, this just means that each mode evolves independently of the others. While this is most convenient from a practical point of view, since it is at the root of the linear perturbations assumption, it is only that, an assumption. Not surprisingly, this will be assumed to hold also when we consider the fluctuations of the inflaton field itself.

Scalar Perturbations

More complicated than tensor perturbations because the scalar potential is mixed with inflaton fluctuations.

Decompose the problem in steps:

1. First ignore it and compute perturbations in $\delta\phi$ in a spatially flat expanding background.
2. Then show that Ψ is small until a mode is beyond the horizon.
3. Last identify a linear combination of Ψ and $\delta\phi$ which is conserved across the horizon. Call it ζ .

This is the easiest way to see how perturbations in the inflaton $\delta\phi$ eventually get converted in perturbations in the potential Ψ .

A discussion of the same issue in other gauge illuminates the meaning of ζ .

Inflaton perturbations in a flat background

Decompose the inflaton into an homogeneous $\phi(t)$ and fluctuating $\delta\phi(\vec{x}, t)$ part:

$$\phi(\vec{x}, t) = \phi(t) + \delta\phi(\vec{x}, t)$$

The time component of the conservation law of the energy-momentum tensor

$$T^{\mu}_{0;\mu} = 0$$

gives the equation of motion for ϕ .

To leading order, this is the equation for the homogeneous field $\phi(t)$.

Focusing on the perturbation δT^{μ}_{ν} in a flat homogeneous background gives

$$\delta T^0_{0,0} + ik_i \delta T^i_0 + 3H\delta T^0_0 - H\delta T^i_i = 0$$

with

$$\begin{aligned}\delta T^i_0 &= \frac{ik_i}{a^3} \dot{\phi} \delta\phi \\ \delta T^0_0 &= -\frac{\dot{\phi} \delta\phi}{a^2} - V' \delta\phi \\ \delta T^i_j &= \delta^i_j \left(\frac{\dot{\phi} \delta\phi}{a^2} - V' \delta\phi \right)\end{aligned}$$

where overdots mean derivation w.r.t. conformal time.

Altogether inflaton fluctuations obey

$$\delta\ddot{\phi} + 2\frac{\dot{a}}{a}\delta\dot{\phi} + k^2\delta\phi = 0 \quad (30)$$

This equation is identical to that for tensor perturbations. We can right away borrow the result of the previous section on tensor perturbations, in particular Eq.(29).

Hence, the power spectrum of fluctuations in $\delta\phi$ is

$$\frac{k^3}{2\pi^2}P_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2 \quad (31)$$

There is no factor of $16\pi G$, since $\delta\phi$ is already canonically normalized.

We now want to check if our approximation of a flat, unperturbed background is valid.

Super horizon perturbations

Claim: Ψ is small as long as perturbations are within the horizon.

In Newtonian gauge, energy-momentum conservation gives

$$\delta T^0_{0,0} + ik_i \delta T^i_0 + 3H\delta T^0_0 - H\delta T^i_i = -3(P + \rho)\partial_0\Psi \quad (32)$$

Naively the R.H.S. is negligible during inflation since $P \approx -\rho$.

Let's check when

$$\Psi \ll \frac{\delta T^0_0}{P + \rho} \quad ?$$

Take the $0 - 0$ component of the Einstein equation, assuming $\Phi \approx -\Psi$

$$k^2\Psi + 3aH(\Psi + aH\Psi) = 4\pi G a^2 \delta T^0_0 \equiv \frac{3H^2 a^2 \delta T^0_0}{2\rho}$$

At horizon crossing, $k \sim aH$,

$$\Psi \sim \frac{\delta T^0_0}{\rho} = \frac{P + \rho}{\rho} \left(\frac{\delta T^0_0}{P + \rho} \right) \ll \frac{\delta T^0_0}{P + \rho}$$

for $P \approx -\rho$.

Thus the approximation of a flat background is valid for $k \gtrsim aH$.

Well after horizon crossing the potential Ψ starts growing and the approximation breaks down at some point.

Fortunately, the following weird quantity is conserved beyond the horizon:

$$\zeta \triangleq -i \frac{k_i \delta T^0_i H}{k^2 (P + \rho)} - \Psi \quad (33)$$

Using

1.

$$\Psi \ll 1$$

for $k \gtrsim aH$

2.

$$\rho + P = \frac{\dot{\phi}^2}{a^2}$$

during slow-roll

3. and

$$\delta T^i_0 = \frac{ik_i}{a^3} \dot{\phi} \delta\phi \quad \text{or} \quad \delta T^0_i = -\frac{ik_i}{a} \dot{\phi} \delta\phi$$

gives

$$\zeta \equiv -\frac{aH\delta\phi}{\dot{\phi}}$$

during inflation.

Well after inflation, during the radiation dominated era,

$$ik_i \delta T_i^0 = 4ak\rho_R \Theta_1 \quad \text{by def.}$$

Since

$$\begin{aligned} \Theta_1 &= \frac{iv}{3} \quad \text{strong Compton/Coulomb} \\ &= \frac{k\Psi}{6aH} \quad \text{from Eq.(27) with } \Psi = -\Phi \end{aligned}$$

we finally have

$$\zeta = -\frac{3aH\Theta_1}{k} - \Psi \equiv -\frac{3}{2}\Psi$$

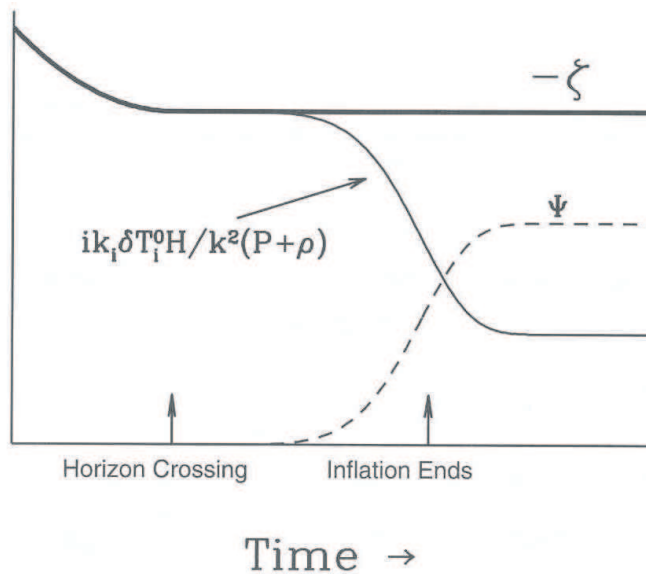
after inflation.

Hence,

$$\Psi|_{\text{post inflation}} = \frac{2}{3}aH \frac{\delta\phi}{\dot{\phi}} \Big|_{\text{horizon crossing}}$$

Equivalently

$$\boxed{P_\Psi = \frac{4}{9} \left(\frac{aH}{\dot{\phi}} \right)^2 P_{\delta\phi} \Big|_{aH \approx k}} \quad (34)$$



Demonstration that ζ is conserved on SH scales

Combining the Einstein equations

$$\begin{aligned} k^2\Psi + 3aH(\dot{\Psi} + aH\Psi) &= 4\pi G a^2 \delta T_0^0 \\ ik_i(\dot{\Psi} + aH\Psi) &= -4\pi G a \delta T_{0i}^0 \end{aligned}$$

where we set $\Phi = -\Psi$, gives

$$k^2\Psi = 4\phi G a^2 \left[\delta T_0^0 - \frac{3Hk_i \delta T_i^0}{k^2} \right] \approx 0$$

on large scales. We conclude, first, that $k_i \delta T_i^0 = \mathcal{O}(k^2)$ and can be neglected in equation (32) and, second, that

$$\zeta \triangleq -i \frac{k_i \delta T_i^0 H}{k^2 (P + \rho)} - \Psi \equiv -\frac{1}{3} \frac{\delta T_0^0}{\rho + P} - \Psi$$

Eliminating Ψ for ζ in the conservation equation (32) gives

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{P + \rho} \left[H(\rho + P) \delta T_i^i - \delta T_0^0 \frac{dP}{dt} \right] \quad (35)$$

Focusing on adiabatic perturbations,

$$\frac{dP}{dt} = \frac{d\rho}{dt} \frac{\delta P}{\delta \rho} \equiv -3H(P + \rho) \frac{\delta P}{\delta \rho}$$

Since $\delta T_0^0 = -\delta \rho$ and $\delta T_i^i = 3\delta P$, with that the RHS of Eq.(35) vanishes.

Hence, for modes satisfying $k^2 \lesssim H^2 a^2$, the quantity ζ is conserved.

Flat slicing

Consider a gauge with spatially flat slicing

$$ds^2 = -(1 + 2A)dt^2 - 2aB_i dt dx^i + \delta_{ij} a^2 dx^i dx^j$$

Then the equation of motion for the inflaton perturbations is that of Eq.(30). The power spectrum for $\delta\phi$ is then as computed, without any approximation.

The next step is to identify a gauge invariant variable. One such is Bardeen's velocity Eq.(25)

$$v = ikB - \frac{ik\dot{\phi}^{(0)}\delta\phi}{(\rho + p)a^2}$$

Another one is Eq.(23)

$$\Phi_H = aHB$$

on a spatially flat slicing. Hence the combination

$$\zeta = -\Phi_H - i\frac{aH}{k}v$$

is gauge invariant and depends only on δ **on a spatially flat slicing**,

$$\zeta = -\frac{aH}{\dot{\phi}^{(0)}}\delta\phi$$

The power spectrum in term of a gauge invariant quantity is then

$$P_\zeta = \left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 P_{\delta\phi}$$

Since it is gauge invariant, we can express it in any other gauge. In Newtonian conformal gauge, after inflation, $\zeta = 3\Phi/2$, so

$$P_\Phi = 4P_\zeta/9 = \frac{4}{9} \left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 P_{\delta\phi}$$

which is the result of the previous section.

A final remark is that in a comoving gauge, in which the velocity is vanishing, $\zeta = -\Phi_H$ has a nice interpretation in terms of curvature perturbations. These are constant beyond the horizon, reflecting the fact that perturbations on a given scale evolve like independent universes with fixed curvature. This interpretation is the basis of a nice approach to the study of density perturbations, initiated by Hawking and overtaken by Ellis, Bruni, Lyth, Liddle, ... This approach however loses its interest when damping and such effects have to be considered. See the book of Liddle & Lyth for an extensive introduction to this formalism.

Inhomogeneities

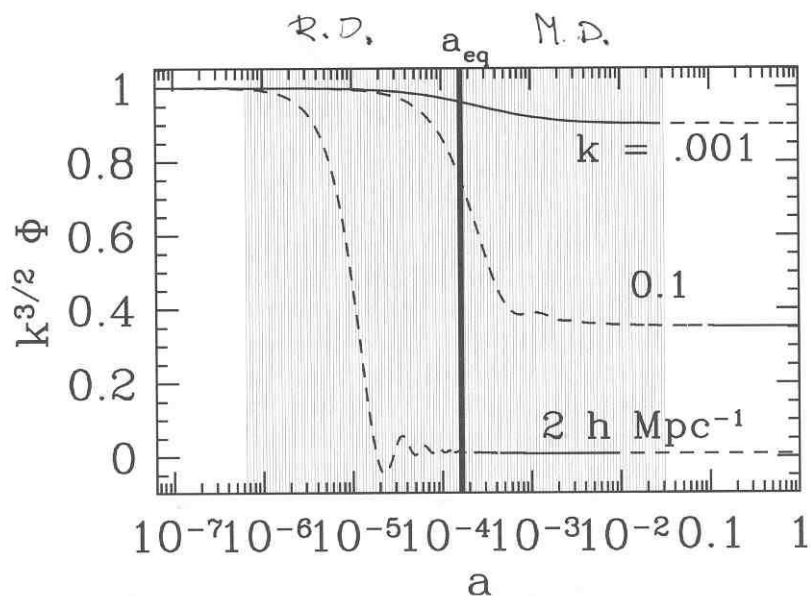
Synopsis

The evolution of perturbations breaks up naturally into three steps. (See figure.)

I. Early on, the modes are outside the horizon and the potential is constant.

II. At intermediate times, a given mode enters the horizon. At about the same time, the universe goes from radiation to matter dominated. The precise evolution of a given mode across the horizon depends very much on whether it enters the horizon before or after matter/radiation equality.

III. At late times all the modes evolve identically again. If the universe stays matter dominated, the potential stays constant.



We observe the distribution of matter much later, when all the modes are within the horizon. To relate them to the initial perturbations, one defines the **transfer function** $T(k)$

$$\Phi(\vec{k}, a) = \frac{9}{10} \Phi_p(\vec{k}) T(k) \frac{D_1(a)}{a} \quad (36)$$

This expression requires some explanations.

First the factor of $9/10$ is there to account for the fact that even the largest scale perturbations evolve across the horizon. We will see that these modes decrease by a factor of $9/10$ (for a matter dominated universe). The definition of $T(k)$ is then

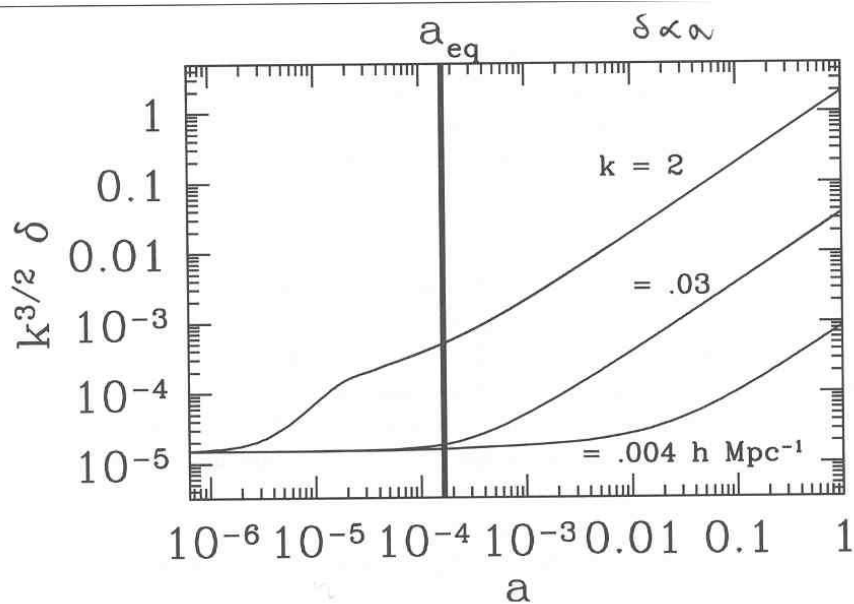
$$T(k) = \frac{\Phi(k, a_{\text{late}})}{\Phi_{\text{Large-scales}}(k, a_{\text{late}})}$$

where a_{late} refers to the scale factor at some late time such that all the modes have entered the horizon.

If the universe is not flat and matter dominated they will evolve for $a > a_{\text{late}}$. The factor of

$$\frac{D_1(a)}{a}$$

known as the **growth function** takes this effect into account. $D_1(a)$ express the growth of *matter perturbations* at late times. See figure.



In a flat matter dominated universe, $\delta \propto a$ and $\Phi \propto \delta/a$

We can express the power spectrum of the matter distribution in terms of the primordial power spectrum generated during inflation, the transfer function and the growth function.

On scales within the horizon, Newtonian gravity holds. The Newtonian potential and matter density perturbation are related by the Poisson equation

$$\Phi = \frac{4\pi G_N \rho_m \delta a^2}{k^2}$$

From $\rho_m = \Omega_m \rho_c / a^3$ and the Friedmann equation

$$\delta(\vec{k}, a) = \frac{2k^2 \Phi(\vec{k}, a) a}{3\Omega_m H_0^2}$$

or

$$\delta(\vec{k}, a) = \frac{3}{5} \frac{k^2}{\Omega_m H_0^2} T(k) D_1(a) \Phi_p(\vec{k})$$

Finally

$$P_\delta(k, a) = \frac{9}{25} \frac{k^4}{\Omega_m^2 H_0^4} T^2(k) D_1^2(a) P_\Phi(k)$$

An often used parameterization of the primordial power spectrum is

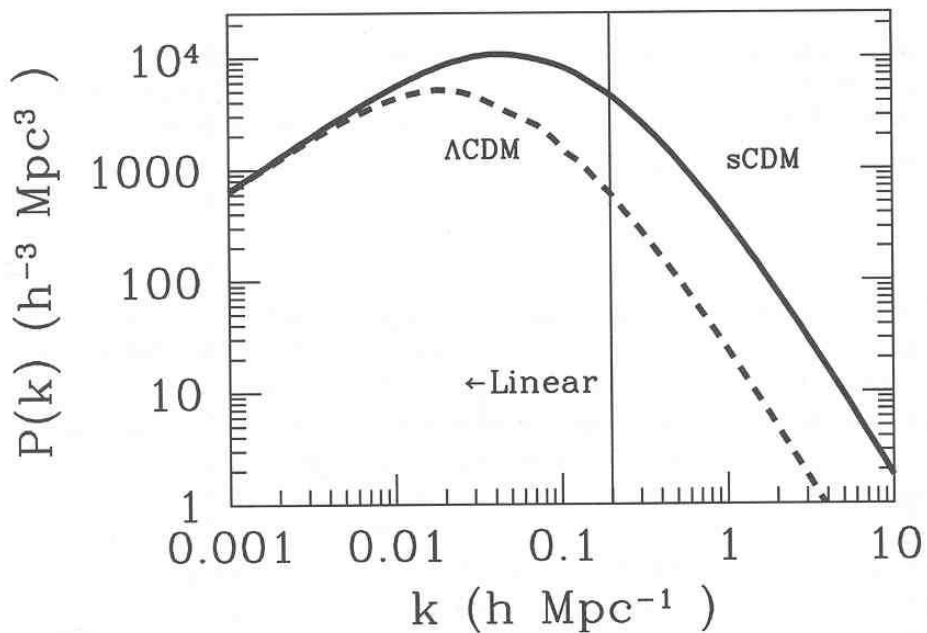
$$P_\Phi(k) = \frac{50\pi^2}{9k^3} \left(\frac{k}{H_0} \right)^{n-1} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)} \right)^2$$

An Harrison-Zel'dovich-Peebles spectrum correspond to $n = 1$. The various factors are conventional.

Finally, with these conventions

$$\boxed{\frac{k^3}{2\pi^2} P_\delta(\vec{k}, a) = \delta_H^2 \left(\frac{k}{H_0} \right)^{n-3} T^2(k) \left(\frac{D_1(a)}{D_1(a=1)} \right)^2} \quad (37)$$

The following figure shows the power spectrum for density perturbations. On large scales, it scales like k , the signature of a scale invariant spectrum of initial curvature perturbations.



On larger scales there is a turn over in the transfer function. This is due to modes which entered the horizon when the universe was radiation dominated. For these modes, the newtonian potential was dictated by radiation perturbations whose growth was inhibited by pressure. The potential was then a decreasing function of time (See figure showing the evolution of the potential.) Furthermore, the earlier the entrance the more the suppression and this until the universe became matter dominated. Hence the shape of the transfer function. Computing the transfer function will be the major goal of this chapter. Fasten your seat belt.

One last comment. Eventually the evolution of density perturbations becomes non-linear. Heuristically, this is when $k^3 P(k) \sim 1$. This occurs typically on scales $k_{nl} \approx 0.2 h \text{Mpc}^{-1}$. The power spectra are those as computed using linear perturbations. This limit is the vertical line on the figure.

Dark Matter Overdensity

Early, before recombination, need only two moments to characterize the photon distribution Θ_0 and Θ_1 .

Relevant Boltzmann Equations:

$$\dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} \quad (38)$$

$$\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\dot{\Phi} \quad (39)$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \quad (40)$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\Phi \quad (41)$$

This is a spin-off of the full fledged Boltzmann equation (76). The subscript r refers to radiation (photons and neutrinos). They both contribute to the gravitational potential and both start with the same initial conditions.

For the potential Φ , we can use either

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2[\rho_{\text{dm}}\delta + 4\rho_r\Theta_{r,0}] \quad (42)$$

or the generalized Poisson equation

$$k^2\Phi = 4\pi G a^2 \left[\rho_{\text{dm}}\delta + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}(i\rho_{\text{dm}}v + 4\rho_r\Theta_{r,1}) \right] \quad (43)$$

This reduces to Poisson's equation on sub-horizon scales $k \gtrsim aH$.

There are different regimes, depending on whether a given scale is super or sub-horizon in size.

I. Large Scales

I.a Super horizon solutions:

For these modes, $k\eta \ll 1$. The first consequence is that velocity decouples from the other equations. We use Eq.(42) to avoid with the dealing the $1/k$ term in Eq.(43).

$$\dot{\Theta}_{r,0} = -\dot{\Phi} \quad (44)$$

$$\dot{\delta} = -3\dot{\Phi} \quad (45)$$

$$3\frac{\dot{a}}{a} \left(\dot{\Phi} + \frac{\dot{a}}{a} \Phi \right) = 4\pi G a^2 [\rho_{dm} \delta + 4\rho_r \Theta_{r,0}] \quad (46)$$

The first two equation require that

$$\delta - 3\Theta_{r,0} = \text{const} = 0$$

The last equality follows from the assumption (prediction of inflation) of adiabatic initial conditions. The time-time Einstein equation can then be written as

$$3\frac{\dot{a}}{a} \left(\dot{\Phi} + \frac{\dot{a}}{a} \Phi \right) = 4\pi G a^2 \rho_{dm} \delta \left[1 + \frac{4}{3y} \right]$$

where

$$y = \frac{\rho_{dm}}{\rho_r} \equiv \frac{a}{a_{EQ}}$$

In the limit of no radiation (or $a \gg a_{EQ}$) or no matter ($a \ll a_{EQ}$) this equation is easy to solve, using

$$\dot{\delta} = -3\dot{\Phi}$$

It has two solutions, one decaying mode

$$\Phi \propto \eta^{-5/2} \quad (\text{RD}) \quad \text{or} \quad \eta^{-5} \quad (\text{MD})$$

and a constant solution. Only the latter is of interest here. To relate it to initial conditions (ie when the universe was radiation dominated) requires more work however.

It is convenient to use y as the time parameter instead of conformal time,

$$\frac{d}{d\eta} = aHy \frac{d}{dy}$$

Combining

$$\dot{\delta} = -3\dot{\Phi}$$

with the Einstein equation for Φ gives the not so cute diff. eq.

$$\Phi'' + \frac{21y^2 + 54y + 32}{2y(y+1)(3y+4)}\Phi' + \frac{\Phi}{y'(y+1)(3y+4)} = 0$$

Amazingly this equation can be solved analytically! (Using $u = \frac{y^3}{\sqrt{1+y}}\Phi$ and some more work.) The solution is

$$\Phi = \frac{\Phi(0)}{10y^3} \left[16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16 \right]$$

with

$$\lim_{y \rightarrow 0} \Phi = \Phi(0) \quad \text{superhorizon, radiation dominated}$$

and

$$\lim_{y \rightarrow \infty} \Phi = \frac{9}{10}\Phi(0) \quad \text{superhorizon, matter dominated}$$

To emphasize, what we have found is the solution for Φ on superhorizon scales, across the radiation dominated/matter dominated eras.

A potentially important feature is that the transition between the two constant regimes is quite long. See later.

I.b Crossing the horizon:

We want to show that Φ stays constant if horizon crossing occurs during the matter dominated era.

A lesser question is to see that Φ constant is a solution inside the horizon. First drop the term in $aH/k \ll 1$ in the Einstein equation. After some simple manipulations of the Boltzmann equations, one gets the equation for CDM perturbation in the MD era:

$$\ddot{\delta} + \frac{2}{\eta}\dot{\delta} - \frac{6}{\eta^2}\delta = 0 \quad (47)$$

The generalization of this equation, valid when radiation perturbations are negligible but including the transition from the radiation to the matter dominated era is called the Meszaros equation. See later.

The equation for cdm perturbations has a decaying solution

$$D_2(\eta) = a^{-3/2} \propto \eta^{-3}$$

and a growing solution

$$D_1(\eta) = a \propto \eta^2$$

Putting the latter $\delta = D_1$ in the Poisson equation

$$\Phi = \frac{3a^2H^2}{2k^2}\delta$$

shows that, in the MD era, Φ is constant on sub-horizon scales.

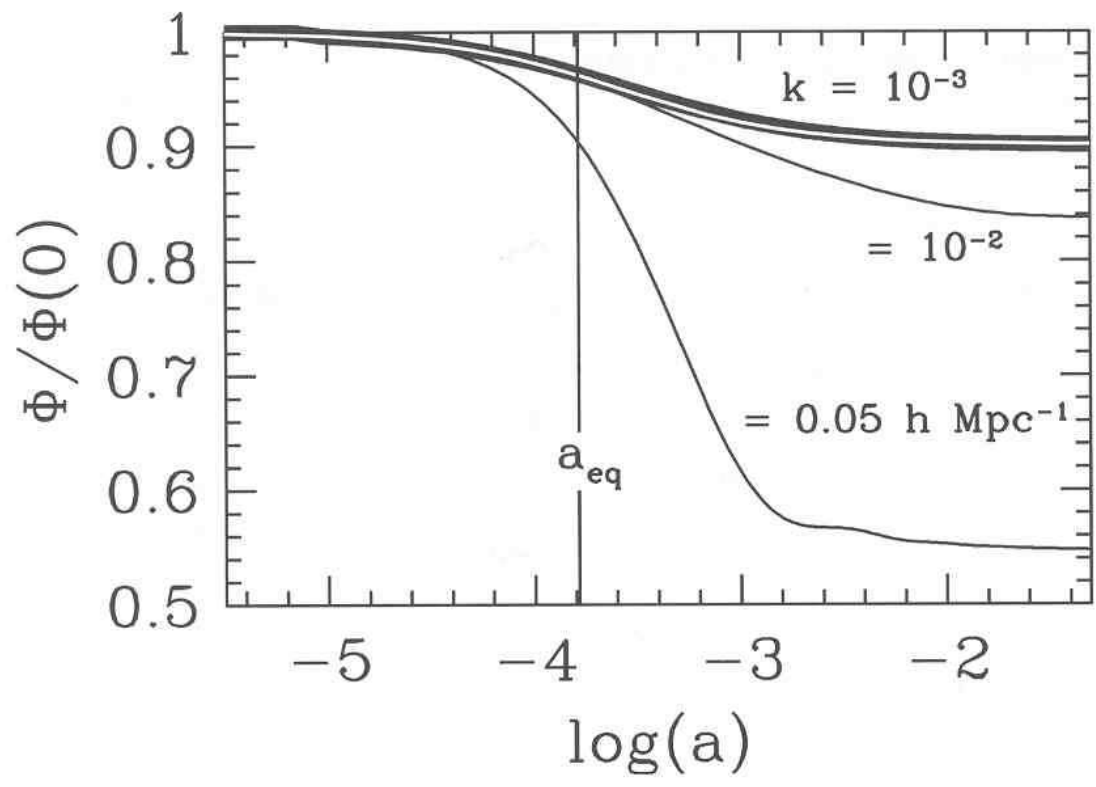
With a bit more of work, one can show that Φ constant is actually a solution across the horizon. The trick is to not solve the equation, just verify that Φ constant is a solution. We leave this as an exercise.

The lesson of this section is that the transfer function is close to unity on scales which are superhorizon at the time of matter-radiation equality. That is

$$T(k) \approx 1 \quad \text{for} \quad k \lesssim a_{\text{EQ}}H_{\text{EQ}}$$

The relevant scale

$$k_{\text{EQ}} = 0.073 \text{Mpc}^{-1} \Omega_m h^2$$



II. Small Scales

The logic is the same as for large scales. The order of events is reversed. First we address the problem of horizon crossing during radiation domination. Then that of the transition from radiation to the matter dominated regime.

II.a Horizon crossing

In the radiation dominated era, the potential is determined by the fluctuation in radiation (photons and neutrinos). The overdensity of dark matter is in turn determined by the potential.

We start from (43) without matter

$$\boxed{\Phi = \frac{6a^2H^2}{k^2} \left[\Theta_{r,0} + \frac{3aH}{k} \Theta_{r,1} \right]} \quad (48)$$

with

$$aH = \frac{1}{\eta}$$

in the RD era.

We can use this equation to eliminate $\Theta_{r,0}$ in the Boltzmann equations (38) and (39). Then we combine the two resulting equation to eliminate $\Theta_{r,1}$. This gives

$$\begin{aligned} \Theta_{r,1} &= -\frac{k\eta}{6} (\eta\dot{\Phi} + \Phi) \\ \dot{\Theta}_{r,1} &= \frac{k\eta}{6} \dot{\Phi} - \frac{k}{6} \Phi \left(1 - \frac{k^2\eta^2}{3} \right) \end{aligned}$$

Deriving the first equation and using the second to eliminate $\dot{\Theta}_{r,1}$ finally gives

$$\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + \frac{k^2}{3}\Phi = 0 \quad (49)$$

This equation can be put in the form of the spherical Bessel equation of order one by defining $\Phi = a^{-1}u = u/\eta$

$$\ddot{u} + \frac{2}{\eta}\dot{u} + \left(\frac{k^2}{3} - \frac{2}{\eta^2} \right) u = 0$$

Since Φ is constant at small times, we only keep the solution regular at small η , $j_1(k\eta/\sqrt{3})$,

$$\Phi = \Phi_p \left[\frac{3 \sin(k\eta/\sqrt{3}) - 3(k\eta/\sqrt{3}) \cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3} \right] \quad (50)$$

This solutions shows that the potential decays at horizon entry and oscillates.

Explanation (tentative): This is to be expected. First, within the horizon, radiation pressure can compete with gravitational collapse and $\Theta_{r,0}$ will oscillate with constant amplitude. Indeed, within the horizon we recover the Newtonian limit of the Einstein equation (48), ie the Poisson equation

$$\Phi \approx \frac{6}{k^2\eta^2}\Theta_{r,0}$$

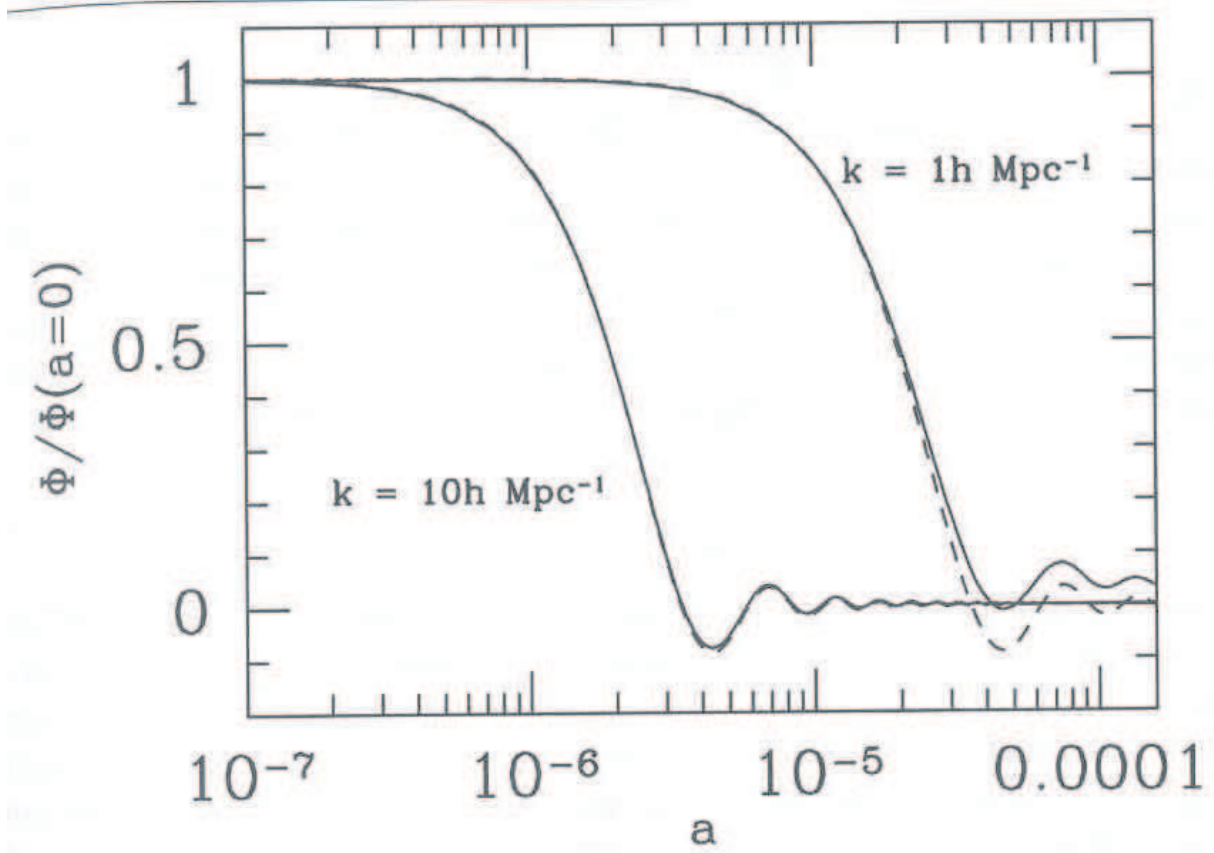
This shows that the potential is subdominant compared to $\Theta_{r,0}$. Consequently Φ can be neglected in the Boltzmann equations (38) and (39). Combining these equations simply gives

$$\ddot{\Theta}_{r,0} + \frac{k^2}{3}\Theta_{r,0} = 0$$

Radiation perturbations are oscillatory waves of constant amplitude, with sound velocity $v_s = 1/\sqrt{3}$. According to the Poisson equation,

$$\Phi \propto \frac{1}{k^2\eta^2}\Theta_{r,0}(k\eta/\sqrt{3})$$

and Φ oscillates with an amplitude decaying as $1/\eta^2$, the latter effect being simply the redshifting due to background expansion.



The last step is to determine the evolution of the matter distribution. From (40) and (41) we get the second order equation

$$\ddot{\delta} + \frac{1}{\eta}\dot{\delta} = \mathcal{S}(k, \eta) \equiv -3\ddot{\Phi} + k^3\Phi - \frac{3}{\eta}\dot{\Phi}$$

where we have put the source term \mathcal{S} on the rhs of the equation for the dark matter fluctuation.

The solutions of this equations are

$$\delta(k, \eta) = C_1 + C_2 \ln(\eta) - \int_0^\eta d\eta' \mathcal{S}(k, \eta) \eta' (\ln |k\eta'| - \ln |k\eta|)$$

where the first two terms correspond to the solutions of the homogeneous equation.

At early times, the integral is small, so we must set $C_2 = 0$ since δ is constant initially and

$$C_1 = \delta(k, \eta = 0) = \frac{3}{2}\Phi_p$$

In the integral, the source term \mathcal{S} decays at horizon crossing and goes rapidly to zero. So the integral is over a limited range. The first term in the integral gives some constant, while the second asymptotes to a constant times $\ln(k\eta)$. Hence, after horizon crossing we expect

$$\boxed{\delta(k, \eta) = A\Phi_p \ln(Bk\eta) \propto \ln(a)} \quad (51)$$

The growth is less rapid than during the matter dominating era $\delta \propto a$ due to the effect of radiation pressure.

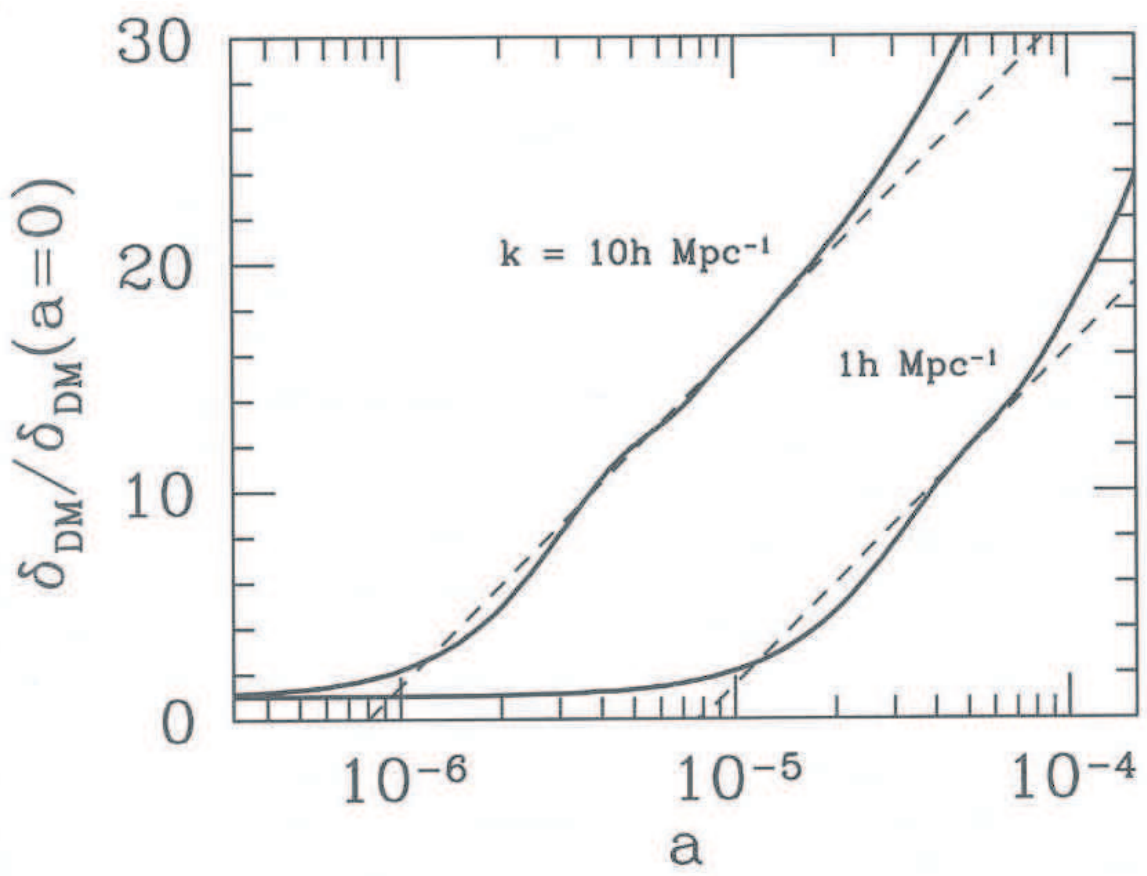
The constant A and B correspond to

$$A\Phi_p = \int_0^\infty d\eta' \mathcal{S}(k, \eta') \eta' \approx 9\Phi_p$$

and

$$A\Phi_p \ln(B) = \frac{3}{2}\Phi_p - \int_0^\infty d\eta' \mathcal{S}(k, \eta') \eta' \ln(k\eta') \approx 9\Phi_p \ln(0.5)$$

where it is a good approximation to extend the integrals to infinity. The numerical values are quoted for your information. (I have not checked these numbers. Dodelson tells that you get slightly different values depending on whether you use the analytic expression for Φ or exact numerical solutions. The order of magnitude are the same though.)



As the universe expands, eventually one gets into the matter dominated era. As the effect of radiation pressure become negligible, the previous solutions will then turn into $\delta \propto a$ growing solutions. This brings us to

I**b**. Sub-horizon evolution

Our interpretation is that radiation pressure prevents radiation perturbations from growing. As a consequence, potential perturbations grow logarithmically. It may be that, eventually, their contribution dominate that of radiation perturbation, that is

$$\rho_m \delta \gtrsim \rho_r \Theta_{r,0}$$

regardless of whether $\rho_r \gtrsim \rho_m$. When this happens, radiation becomes irrelevant and potential and matter perturbations evolve together. There solutions should then be matched to the the log solution of the previous section.

Since we follow the evolution of perturbation across the matter/radiation equality period, we use again the variable $y = a/a_{EQ}$. Dropping the radiation perturbations equations, the relevant equations (expressed using y) are

$$\delta' + \frac{ikv}{aHy} = -3\Phi' \quad (52)$$

$$v' + \frac{v}{y} = \frac{ik\Phi}{aHy} \quad (53)$$

$$k^2\Phi = \frac{3y}{2(1+y)} a^2 H^2 \delta \quad (54)$$

We dropped terms in $aH/k \ll 1$ since we are well within the horizon.

Taking the derivative of the first equation, using the second, gives

$$\delta'' - \frac{ik(2+3y)v}{2aHy^2(1+y)} = -3\Phi'' + \frac{k^2\Phi}{a^2H^2y^2}$$

It is natural (*ie* confirmed by numerical resolution of the equations) to expect the first term to be smaller than the second one, which is $\propto k^2/a^2H^2 \gg 1$. So we drop the first term. Using the third equation (Poisson), we finally get the

Meszaros equation

$$\boxed{2y(y+1)\delta'' + (2+3y)\delta' - 3\delta = 0} \quad (55)$$

dictates the evolution of sub-horizon cold dark matter perturbations, when radiation perturbations have become negligible.

Solutions of the Meszaros equation:

1. Growing mode:

$$D_1(y) = y + \frac{2}{3}$$

This solution matches with the behaviour of sub-horizon cdm perturbations deep in the matter era $\delta \propto a$ studied in a previous section. (See Eq.(47).)

II. Nice trick (I had forgotten this one). Knowing the solution $D_1(y)$, define $\delta = u(y)D_1(y)$, allows to easily find the other solution. The Meszaros equation becomes

$$(1 + 3y/2)u'' + \frac{u'}{y(y+1)} \left[\frac{21}{4}y^2 + 6y + 1 \right] = 0$$

This equation is first order and can be integrated to get u' .

This gives

$$u' = \frac{C}{y(y + 2/3)^2 \sqrt{1 + y}}$$

Integrating again gives the decaying solution of the Meszaros eq.

$$D_2 = (y + 2/3) \ln \left[\frac{\sqrt{1 + y} + 1}{\sqrt{1 + y} - 1} \right] - 2\sqrt{1 + y} \propto \frac{1}{y^{3/2}} \quad (\text{large } y)$$

Large y behaviour means CDM era. The solutions then correspond to those of Eq.(47). Using

$$\frac{d}{dy} \equiv \frac{1}{2\eta} \frac{d}{d\eta}$$

it is a simple exercise to check that the Meszaros equation at large y reduces to Eq.(47).

To match with the log. growing solution (51) we need both solutions,

$$\delta(k, y) = C_1 D_1(y) + C_2 D_2(y) \quad y \gg y_H$$

where $a_{EQ} y_H$ is the scale factor at horizon crossing for the mode k .

The matching

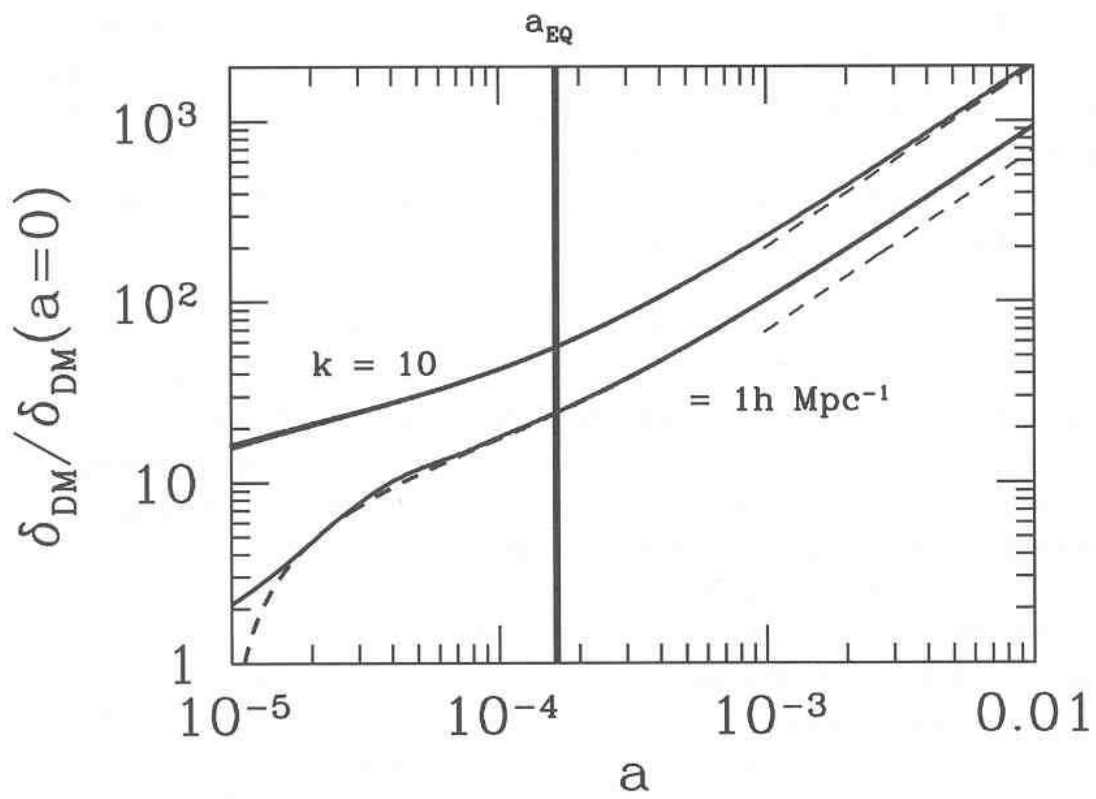
$$A \Phi_P \ln(B y_m / y_H) = C_1 D_1(y_m) + C_2 D_2(y_m) \quad (56)$$

$$A \frac{\Phi_P}{y_m} = C_1 D_1'(y_m) + C_2 D_2'(y_m)$$

is at a time such that

$$y_H \ll y_m \ll 1$$

ie for modes which are within the horizon but at a time less than matter/radiation equality so that the log. solution is still (approximatively) valid.



Numerical results and fitting functions

In this section we derive an analytic approximation to the the small scale behaviour of the transfer function $T(k)$ well after the time of matter/radiation equality.

Since $y \gg 1$, we are only interested in determining the coefficient of the growing solution C_1 . Eliminating C_2 from Eqs.(56) gives

$$\begin{aligned} C_1 &= \frac{D'_2(y_m)A \ln(By_m/y_H) - D_2(y_m)(A/y_m)}{D_1(y_m)D'_2(y_m) - D'_1(y_m)D_2(y_m)} \\ &\approx \frac{3A\Phi_p}{2} \ln\left(\frac{4Be^{-3}}{y_H}\right) \quad \text{for } y_m \ll 1 \end{aligned}$$

In the last expression we used the fact that $y_m \ll 1$. It so happens that all reference to y_m drops in this approximation.

The late time behaviour of the small scale CDM perturbations is then approximately given by

$$\delta(\mathbf{a}, \vec{k}) = \frac{3A\Phi_p(\vec{k})}{2} \ln\left(\frac{4Be^{-3}a_{EQ}}{a_H}\right) D_1(\mathbf{a})$$

Comparing with the definition of the transfer function Eq(36) finally gives

$$T(k) = \frac{5A\Omega_m H_0^2}{2k^2 a_{EQ}} \ln\left(\frac{4Be^{-3}\sqrt{2}k}{k_{EQ}}\right) \quad k \gg k_{EQ}$$

where we used the approximation that

$$\frac{a_{EQ}}{a_H} = \frac{\sqrt{2}k}{k_{EQ}}$$

valid on small scales $k \gg k_{EQ}$

This comes from $a_H H(a_H) = k$ which gives the scale factor when a given mode k crossed the horizon. Accordingly, at equality $a_{EQ} H_{EQ} = k_{EQ}$. Using

$$H_{EQ}^2 = H_0^2 (\Omega_m / a_{EQ}^3 + \Omega_r / a_{EQ}^4) = 2H_0^2 \Omega_r / a_{EQ}^4$$

or

$$k_{eq} = \sqrt{2} H_0 \Omega_m^{1/2} a_{eq}^{-1}$$

For $k \gg k_{EQ}$,

$$H^2 = H_0^2 \frac{\Omega_r}{a^4}$$

which gives

$$\frac{a_{EQ}}{a_H} = \frac{\sqrt{2} k}{k_{EQ}}$$

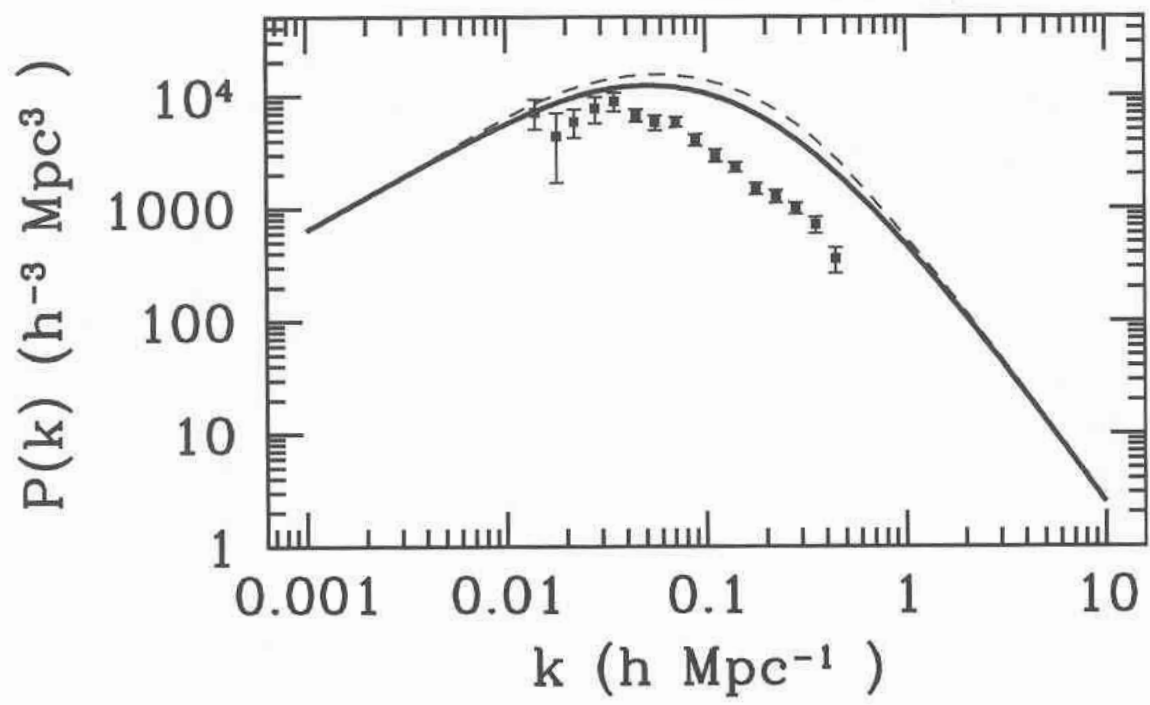
Putting numbers in, Dodelson quotes

$$T(k) = \frac{12k_{eq}^2}{k^2} \ln \left(\frac{k}{8k_{eq}} \right)$$

As anticipated before, if not for the period of logarithmic growth of sub-horizon modes during the radiation era, perturbations in the potential would simply drop like $1/k^2$ for modes which entered the horizon before the epoch of equality.

This analytic approximation works pretty well for small scales $k \gg k_{eq}$ as the figure reveals. For information, a fitting formula commonly quoted is the that due to Bardeen, Bond, Kaiser and Szalay (BBKS)

$$T(x = k/k_{eq}) = \frac{\ln(1 + 0.171x)}{0.171x} \left[1 + 0.284x + (1.18x)^2 + (0.399x)^3 + (0.490x)^4 \right]^{-0.25}$$



Growth Function

This is about the late time behaviour of perturbations, when all modes have entered the horizon. If $\Omega_m = 1$, then we can just take the results we got from the resolution of the Meszaros equation. Otherwise, we have to step back a little. An important point is that the growth of solution will be independent of k (as the equation doesn't depend on k): all modes grow the same as soon as they are within the horizon in the matter dominated era. Basically this comes from the fact that in the absence of pressure (cdm means pressureless), there is nothing to smooth out the inhomogeneities.

There are two steps we need to correct. First the coefficient of δ in the Poisson equation (54) must be replaced if $\Omega_m \neq 1$ (we work in the limit $y \gg 1$)

$$\frac{k^2}{a^2} \Phi = \frac{3H_0^2 \Omega_m}{2a^3} \delta$$

Also the derivative with respect to y in the Meszaros equation must now take into account the fact that H scales differently than $a^{-3/2}$ if $\Omega_m \neq 1$. Working things out gives

$$\frac{d^2 \delta}{da^2} + \left(\frac{d \ln H}{da} + \frac{3}{a} \right) \frac{d\delta}{da} - \frac{3\Omega_m H_0^2}{2a^5 H^2} \delta = 0$$

All factors of a_{eq} have dropped in the large y limit, so better derive wrt to the scale factor.

This equation has a decaying solution $\delta \propto H$ regardless of the equation of state of the universe. To get the other, growing solution, define as before $\delta = uH$.

The equation becomes

$$\frac{d^2 u}{da^2} + 3 \left[\frac{d \ln H}{a} + \frac{1}{a} \right] \frac{du}{da} = 0$$

which gives

$$\frac{du}{da} \propto (aH)^{-3}$$

and finally

$$D_1(a) \frac{5\Omega_m H(a)}{2 H_0} \int_0^a \frac{da}{(aH(a)/H_0)^3}$$

where the integration constant was fixed imposing that $D_1(a) = a$ at early times ($z \gtrsim 10$).

It is perhaps worth understanding where these solutions come from. The Meszaros equation dictates the behaviour of cdm perturbations in an expanding universe. One can define a local expansion parameter a , such that $\rho \propto a^{-3}$ by continuity and a satisfies a locally defined Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2}$$

where all the parameters are position dependent. Times flows differently at different positions according to

$$t = C + \int_0^a \frac{da}{aH}$$

From the continuity of ρ , $\delta = -3\delta a/a$ and a depends on two parameters C , the origin of time, and K the local curvature. Hence either

$$\delta a = \frac{\partial a}{\partial C} \delta C$$

or

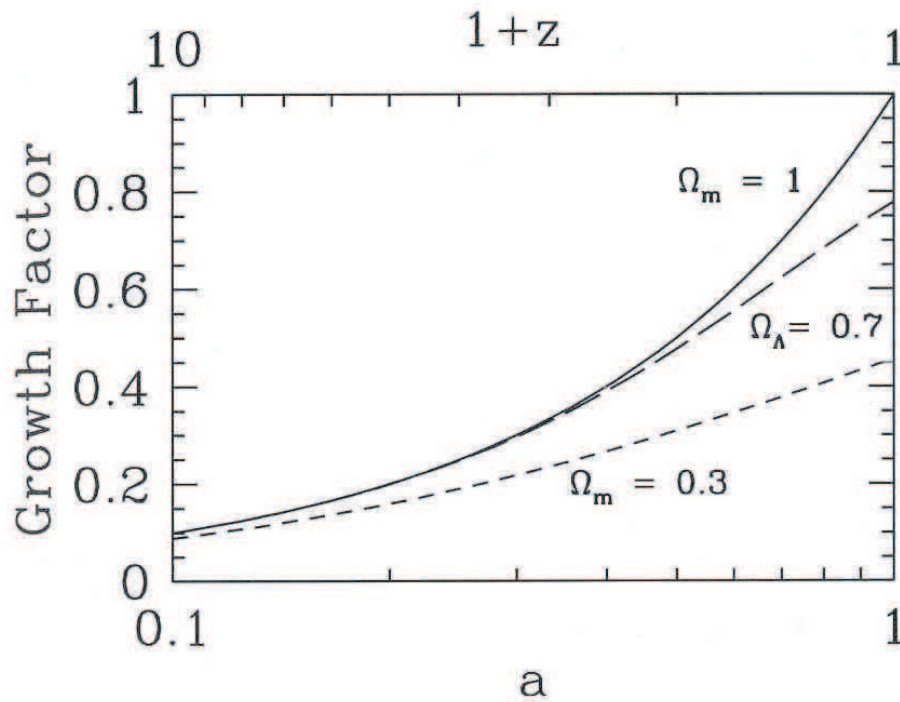
$$\delta a = \frac{\partial a}{\partial K} \delta K$$

Using the Friedmann equation, this argument gives the two solutions of the equation for the evolution of cold dark matter. The decaying one

$$D_2 = \frac{1}{a} \frac{da}{dt} \frac{\partial t}{\partial C} \equiv H$$

and the growing one

$$D_1 = \frac{1}{a} \frac{da}{dt} \frac{\partial t}{\partial K} \equiv H \int^a \frac{da}{(aH)^3}$$



The solutions to these equations are shown on the figure. In an open or dark energy dominated universe, the growth of perturbations is suppressed at late times. Consequently, whatever structure that exists in such a universe must have formed earlier than in a flat, matter dominated universe.

Beyond Dark Matter

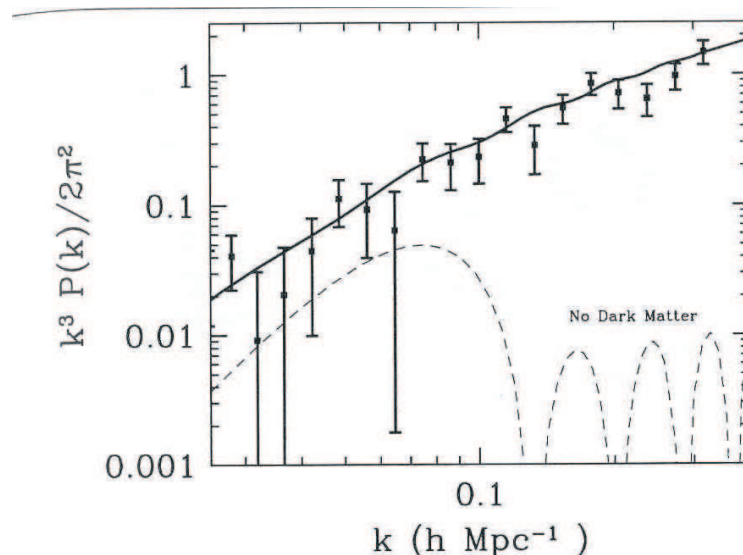
Baryons

About 4% of energy density is in the form of baryons.

There have essentially two effects on the power spectrum.

I. Suppression on small scales. Baryons are tightly coupled to photons before recombination. Unlike that of cdm, their overdensity does not grow. After decoupling, they fall into the potential made by cdm, but this potential is not as deep as in the pure cdm case.

II. There are also small oscillations on small scales. These are tiny effects. They would be larger in a pure baryonic universe. See other figure.



Massive neutrinos

Neutrinos are massive. They contribute some sort of dark matter cold hot dark matter. If $m_\nu \approx 0.05\text{eV}$, $\Omega_\nu \approx 10^{-3}$.

Perturbations on scales smaller than the free streaming scale of a light neutrino are suppressed. For a neutrino mass of order one eV, the free-streaming scale is of the order of k_{eq} .

Note that the free-streaming scale for lighter neutrinos is larger, but this is counterbalanced by the fact that lighter neutrinos contribute less to the energy density. (Otherwise massless neutrinos would wash out any perturbations.)

Dark energy

There is substantial evidence that the universe is flat, $\Omega = 1$ and accelerating $\Omega_{\text{de}} \approx 0.7$. Altogether this gives $\Omega_{\text{m}} \approx 0.3$.

I. Since $a_{\text{eq}} \propto \Omega_{\text{m}}$, less CDM implies that equality took place later and so that the turnover in the power spectrum occurs on a larger scales $k_{\text{eq}} \propto \Omega_{\text{m}}$.

II. Because of the Poisson equation, for fixed potential, the CDM overdensity is *larger* for a smaller Ω_{m} (more room for perturbations, essentially). Therefore the amplitude of the power spectrum increases for decreasing Ω_{m} .

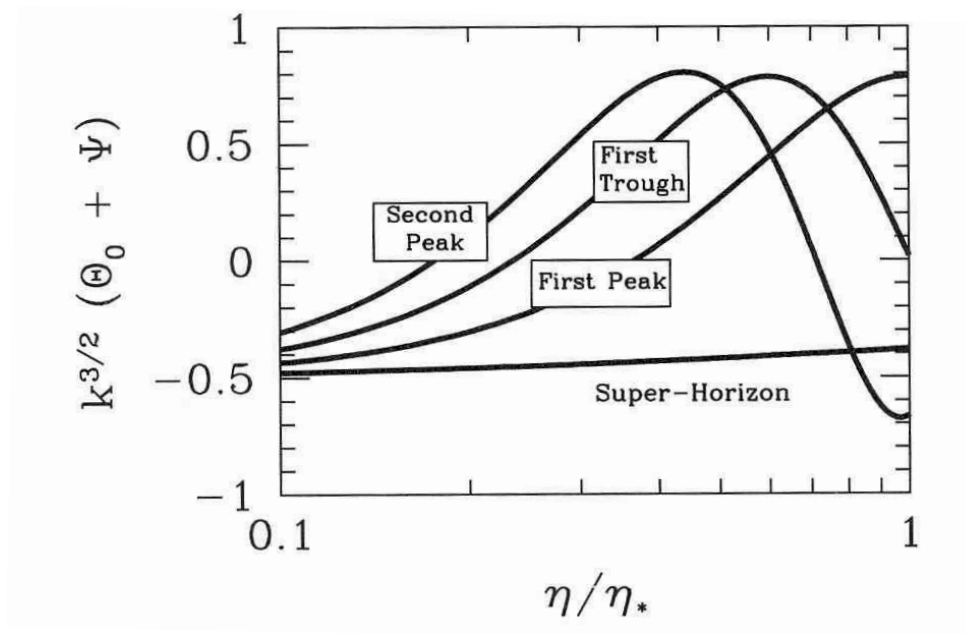
III. At late times, the amplification of perturbations is controlled by the growth factor. Growth is suppressed at late times in a universe with dark energy. So any structure that is observed today was formed earlier.

Anisotropies

Primordial perturbations set up during inflation manifest themselves in radiation and matter distribution.

Before recombination ($z \approx 1100$): photons are tightly coupled to the electrons and baryons. (The photon-baryon fluid).

After recombination: free streaming toward us.



Overview

The temperature we see today is a combination of the intrinsic temperature perturbation Θ and of the gravitational potential Ψ because photons had to climb out the potential they were in at the time of recombination.

The relevant combination is

$$k^{3/2}(\Theta + \Psi)$$

The factor of $k^{3/2}$ is there to account the fact that (in scale invariant scenarios) the amplitude of fluctuations scales like k^{-3} .

Super-horizon modes hardly evolve. Observing large scale anisotropies thus reveals the original fluctuations.

Shorter wave-length modes that could have entered the horizon before recombination show a succession of peaks and troughs known as acoustic peaks. The first peak correspond to the mode that entered last at recombination. It had just the time to undergo half an oscillation at the time of recombination and correspondly there are overdensity regions on this scale. The next speak in the power spectrum went through a full period of oscillation and corresponds to depletion or underdensity regions.

This is the basic origin of the peak structure. There are further, more subtle effects. First the odd and even peak alternate: the latters seem to be enhanced wrt to the formers. There is also an overall damping of oscillations on smaller scales (larger l).

Roughly speaking, the equation governing temperature perturbations takes the form

$$\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F$$

where F is a driving force due to gravity and c_s is the speed of sound. Dots refer to derivative wrt to conformal time. This is the equation of a forced harmonic oscillator.

As we add more baryons to the universe, the speed of sound goes down (why?). This implies that the frequency goes down and so that the peaks are displaced toward larger k . Further, adding baryons enhances the disparity between the odds and even peaks: adding more baryons enhances attraction in overdense regions, making it harder for photons to escape.

The damping is due to imperfect coupling between baryons and photons on smaller scales. Photon travel a finite distance between scatters. In the course of a Hubble time H^{-1} , a photon scatters about $n_e \sigma_T H^{-1}$ times. Correspondingly, the distance travelled during the same time is about

$$\lambda \sim \frac{1}{\sqrt{n_e \sigma_T H}}$$

Perturbations on scales smaller than λ are expected to be washed-out.

The last step is to relate perturbations in three-dimensional space to those observed on the surface of last scattering. Roughly speaking, to a comoving scale k corresponds an angular separation

$$\theta \approx \frac{k^{-1}}{\eta_0 - \eta_*}$$

where $\eta_0 - \eta_*$ is the comoving distance to the surface of last scattering.

The last thing we will have to understand is the effect of late time evolution of the gravitational potential (recombination is not too far from matter-radiation equality), the so-called integrated Sachs-Wolfe effect.

Large scale anisotropies

The large scale solution to the Einstein Boltzmann equations give

$$\Theta_0 = -\Phi + \text{const}$$

The initial conditions are such that initially

$$\Theta_0 = \frac{\Phi_p}{2}$$

Hence the integration constant equals $3\Phi_p/2$.

At recombination the large scale photon perturbations satisfy

$$\Theta_0(k, \eta_*) = -\Phi(k, \eta_*) + \frac{3\Phi_p(k)}{2} = \frac{3}{5}\Phi_p(k) = \frac{2}{3}\Phi(k, \eta_*)$$

The observed anisotropy is $\Theta_0 + \Psi$. Moreover $\Psi = -\Phi$ to a good approximation. Hence

$$(\Theta_0 + \Psi)(k, \eta_*) = \frac{1}{3}\Psi(k, \eta_*)$$

Another way to express large scale perturbations is to relate them to density perturbations. Initially $\delta = 3\Phi/2$. From $\dot{\delta} = -3\dot{\Phi}$ one gets

$$\delta(\eta_*) = \frac{3}{2}\Phi_p - 3(\Phi(\eta_*) - \Phi_p) = 2\Phi(\eta_*)$$

which gives

$$(\Theta_0 + \Psi)(k, \eta_*) = -\frac{1}{6}\delta(\eta_*)$$

Note that overdense regions are cooler on the sky. There are intrinsically hotter, but climbing the potential overweight this and gives bluer photons.

Acoustic oscillations

Tight coupling limit of Boltzmann equations

This limit corresponds to the scattering rate being much larger than the expansion rate: $\tau \gg 1$ where τ is the optical depth. We show that only the first two moments Θ_0 and Θ_1 are important in this regime. This is the sense in which the photons can be treated as a fluid, which depends on only two variables (density and velocity fields).

Consider

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau} \left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}\mathcal{P}_2(\mu)\Pi \right] \quad (57)$$

We multiply by the Legendre polynomials $\mathcal{P}_l(\mu)$ and integrate over $\mu = \cos(\theta)$ with

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^1 \frac{d\mu}{2} \mathcal{P}_l(\mu)\Theta(\mu)$$

Using the recurrence relation

$$(l+1)\mathcal{P}_{l+1} = (2l+1)\mu\mathcal{P}_l - l\mathcal{P}_{l-1}$$

and the fact that Φ and Ψ have little μ dependence, for $l > 2$ *

$$\dot{\Theta}_l - \frac{kl}{2l+1}\Theta_{l-1} + \frac{k(l+1)}{2l+1}\Theta_{l+1} = \dot{\tau}\Theta_l$$

The first term is smaller than the RHS for $\tau \gg 1$,

$$\dot{\Theta}_l \sim \frac{\Theta_l}{\eta} \ll \dot{\tau}\Theta_l \sim \frac{\tau}{\eta}\Theta_l$$

(check with expected behaviour of the optical depth) and can be dropped. Assume that Θ_{l+1} is negligible. The second term then gives

$$\Theta_l \sim \frac{k\eta}{\tau}\Theta_{l-1} \ll \Theta_{l-1}$$

on horizon scales and for fast scattering. Thus it is self-consistent to assume that Θ_{l+1} is even smaller.

*We also neglect the possibility of a large Π . See later.

The equations for the radiation monopole and dipole perturbations become

$$\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \quad (58)$$

$$\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = \frac{k}{3}\Psi + \dot{\tau} \left[\Theta_1 - i\frac{v_b}{3} \right] \quad (59)$$

In the tight coupling regime, the equation for baryon velocity

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi + \frac{\dot{\tau}}{R} [v_b + 3i\Theta_1] \quad (60)$$

implies that, to leading order

$$v_b = -3i\Theta_1$$

Putting this zeroth order approximation back in Eq.(60) gives

$$v_b \approx -3i\Theta_1 + \frac{R}{\dot{\tau}} \left[-3i\dot{\Theta}_1 - 3i\frac{\dot{a}}{a}\Theta_1 + ik\Psi \right]$$

This allows to eliminate the baryon velocity from the equation for the dipole moment,

$$\dot{\Theta}_1 + \frac{\dot{a}}{a} \frac{R}{1+R} \Theta_1 = \frac{k}{3(1+R)} \Theta_0 + \frac{k}{3} \Psi$$

From the two first order equations for the monopole and dipole, we can get a second order equation for the monopole

$$\boxed{\ddot{\Theta}_0 + \frac{\dot{a}}{a} \frac{R}{1+R} \dot{\Theta}_0 + k^2 c_s^2 \Theta_0 = -\frac{k^2}{3} \Psi - \frac{\dot{a}}{a} \frac{R}{1+R} \dot{\Phi} - \ddot{\Phi} \equiv F(k, \eta)} \quad (61)$$

where the speed of sound

$$c_s = \sqrt{\frac{1}{3(1+R)}}$$

and

$$R = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$

Tight coupling solutions

Rewrite the equation for Θ_0 as

$$\left\{ \frac{d^2}{d\eta^2} + \frac{\dot{a}}{a} \frac{R}{1+R} \frac{d}{d\eta} + k^2 c_s^2 \right\} [\Theta_0 + \Phi] = \frac{k^2}{3} \left[\frac{1}{1+R} \Phi - \Psi \right]$$

This emphasizes the relationship between temperature perturbations and the gravitational potential.

We first solve the equation in the limit in which the damping term is small. This amounts to neglecting the effect of baryons.

The solution of the homogeneous equation are then simply

$$S_1(k, \eta) = \sin(kr_s(\eta)) \quad \text{and} \quad S_2(k, \eta) = \cos(kr_s(\eta))$$

where the sound horizon r_s is given by

$$r_s(\eta) = \int_0^\eta d\eta' c_s(\eta')$$

The solution of the inhomogeneous equation is given by

$$\Theta_0 + \Phi = C_1 S_1 + C_2 S_2 + \frac{k^2}{3} \int_0^\eta d\eta' [\Phi(\eta') - \Psi(\eta')] \frac{S_1(\eta') S_2(\eta) - S_1(\eta) S_2(\eta')}{S_1(\eta') \dot{S}_2(\eta') - \dot{S}_1(\eta') S_2(\eta')}$$

The initial conditions of constant Θ_0 and Φ give

$$C_2 = \Theta_0 + \Phi \quad \text{and} \quad C_1 = 0$$

and

$$\begin{aligned} \Theta_0(\eta) + \Phi(\eta) &= [\Theta_0(0) + \Phi(0)] \cos(kr_s) \\ &+ \frac{k}{\sqrt{3}} \int_0^\eta d\eta' [\Phi(\eta') - \Psi(\eta')] \sin [k(r_s(\eta) - r_s(\eta'))] \end{aligned} \quad (62)$$

where we replaced the denominator in the integrand above by $-k/\sqrt{3}$.

Comments:

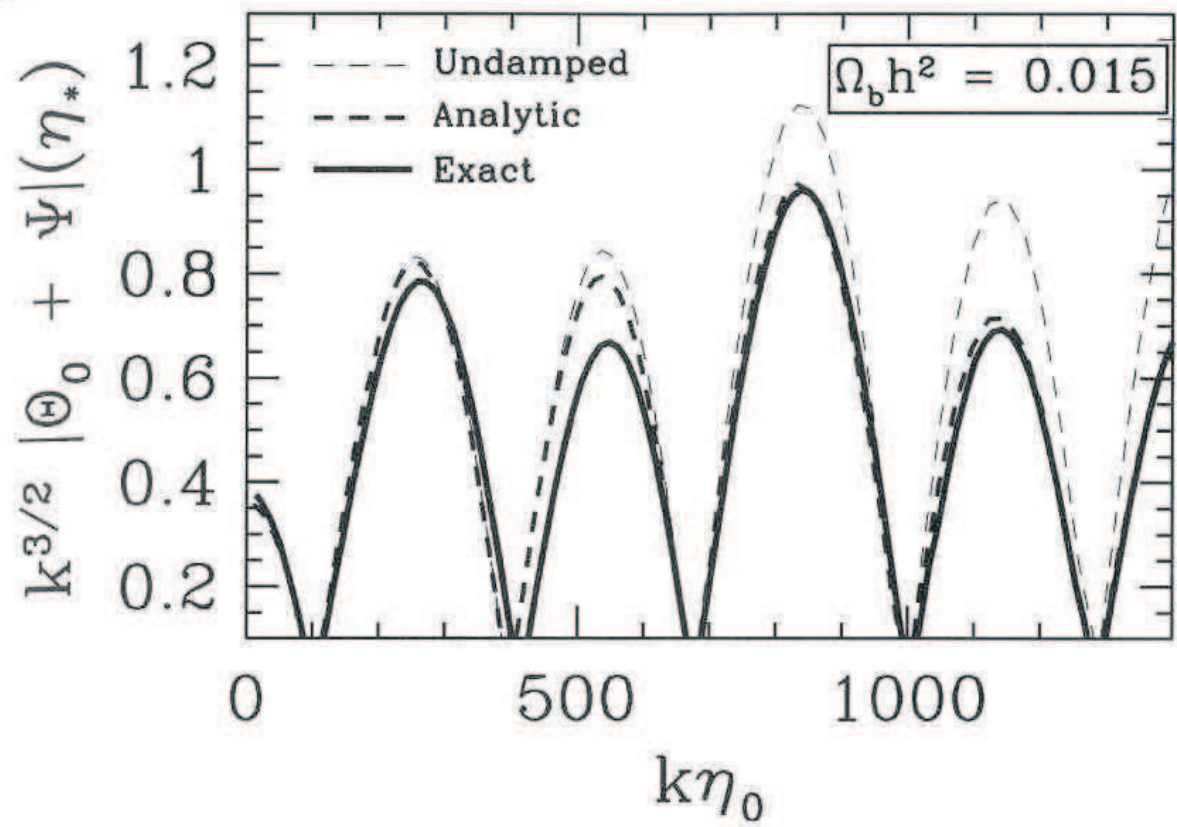
- This analytical approximation does a good job of approximating the position of the peaks. The height as well, but not quite. Needs damping. (See figure next page)
- Separates the calculation into two. Get the gravitational potentials first, integrate for the temperature fluctuations next.
- The cosine mode is there because of causality, a clear indication in favour of inflation (but not a proof of).
- In the limit in which the first term dominates the solution, the location of the peaks is at

$$k_p = n\pi/r_s$$

We can also get an approximate solution for the dipole, by differentiating the solution for the monopole and using Eq.(58).

$$\Theta_1(\eta) = \frac{1}{\sqrt{3}} [\Theta_0(\eta) + \Phi(\eta)] \sin(kr_s) \quad (63)$$
$$- \frac{k}{\sqrt{3}} \int_0^\eta [\Phi(\eta') - \Psi(\eta')] \cos(k(r_s(\eta) - r_s(\eta')))$$

which shows that dipole perturbations are out of phase with monopole perturbations.



Diffusion Damping

Diffusion is required to match the precision of numerical solutions. Diffusion introduces a small quadrupole moment. Things are simplified by the fact that we only consider small scales, on which the gravitational potentials are small because of radiation pressure. So we can neglect them altogether. We still neglect higher modes and check that this is a self-consistent approximation.

In these approximations, we have

$$\dot{\Theta}_0 + k\Theta_1 = 0 \quad (64)$$

$$\dot{\Theta}_1 + k \left(\frac{2}{5}\Theta_2 - \frac{1}{3}\Theta_0 \right) = \dot{\tau} \left(\Theta_1 - \frac{iv_b}{3} \right) \quad (65)$$

$$\dot{\Theta}_2 - \frac{2k}{5}\Theta_1 = \frac{9}{10}\dot{\tau}\Theta_2 \quad (66)$$

This need to be supplemented by an equation for the baryon velocity. Here we take into account the fact that damping occurs on small scales, so that we can neglect expansion. Writing

$$(v_b, \Theta_0, \Theta_1) \propto \exp i \int \omega d\eta$$

with $\omega \sim k$ first gives

$$\dot{v}_b = i\omega v_b \gg \frac{\dot{a}}{a} v_b \sim \frac{1}{\eta} v_b$$

since $k\eta \gg 1$. Then the equation for v_b becomes

$$v_b + 3i\Theta_1 = \frac{R}{\dot{\tau}} \left[\dot{v}_b + \frac{\dot{a}}{a} v_b \right] \approx \frac{R}{\dot{\tau}} i\omega v_b$$

Solving for v_b up to second order in $R/\dot{\tau}$ gives

$$v_b \approx -3i\Theta_1 \left(1 + \frac{i\omega R}{\dot{\tau}} - \left(\frac{\omega R}{\dot{\tau}} \right)^2 \right)$$

Similarly

$$\Theta_2 \approx -\frac{4k}{9\dot{\tau}}\Theta_1 \ll \Theta_1$$

since $\dot{\Theta}_2 \ll \dot{\tau}\Theta_2$ and

$$i\omega\Theta_0 \approx -k\Theta_1$$

Putting all these together in Eq.(65) gives the following equation for the dispersion relation

$$\omega^2(1+R) - \frac{k^2}{3} + \frac{i\omega}{\dot{\tau}} \left[\omega^2 R^2 + \frac{8k^2}{27} \right] = 0$$

To leading order in $1/k\dot{\tau}$,

$$\omega^2 = k^2 c_s^2$$

Writing $\omega = \omega + \delta\omega$ gives

$$\delta\omega = -i \frac{k^2}{2(1+R)\dot{\tau}} \left[c_s^2 R^2 + \frac{8}{27} \right]$$

and

$$\Theta_0, \Theta_1 \sim \exp \left\{ ik \int d\eta c_s \right\} \exp \left\{ \frac{-k^2}{k_D^2} \right\}$$

where

$$\frac{1}{k_D^2} = \int_0^\eta d\eta' \frac{1}{6(1R)n_e\sigma_T a(\eta')} \left[\frac{R^2}{1+R} + \frac{8}{9} \right] \sim \frac{\eta}{n_e\sigma_T a}$$

A first estimate of the damping scale in the prerecombination regime is to consider that all electrons, except those bound in helium are free. The mass fraction of helium, noted X_4 , is approximately 0.24. The ratio of helium to the total number of nuclei is then $X_4/4$ and each helium atom has two electrons. When counting the number of free electrons before hydrogen recombination we must multiply our previous estimate by $1 - X_4/2$,

$$n_e \sigma_T a = 2.3 \times 10^{-5} \text{Mpc}^{-1} \Omega_B h^2 a^{-2} \left(1 - \frac{X_4}{2}\right)$$

(

It is left as an exercise to derive an estimate for the damping scale

$$k_D^{-2} \approx 3 \times 10^6 \text{Mpc}^2 a^{5/2} \frac{1}{\Omega_b h^2} \left(1 - \frac{X_4}{2}\right)^{-1} \frac{1}{\Omega_m h^2}$$

From inhomogeneities to anisotropies

The purpose of this section is to relate inhomogeneities in radiation to anisotropies seen on the sky.

Free streaming

The strategy is a bit involved. To summarize, we integrate formally the equation for temperature inhomogeneities Eq(57) and eventually express the solution in term of the monopole, dipole and baryon velocity field. Hold your breath.

Rewrite Eq.(57) as

$$\dot{\Theta} + (ik\mu - \dot{\tau})\Theta \equiv e^{-ik\mu\eta + \tau} \frac{d}{d\eta} [\Theta e^{ik\mu\eta - \tau}] = \tilde{\mathcal{S}}$$

where

$$\tilde{\mathcal{S}} = -\dot{\Phi} - ik\mu\Psi - \dot{\tau} \left[\Theta_0 + \mu v_b - \frac{1}{2} \mathcal{P}_2(\mu)\Pi \right]$$

Integrating this equation gives, formally,

$$\Theta(\eta_0) = \Theta(\eta_{\text{init}}) e^{ik\mu(\eta_{\text{init}} - \eta_0)} e^{-\tau(\eta_{\text{init}})} + \int_{\eta_{\text{init}}}^{\eta_0} d\eta \tilde{\mathcal{S}}(\eta) e^{ik\mu(\eta - \eta_0) - \tau(\eta)} \quad (67)$$

using $\tau(\eta_0) = 0$.

If the initial time is early enough (ie before recombination) $\tau(\eta_{\text{init}})$ is large. In this limit, the first term in Eq.(67) vanishes (Compton scattering erases the primordial inhomogeneities).

The equation becomes

$$\Theta(k, \mu, \eta_0) = \int_0^{\eta_0} d\eta \tilde{S}(k, \mu, \eta) e^{ik\mu(\eta-\eta_0) - \tau(\eta)}$$

We can turn this into an equation for the moments today, multiplying by the Legendre polynomials and integrating over μ .

$$\Theta_l(k, \eta_0) = \frac{1}{(-i)^l} \int_{-1}^1 \frac{d\mu}{2} \int_0^{\eta_0} d\eta \mathcal{P}_l(\mu) e^{ik\mu(\eta-\eta_0)} e^{-\tau(\eta)} \tilde{S}(k, \mu, \eta)$$

Now \tilde{S} contains various terms, some of which don't depend on μ , like $\dot{\Phi}$. For these, the integral over μ simply gives spherical Bessel functions

$$\int_{-1}^1 \frac{d\mu}{2} \mathcal{P}_l(\mu) = \frac{1}{(-i)^l} j_l(k(\eta - \eta_0))$$

Terms which are linear in μ can be treated by integrating by part, replacing μ by $\frac{-i}{k} \frac{d}{d\eta}$ acting on $\exp(ik\mu\eta)$. For instance

$$-ik \int_0^{\eta_0} d\eta \mu \Psi e^{ik\mu(\eta-\eta_0) - \tau(\eta)} = \int_0^{\eta_0} d\eta e^{ik\mu(\eta-\eta_0)} \frac{d}{d\eta} [\Psi e^{-\tau(\eta)}]$$

Notice that the term from the integration by part can be dropped. The contribution at small η is damped by $\exp(-\tau(0))$. The contribution today is not small but it is independent of μ . Hence it only contributes to the monopole *here and now* which, by definition, is unobservable.

Putting everything together, we can do the integral over μ and the solution becomes

$$\Theta_l(k, \eta_0) = \int_0^{\eta_0} \mathcal{S}(k, \eta) j_l[k(\eta - \eta_0)]$$

with *

$$\begin{aligned} \mathcal{S}(k, \eta) = & e^{-\tau} \left[-\dot{\Phi} - \dot{\tau} \left(\Theta_0 + \frac{1}{4}\Pi \right) \right] \\ & + \frac{d}{d\eta} \left[e^{-\tau} \left(\psi - \frac{iv_b \dot{\tau}}{k} \right) \right] - \frac{3}{4k^2} \frac{d^2}{d\eta^2} \left[e^{-\tau} \dot{\tau} \Pi \right] \end{aligned}$$

The combination

$$\boxed{g(\eta) = -\dot{\tau} e^{-\tau}} \quad (68)$$

is called the *visibility function*. It is the probability density of last scattering at time η . Note that

$$\int_0^{\eta_0} g(\eta) d\eta = 1$$

If we drop the polarization term Π (small in practice, see later?) the source term becomes

$$\begin{aligned} \mathcal{S}(k, \eta) \approx & g(\eta) [\Theta_0(k, \eta) + \Psi(k, \eta)] + \frac{d}{d\eta} \left(\frac{iv_b(k, \eta)g(\eta)}{k} \right) \\ & + e^{-\tau} [\dot{\Psi}(k, \eta) - \dot{\Phi}(k, \eta)] \end{aligned}$$

*Using $j_l(x) = (-)^l j_l(-x)$

Putting it back in the integral over time (and integrating by part) gives

$$\begin{aligned}\Theta_l(k, \eta_0) &= \int_0^{\eta_0} d\eta g(\eta) [\Theta_0(k, \eta) + \Psi(k, \eta)] j_l [k(\eta_0 - \eta)] \\ &\quad - \int_0^{\eta_0} d\eta g(\eta) \frac{iv_b(k, \eta)}{k} \frac{d}{d\eta} j_l [k(\eta_0 - \eta)] \\ &\quad + \int_0^{\eta_0} d\eta e^{-\tau} [\dot{\Psi}(k, \eta) - \dot{\Phi}(k, \eta)] j_l [k(\eta_0 - \eta)]\end{aligned}$$

Two kinds of term. The last one contains $e^{-\tau}$, hence contributes as long as τ is small, that is all the time after last scattering. Also this term vanishes if the potentials are constant. This is what happens in many models, for instance in a purely matter dominated universe. Here, these terms are not fundamental. However this is a gauge dependent statement. In many application this so-called integrated Sachs-Wolfe effect plays a more fundamental role. See later.

The first two terms are peaked at last scattering, which simplifies evaluating the integrals. For not too small scales, the functions that multiply the visibility function are slowly varying. (See figure). Using the normalization of the visibility function gives

$$\begin{aligned}\Theta_l(k, \eta_0) &\approx [\Theta_0(k, \eta_*) + \Psi(k, \eta_*)] j_l [k(\eta_0 - \eta_*)] \tag{69} \\ &\quad + 3\Theta_1(k, \eta_*) \left[j_{l-1}([k(\eta_0 - \eta_*)]) - \frac{(l+1)j_l(k(\eta_0 - \eta_*))}{k(\eta_0 - \eta_*)} \right] \\ &\quad + \int_0^{\eta_0} d\eta e^{-\tau} [\dot{\Psi}(k, \eta) - \dot{\Phi}(k, \eta)] j_l [k(\eta - \eta_0)]\end{aligned}$$

using

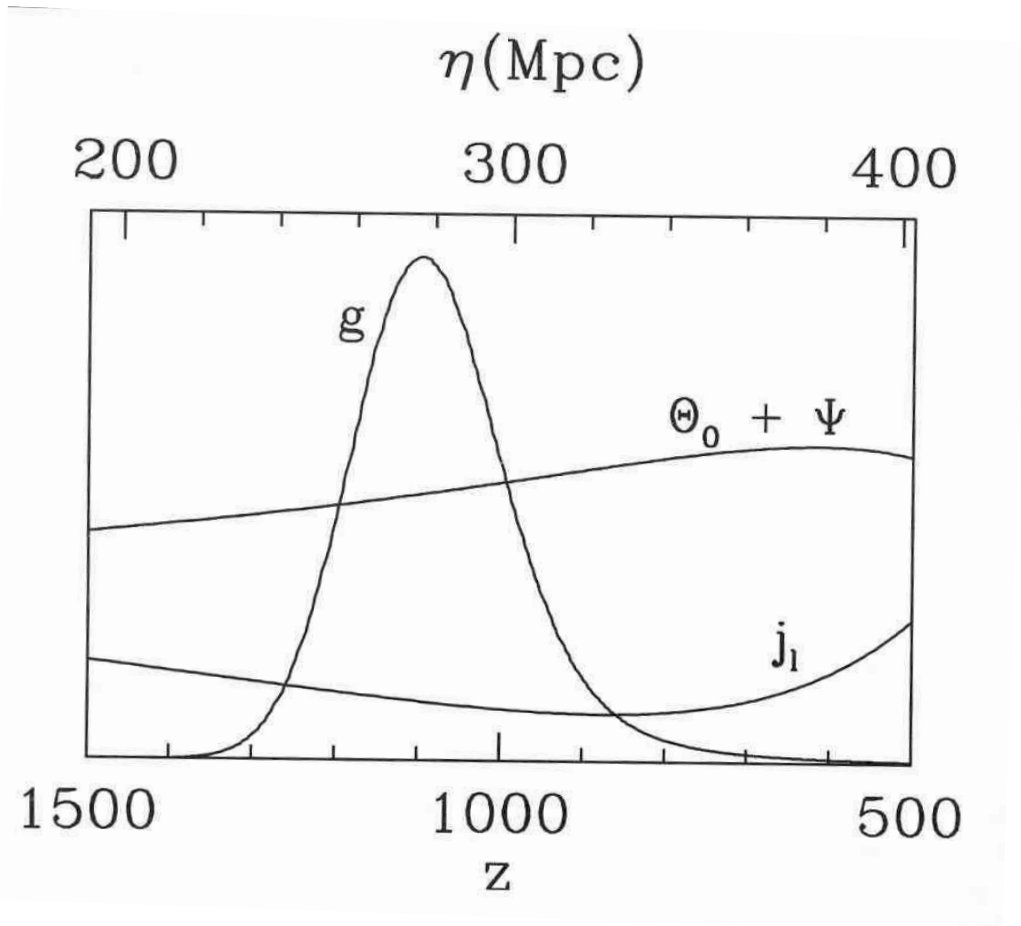
$$\frac{d}{dx} j_l = j_{l-1} - \frac{l+1}{x} j_l$$

and the tight coupling approximation

$$v_1 \approx -3i\Theta_1$$

at last scattering.

For small scales, it is possible to improve things, but I don't cover this here.



This Eq.(69) is our final result. The first term is what should be expected. It is simply the projection over the sphere of the monopole (plus gravitational potential at last scattering). On small scales

$$\lim_{l \rightarrow \infty} j_l(x) = \frac{1}{l} \left(\frac{x}{l} \right)^{l-1/2}$$

ie it is small for large l when $x < l$ so that $\Theta_l(k, \eta_0)$ is very small for $l > k\eta_0$.

Relation to temperature power spectrum?

Temperature field of the universe:

$$T(\vec{x}, \hat{p}, \eta) = T(\eta) [1 + \Theta(\vec{x}, \hat{p}, \eta)]$$

We measure the temperature here and now. Our only handle is the direction in which the temperature field is observed.

In spherical harmonics

$$\Theta(\vec{x}, \hat{p}, \eta) = \sum_{l=1}^{\infty} \sum_{m=-l}^l a_{lm}(\vec{x}, \eta) Y_{lm}(\hat{p})$$

or, from the temperature perturbation in comoving momentum space,

$$a_{lm}(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d\Omega Y_{lm}^*(\hat{p}) \Theta(\vec{k}, \hat{p}, \eta)$$

The mean of the a_{lm} coefficient vanishes by construction of temperature perturbations but they have a non-zero variance. It is assumed (and somewhat a necessity of inflation) that the distribution is Gaussian, so that the distribution is entirely characterized by the variance.

$$\langle a_{lm} \rangle = 0 \quad \text{and} \quad \langle a_{lm} a_{l'm'}^* \rangle = \delta_{ll'} \delta_{mm'} C_l$$

The average is supposed to be drawn from a distribution of perturbations, for instance as predicted by inflation. In practice, we have only one universe to sample. For a given l , we have however a sample of $2l + 1$ a_{lm} 's. It is thus crucial that the variance is the same for all the a_{lm} . This **cosmic variance**, which limits the statistical significance of our sample at low l 's, scales like the square root of the number of samples

$$\frac{\Delta C_l}{C_l} = \sqrt{\frac{2}{2l+1}}$$

We can now obtain the C_l in terms of $\Theta_l(k)$. The correlator

$$\langle \Theta(\vec{k}, \hat{p}) \Theta^*(\vec{k}', \hat{p}') \rangle$$

is a complicated object, which depends on the initial perturbations, for instance as created during inflation, and of their evolution toward us. To separate these questions, we write

$$\begin{aligned} \langle \Theta(\vec{k}, \hat{p}) \Theta^*(\vec{k}', \hat{p}') \rangle &= \langle \delta(\vec{k}) \delta^* \delta(\vec{k}') \rangle \frac{\Theta(\vec{k}, \hat{p})}{\delta(\vec{k})} \frac{\Theta^*(\vec{k}', \hat{p}')}{\delta(\vec{k}')} \\ &= (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P(k) \frac{\Theta(\vec{k}, \hat{p})}{\delta(\vec{k})} \frac{\Theta^*(\vec{k}', \hat{p}')}{\delta(\vec{k}')} \end{aligned}$$

The ratio

$$\frac{\Theta(\vec{k}, \hat{p})}{\delta(\vec{k})} = \frac{\Theta(k, \vec{k} \cdot \hat{p})}{\delta(k)}$$

is independent of the initial conditions (fixed for instance by inflation) so can be taken out of the average and depend only on k and the scalar product $\vec{k} \cdot \hat{p}$.

Altogether

$$\begin{aligned}
C_l &= \int \frac{d^3k}{(2\pi)^3} P(k) \int d\Omega Y_{lm}^*(\hat{p}) \frac{\Theta(k, \hat{k} \cdot \hat{p})}{\delta(k)} \int d\Omega' Y_{lm}^*(\hat{p}') \frac{\Theta^*(k', \hat{k}' \cdot \hat{p}')}{\delta(k')} \\
&= \int \frac{d^3k}{(2\pi)^3} P(k) \sum_{l'} (-i)^{l'} i^{l''} (2l' + 1)(2l'' + 1) \frac{\Theta_{l'}(k) \Theta_{l''}^*(k)}{|\delta(k)|^2} \\
&\quad \times \int d\Omega \mathcal{P}_{l'}(\hat{k} \cdot \hat{p}) Y_{lm}^*(\hat{p}) \int d\Omega' \mathcal{P}_{l''}(\hat{k}' \cdot \hat{p}') Y_{lm}(\hat{p}')
\end{aligned}$$

Using

$$\int d\Omega \mathcal{P}_l(\hat{k} \cdot \hat{p}) Y_{lm}(\hat{p}) = \delta_{ll} 4\pi \frac{Y_{lm}(\hat{k})}{2l + 1}$$

finally gives

$$\boxed{C_l = \frac{2}{\pi} \int_0^\infty dk k^2 P(k) \left| \frac{\Theta_l(k)}{\delta(k)} \right|^2} \quad (70)$$

The anisotropy spectrum today

The Sachs-Wolfe effect

We consider first large scale anisotropies. On scales larger than the horizon at recombination, only the monopole contributes to the anisotropy, see Eq.(63).

The large solution of the combined Einstein/Boltzmann equations gave

$$\Theta_0(\eta_*) + \Psi(\eta_*) = \frac{\Psi(\eta_*)}{3} \approx -\frac{\Phi(\eta_*)}{3}$$

taking into account the fact that anisotropic stress is small at late times.

We need to evaluate the ratio $\Theta(\eta_0)/\delta(\eta_0)$. To do so, we first relate the potential at the time of decoupling to the potential today using

$$\Phi(\eta_0) = D_1(\eta_0)\Phi(\eta_*)$$

(remember, $D_1(\eta_0) = 1$ in a flat, matter dominated universe.)

Last, the potential today can be related to the dark matter perturbation today (as computed using linear perturbation theory !) using the Poisson equation

$$\Phi(\eta_0, k) = \frac{3\Omega_m H_0^2}{2k^2} \delta(\eta_0, k)$$

Finally, in all its glory, we have the famous result of Sachs and Wolfe for the anisotropies on large scales

$$C_l^{SW} \approx \frac{\Omega_m^2 H_0^4}{2\pi D_1^2(a=1)} \int_0^\infty \frac{dk}{k^2} j_l^2(k(\eta_0 - \eta_*)) P(k) \quad (71)$$

The power spectrum on large scales is given by

$$P(k) = 2\pi^2 \delta_H^2 \frac{k^h}{H_0^{n+3}}$$

since the transfer function equal one in this regime. The integral over the Bessel functions can be done analytically.

$$C_l^{SW} \approx 2^{n-4} \pi^2 (\eta_0 H_0)^{1-n} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)} \right)^2 \frac{\Gamma(l + \frac{n}{2} - \frac{1}{2}) \Gamma(3 - n)}{\Gamma(l + \frac{5}{2} - \frac{n}{2}) \Gamma^2(2 - \frac{n}{2})}$$

If the spectrum is of the Harrison-Zel'dovich-Peebles type, $n = 1$,

$$l(l+1)C_l^{SW} = \frac{\pi}{2} \left(\frac{\Omega_m}{D_1(a=1)} \right)^2 \delta_H^2$$

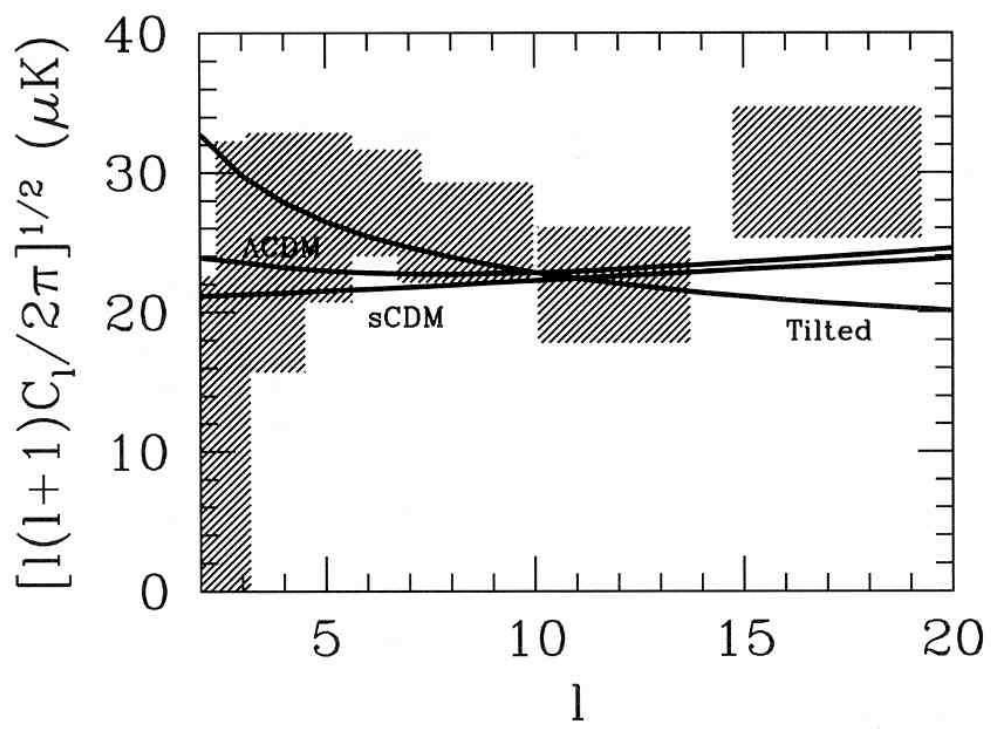
is constant. This is the Sachs-Wolfe plateau.

Best fits to COBE give, for instance, for $n = 1$

$$\delta_H = 1.9 \times 10^{-5} \quad \text{for} \quad \Omega_m = 1$$

and

$$\delta_H = 4.6 \times 10^{-5} \quad \text{for} \quad \Omega_m = 0.3, \Omega_\Lambda = 0.7$$



Small Scales

On smaller scales, the spectrum depends also on the dipole and on the integrated Sachs-Wolfe effect.

Monopole

Free-streaming of the monopole gives anisotropies on scales $l \sim k\eta_0$. There are two interesting effects. First there are many k -modes which contribute to the anisotropy on a given l -scale. This effect turns the zero of the spectrum into troughs. There is also a shift in the minima of the spectrum because the Bessel function in Eq(69). A better approximation to the position of the first peak is

$$l_p \approx 0.75 \frac{\pi\eta_0}{r_s}$$

Dipole

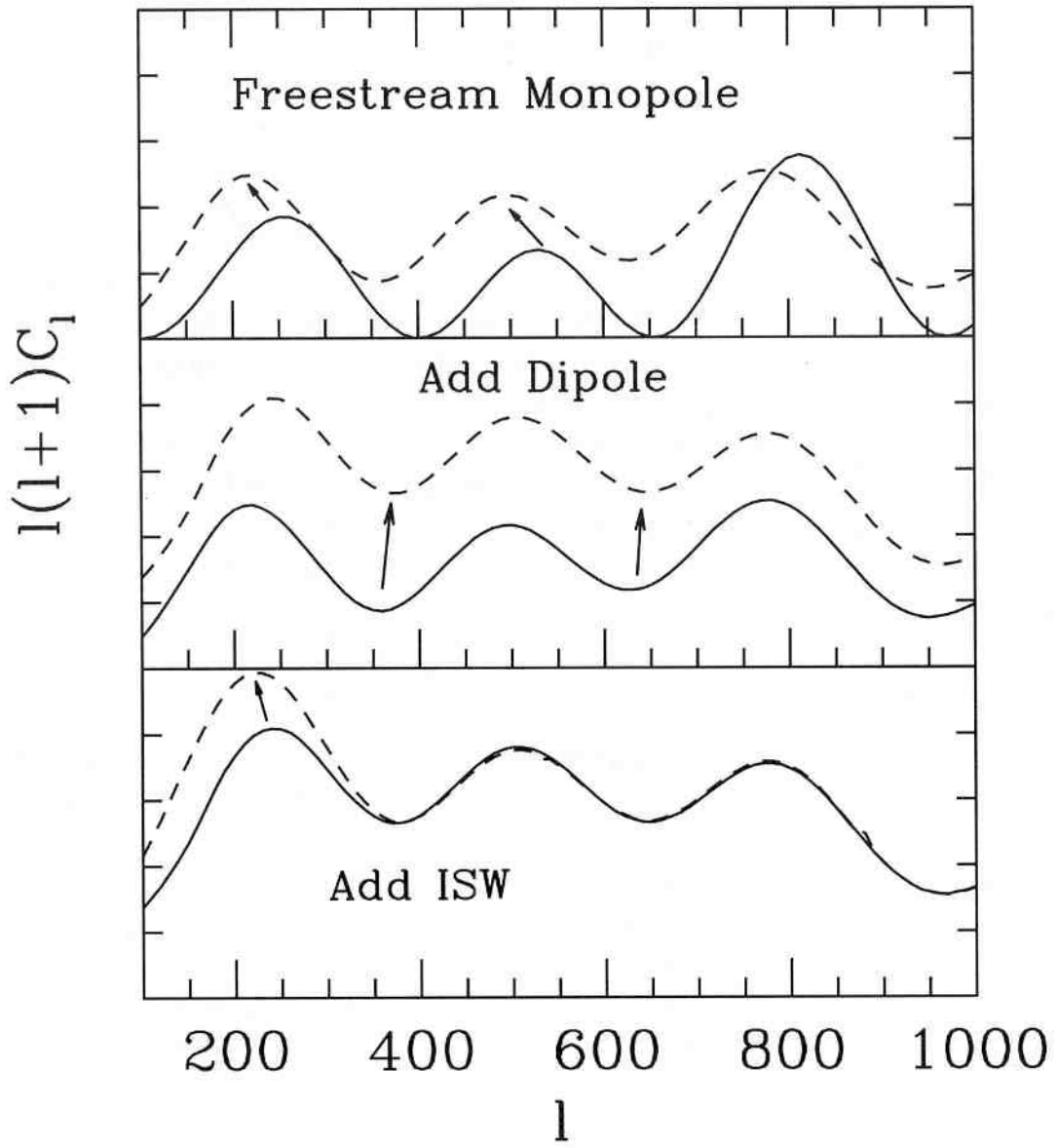
The dipole is slightly smaller and also is out of phase wrt to the monopole. Adding the dipole fills in the troughs. One important point is that the monopole and dipole don't interfere in the spectrum. The mixed term vanishes when integrated over k . This is why the dipole is less important than naively expected.

Integrated Sachs-Wolfe effect

This is most important on the scale of the horizon at recombination. An important instance is that of residual radiation domination at recombination. This leads to an ISW right after recombination, a so called early ISW effect. It is important because if the integral in Eq.(69) is dominated by the region $\eta \approx \eta_*$, the ISW effect adds up coherently with the monopole effect (same Bessel functions). Integrating by part in Eq.(69) gives

$$\Theta_l(k, \eta_0)^{\text{earlyISW}} = [(\Psi(k, \eta_0) - \Psi(k, \eta_*)) - (\Phi(k, \eta_0) - \Phi(k, \eta_*))] j_l(k(\eta_0 - \eta_*))$$

which adds up with the first term of Eq.(69).



Cosmological Parameters

- Curvature density $= 1 - \Omega_m - \Omega_\Lambda$
- Normalization C_{10}
- Primordial tilt n
- Tensor modes r
- Reionization, parameterized by τ
- Baryon density $\Omega_b h^2$
- Matter density $\Omega_m h^2$
- Cosmological constant Ω_Λ

Curvature

In an open universe, for instance, the all other things being kept constant, the position of the first peak is moved to higher multipoles. The opposite in a closed universe.

This effect depends on the comoving angular diameter distance to the last scattering surface. This is $\eta_0 - \eta_*$ in a flat universe. Otherwise

$$d_A(z^*)(1+z^*) = \frac{1}{H_0 \sqrt{|\Omega_k|}} \begin{cases} \sinh(\sqrt{\Omega_k} H_0 (\eta_0 - \eta_*)) & \Omega_k > 1 \\ \sin(\sqrt{-\Omega_k} H_0 (\eta_0 - \eta_*)) & \Omega_k < 1 \end{cases} \quad (72)$$

The angular distance scales by a factor of $(1 - \Omega_k)^{-0.45}$ where $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$.

Degenerate parameters

Normalization: The parameter simply C_{10} moves the curves up and down.

Tilt: if $n < 1$, the small scales anisotropies are smaller than if $n = 1$. The effect of tilt is thus more pronounced on small scales.

Reionization: the universe was reionized at late times. This is seen in the absorption lines of high redshift quasars. Hydrogen ionized gas up to $z \sim 6$. Early reionization washes away the spectrum.

Tensors: the amplitude of gravitational modes dies away as they enter the horizon. Only important on scales larger than the horizon at recombination. If present, they lift the large scales part of the spectrum. For fixed large scales, the presence of tensor modes lowers the amplitude of the peaks.

Distinct imprint: Tegmark's movie captions

<http://www.hep.upenn.edu/~max/cmb/movie>

Curvature

Spatial curvature was completely irrelevant at $z > 1000$, when the acoustic oscillations were created. Ω_k therefore doesn't change the shape of the peaks at all - it merely shifts them sideways, since the conversion from the physical scale of the wiggles (in meters) into the angular scale (in degrees, or multipole l) depends on whether space is curved. If space has negative curvature (positive Ω_k), like a Pringles potato chip or a saddle, then the angle subtended on the sky decreases, shifting the peaks to the right. If space has positive curvature, like a balloon, then the peaks shift to the left. Meanwhile, you'll see that the galaxy power spectrum shifts both horizontally (in the opposite direction) and vertically. Let's start with the vertical part. Curvature slows the growth of fluctuations and therefore lowers the curve. The horizontal shift wouldn't occur if the horizontal axis had physical distance units on it (meters or Mpc, say). But it uses astronomical distance units (Mpc/h), so changing the Hubble parameter h shifts the curve. Did somebody say h ? Weren't we just changing the curvature? No! Of the 12 parameters shown, only 11 are independent. So when we change the curvature with Λ and all the matter densities fixed, then the Hubble constant automatically changes too. Finally, if you've patiently read this far, let's return to the CMB (top) panel. In addition to shifting the peaks sideways, curvature also causes fluctuations in the gravitational field to decrease over time. This means that if a photon flies through a potential well on the way to us, its blueshift from falling in will exceed its redshift from climbing out. Since these extra fluctuations in the photon temperature (known as the late ISW effect) happen at late times, they show up on large scales (to the left in the power spectrum).

Baryons

This movie keeps the total matter density constant, but increases the fraction of ordinary matter, baryons (see the two dials shift in sync). Density oscillations caused by the competition between pressure and gravity in the early Universe give rise to wiggles "acoustic peaks") in both the CMB and galaxy power spectra - the more baryons, roughly speaking. However, it's actually mainly the odd-numbered peaks (1, 3, etc) that get boosted, which helps distinguish baryons from cold dark matter. Indeed, you can see that the 2nd peak actually shrinks for a while. This is why the Boomerang and Maxima measurements prefer lots of baryons - they don't see much of a 2nd peak. For more details on the baryon effect, see Wayne's CMB tutorial. Note that both the CMB and galaxy fluctuations are unaffected on the largest (leftmost) scales. Moreover, note that the galaxy power spectrum doesn't merely get wigglier, but gets suppressed as well.

Lambda

The cosmological constant was completely irrelevant at $z > 1000$, when the acoustic oscillations were created. It therefore doesn't change the shape of the peaks at all - it merely shifts them sideways, since it affects the conversion from the physical scale of the wiggles (in meters) into the angular scale (in degrees, or multipole l). Not that this is analogous to the effect of increasing curvature, but goes in the opposite direction. In addition to shifting the peaks sideways, a cosmological constant also causes fluctuations in the gravitational field to decrease over time, like spatial curvature in the previous movie. This means that if a photon flies through a potential well on the way to us, its blueshift from falling in will exceed its redshift from climbing out. Since these extra fluctuations in the photon temperature (known as the late ISW effect) happen at late times, they show up on large scales (to the left in the power spectrum). The effect of Lambda on the galaxy power spectrum (bottom) is also similar to that of Ω_k : it slides the curve vertically (by slowing the growth of fluctuations) and horizontally (by changing the Hubble parameter h).

Hot dark matter

If a CMB theorist gloats that he or she can measure the neutrino density, make sure to point out that galaxy surveys are much more sensitive. As you can see, the CMB power spectrum (top) changes only very weakly as you replace cold dark matter by hot. This is because the neutrinos were already quite cold (nonrelativistic) the time the CMB fluctuations formed. The effect on the galaxy power spectrum (bottom) is seen to be much more dramatic, effectively erasing small-scale power. This will happen even if the neutrinos are fairly cold (moving much slower than the speed of light). A speed of a few hundred kilometers per second is sufficient to escape from a Galaxy halo, thereby erasing structure on that scales.

Scalar tilt

Changing the spectral index of scalar fluctuations simply tilts both power spectra, altering the ratio of small-scale and large-scale power. This is one of the few parameters that has a similar effect of both curves.

Appendix

Units and cosmological parameters

$$\begin{aligned}
 M_{\text{Pl}} &= \frac{1}{\sqrt{8\pi G_{\text{N}}}} = 4.3 \times 10^{-6} \text{g} = 2.4 \times 10^{-18} \text{GeV} && \text{reduced Planck mass} \\
 T_{\text{Pl}} &= 2.7 \times 10^{-43} \text{s} && \text{Planck time} \\
 L_{\text{Pl}} &= 8.1 \times 10^{-33} \text{cm} && \text{Planck length} \\
 H_0 &= 100h \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1} \approx \frac{h}{3000} \text{Mpc}^{-1} \\
 H_0^{-1} &= 9.78h^{-1} \text{Gyr} = 2998h^{-1} \text{Mpc} \\
 \rho_{\text{c}} &= 3M_{\text{Pl}}^2 H^2 = 2.775h^{-1} \times 10^{11} \frac{M_{\odot}}{(\text{h}^{-1} \text{Mpc})^3} && \text{critical density} \\
 &= 1.88h^2 \times 10^{-29} \text{g} = (3 \times 10^{-3} \text{eV})^4 h^2 \\
 \Omega_{\gamma,0} h^2 &= 2.48 \times 10^{-5} && \text{photon density} \\
 \Omega_{\text{R},0} h^2 &= 4.17 \times 10^{-5} && \text{three massless neutrinos} \\
 1 + z_{\text{eq}} &= \frac{\Omega_0}{\Omega_{\text{R},0}} = 24000 \Omega_0 h^2 \\
 \frac{1}{a_{\text{eq}} H_{\text{eq}}} &= 14 \Omega_0^{-1} h^{-2} \text{Mpc} && \text{comoving Hubble length at equality}
 \end{aligned}$$

Distances

An object at comoving distance χ is today at proper or physical distance

$$d_p = a\chi$$

Can this distance be measured? How is it related to observations of distant objects? Because looking far is looking in the past, in an expanding universe the notion of distance is not quite immediate. Here we consider two practical ways of measuring distances and compare them to the definition of d_p .

Luminous distance

Definition: the **absolute luminosity** L of an object is the energy it radiates per unit of time. This energy could be in a range of frequencies or integrated over the whole spectrum of radiation. We consider only the latter here. Its **apparent luminosity** l is the energy received per unit area per unit of time.

Suppose you know the absolute luminosity L of a distant object (galaxy, quasar,...) and measure its apparent luminosity F equal to the flux of energy per steradian. In Euclidean space

$$F = \frac{L}{4\pi d_L^2}$$

We can thus define the luminous distance by

$$d_L = \left(\frac{L}{4\pi F} \right)^{\frac{1}{2}}$$

This definition applies to an expanding universe, provided we understand of the apparent luminosity is affected by expansion.

Using comoving distance, in a flat static universe, the flux we observe is

$$F = \frac{L}{4\pi\chi^2}$$

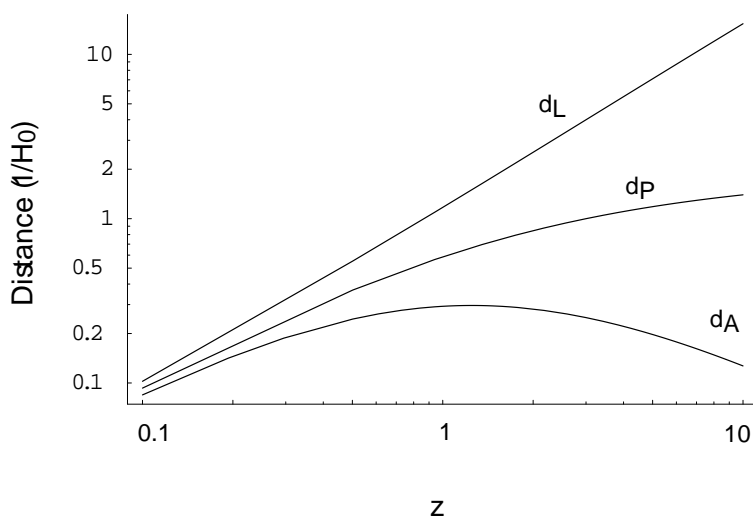
In an expanding universe, this result is modified by two effects. First because of expansion the energy of a received photon is redshifted by a factor of $a = 1/(1+z)$, the scale factor at the time of emission. Next, the rate of photon reception is decreased compared to the rate of emission by another factor of a . Altogether, the flux of energy received is

$$F = \frac{La^2}{4\pi\chi^2}$$

This is legitimate to use

$$d_L = \frac{\chi}{a} = (1+z)\chi$$

as the luminosity distance. This is a factor of $1/a$ larger than the physical distance today $d_P(a) = \chi(a)$. Objects look dimmer in an expanding universe.



Angular diameter distance

As an alternative definition of distance, suppose we know the physical size D of an object. In Euclidean space, the apparent diameter δ is

$$\delta = \frac{D}{d_A}$$

if the object is at distance d_A . Thus we define

$$d_A = \frac{D}{\delta}$$

and asked how D and δ are affected in an expanding universe.

In a flat expanding universe, the comoving size is D/a and the comoving distance χ . Hence

$$\theta = \frac{D}{a\chi}$$

and

$$d_A = a\chi = \frac{\chi}{1+z}$$

Consider for instance a flat matter dominated universe. Then the comoving distance to a an object at redshift z is

$$\chi = \int_{t(a)}^{t_0} \frac{dt}{a(t)} = \int_a^1 \frac{da}{a^2 H} = \frac{2}{H_0} \left[1 - \frac{1}{\sqrt{1+z}} \right]$$

The comoving distance is $\chi = z/H_0$ for small z but then asymptots to $2/H_0$ for large z . This is the size of the particle horizon today. A consequence of this is that, if iniatially the angular distance increases at small redshift, it start to decreases around $z \sim 1$: the apparent diameter of an object increases with distance! Note that this is peculiar conclusion arises because an object which has intrinsic physical size l has a comoving size which **increases** with redshift $l(1+z)$.

In a curved universe,

$$d_A(z) = \frac{1}{(1+z)H_0\sqrt{|\Omega_k|}} \begin{cases} \sinh(\sqrt{\Omega_k}H_0\chi) & \Omega_k > 1 \\ \sin(\sqrt{-\Omega_k}H_0\chi) & \Omega_k < 1 \end{cases} \quad (73)$$

where $\Omega_k = 1 - \Omega = -K/H_0^2$.

Bessel functions

Bessel functions $J_n(z)$ and $Y_n(z)$ are linearly independent solutions of the differential equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - n^2)y = 0$$

For integer n , the $J_n(z)$ are regular at $z = 0$, while the $Y_n(z)$ have a logarithmic divergence at $z = 0$.

The Hankel functions (or Bessel functions of the third kind) $H_n^{(1,2)}(z) = J_n(z) \pm iY_n(z)$ give an alternative pair of solutions to the Bessel differential equation.

Stress-energy tensor

Heuristic argument. Consider a conserved current j^μ . Consider a bunch of particles of charge q with momentum centered around some \vec{p} . In their rest frame, their density is

$$\rho(\vec{r}) = qf \frac{d^3p}{(2\pi)^3}$$

and

$$\vec{j}(\vec{r}) = 0$$

In an arbitrary frame then

$$j^\mu = qf P^\mu \frac{d^3p (2\pi)^3}{E}$$

The last term is a scalar under boosts. Integrating over momentum gives the total current

$$J^\mu = \int \frac{d^3p}{(2\pi)^3} \frac{P^\mu}{P^0}$$

To get the stress-energy tensor, simply replace q by P_ν ,

$$T^\mu{}_\nu = \int \frac{d^3p}{(2\pi)^3} \frac{P^\mu P_\nu}{P^0}$$

Then

$$T^0{}_0 = - \int \frac{d^3p}{(2\pi)^3} E f = -\rho$$

From the definition of the stress-energy tensor

$$T^\mu{}_\nu = (\rho + P)u^\mu u_\nu + P\delta^\mu{}_\nu + \Sigma^\mu{}_\nu$$

Recombination

As long as the reaction



remain in equilibrium, the chemical potentials satisfy

$$\mu_e + \mu_p = \mu_H$$

and the ratio

$$\frac{n_e n_p}{n_H}$$

is independent of the chemical potentials. It is more convenient to work with mass fractions, since those don't scale with the expansion of the universe. Defining the free electron fraction

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}$$

where we use neutrality to set $n_e = n_p$, then

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_\gamma \eta} \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-Q/T}$$

where $Q = m_p + m_e - m_H = 13.6\text{eV}$ and $\eta = n_B/n_\gamma = (n_p + n_H)/n_\gamma \sim 10^9$ is the baryon number of the universe (neglecting a small contribution from helium). Finally we made the standard approximation of replacing the ratio m_p/m_H by one in the prefactor.

In this Saha equation there is an entropic prefactor

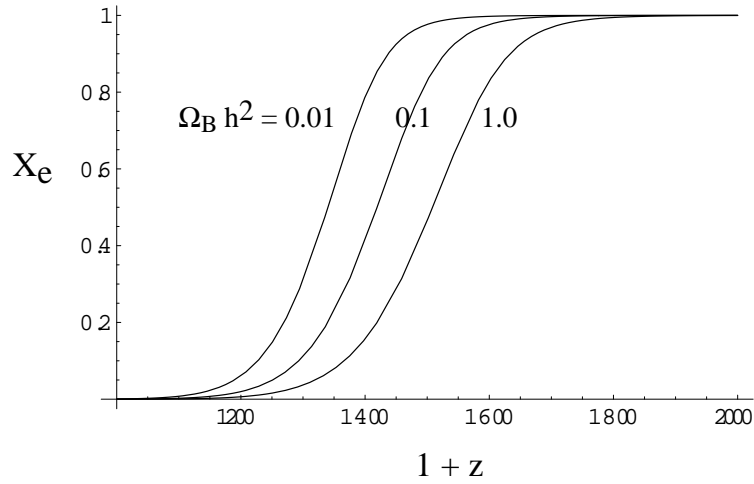
$$10^9 \left(\frac{m_e}{T} \right)^{3/2} \sim 10^{15}$$

When the temperature is of order Q , this implies $X_e \approx 1$, so that all hydrogen is ionized. Only when the temperature drops well below Q can X_e be small.

We can estimate the recombination temperature and redshift by deciding that it occurs when 90 percent of the electrons are bound into hydrogen atoms. This turns out to depend on the baryon number. (See figure). A good order of magnitude is

$$T_{\text{rec}} = 0.3\text{eV}$$

corresponding to $1 + z_{\text{rec}} = 1300$



As X_e falls, the rate of interaction also falls, so that equilibrium is more difficult to maintain. To follow the electron fraction, we then need to solve the Boltzmann equation.

This equation reads

$$\frac{d(n_e a^3)}{a^3 dt} = n_e^{(0)} n_p^{(0)} \langle \sigma v \rangle \left[\frac{n_H}{n_H^{(0)}} - \frac{n_e^2}{n_e^{(0)} n_p^{(0)}} \right]$$

Using

$$\frac{d(n_b a^3)}{dt} = 0$$

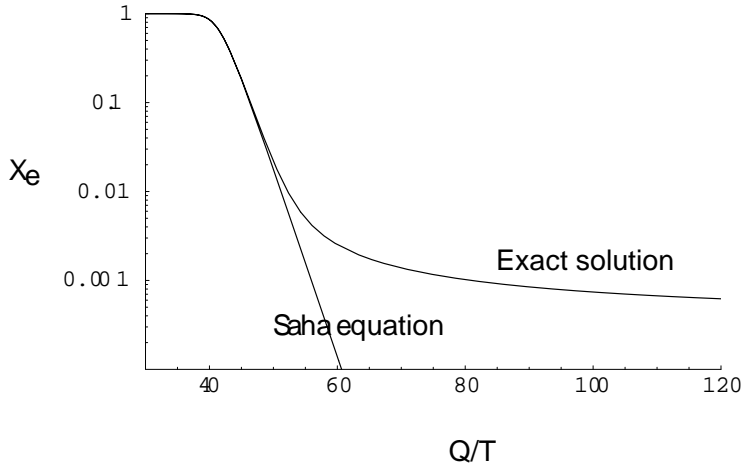
we can express the Boltzmann equation in term of X_e

$$\frac{dX_e}{dt} = \left[(1 - X_e) \langle \sigma v \rangle_{(2)} \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-Q/T} - X_e^2 \langle \sigma v \rangle_{(2)} n_b \right]$$

The subscript (2) refers to the fact that recombination to the ground state ($n = 1$) is essentially irrelevant. Ground state recombination leads to the production of a photon which in turn can ionize a neutral atom, with zero net effect (but for a slight effect due to redshifting). Recombination can proceed through capture into an excited ($n = 2$) state and its subsequent decay into two photons. A good approximation is

$$\langle \sigma v \rangle_{(2)} = 9.78 \frac{\alpha^2}{m_e^2} \left(\frac{Q}{T} \right)^{1/2} \ln \left(\frac{Q}{T} \right)$$

A numerical resolution of the Boltzmann equation is shown below, in comparison with the equilibrium solution (Saha) for $\Omega_m = 1, h = 0.5$ and $\Omega_B = 0.06$.



At the time of recombination, the photons will also decouple from electrons. This decoupling occurs roughly when the optical depth becomes less than one. This translates into asking that the rate for photons to Compton scatter off electrons becomes smaller than the expansion rate,

$$n_e \sigma_T \ll H$$

where

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = .665 \times 10^{-24} \text{cm}^2$$

is the Thomson cross-section. For a universe with a mix of matter and radiation,

$$\frac{n_e \sigma_T}{H} = 113 X_e \left(\frac{\Omega_B h^2}{0.02} \right) \left(\frac{0.15}{\Omega_m h^2} \right) \left(\frac{1+z}{1000} \right)^{3/2} \left[1 + \frac{1+z}{3600} \frac{0.15}{\Omega_m h^2} \right]^{-1/2}$$

During recombination, the free electron fraction drops below 10^{-3} . So decoupling takes place during recombination.

It is interesting to ask when would decoupling take place if the universe was ionized throughout its history. Assuming $X_e = 1$, gives

$$1 + z_{\text{dec}} = 43 \left(\frac{0.02}{\Omega_B h^2} \right)^{2/3} \left(\frac{\Omega_m h^2}{0.15} \right)^{1/3}$$

This quantity is relevant because the universe eventually was reionized when the first stars appeared. If reionization took place earlier than the estimate $z \sim 40$, then all the information would have been washed-out. Fortunately, apparently $z_{\text{ri}} \sim 6$, so we are on the safe side (*ie* the photons scatter very rarely).

Résumé: Boltzmann & Einstein

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2[\rho_m\delta_m + 4\rho_r\Theta_{r,0}] \quad (74)$$

$$k^2(\Phi + \Psi) = -32\pi G a^2\rho_r\Theta_{r,2} \quad (75)$$

where

$$\rho_m\delta_m = \rho_{dm}\delta + \rho_b\delta_b \quad \text{and} \quad \rho_mv_m = \rho_{dm}v + \rho_bv_b$$

$$\rho_r\Theta_{r,0(1)} = \rho_\gamma\Theta_{0(1)} + \rho_\nu\mathcal{N}_{0(1)}$$

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^1 \frac{d\mu}{2} \mathcal{P}_l(\mu)\Theta(\mu)$$

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau} \left[\theta_0 - \theta + \mu v_b - \frac{1}{2}\mathcal{P}_2(\mu)\Pi \right] \quad (76)$$

$$\Pi = \Theta_2 + \Theta_{p2} + \Theta_{p0} \quad (77)$$

$$\dot{\Theta}_p + ik\mu\Theta_p = -\dot{\tau} \left[-\Theta_p + \frac{1}{2}(1 - \mathcal{P}_2(\mu))\Pi \right] \quad (78)$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \quad (79)$$

$$\dot{v} + \frac{\dot{a}}{a}v = -ik\Psi \quad (80)$$

$$\dot{\delta}_b + ikv_b = -3\dot{\Phi} \quad (81)$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi + \frac{\dot{\tau}}{R} [v_b + 3i\Theta_1] \quad (82)$$

$$\dot{\mathcal{N}} + ik\mu\mathcal{N} = -\dot{\Phi} - ik\mu\Psi \quad (83)$$

Conventions and some important definitions

Conformal Newtonian gauge:

$$ds^2 = -(1 + 2\Psi)dt^2 + a(t)^2(1 + 2\Phi)d^2\vec{x}$$

Comoving momentum:

$$P^\mu = \frac{dx^\mu}{d\lambda} = (E, \vec{P})$$

Proper momentum:

$$p^2 = g_{ij}P^iP^j = g^{ij}P_iP_j$$

$$p^i = \hat{p}^i \frac{p}{a}$$

$$P_i = \hat{p}_i a p$$

with $\delta_{ij}\hat{p}^i\hat{p}^j = 1$. Also $\vec{k} \cdot \vec{p} = \delta_{ij}k_i p_j = k p \mu$

Stress-energy tensor in curved spacetime:

$$T^\mu{}_\nu = g \int \prod_{i=1}^3 \frac{dP_i}{2\pi} \frac{1}{\sqrt{-g}P^0} P^\mu P_\nu f(\vec{x}, t)$$

i.e.

$$T^0{}_0 = -g \int \frac{d^3p}{(2\pi)^3} P^0 (\equiv E) f \equiv -\rho$$

$$T^0{}_i = g \int \frac{d^3p}{(2\pi)^3} a p_i f$$

$$\begin{aligned} ik_i T^0{}_i &= a k g \int \frac{d^3p}{2\pi^2} \left(-p \frac{\partial f}{\partial p} \right) \int_{-1}^{+1} \frac{d\mu}{2} \mu i \Theta \\ &= a k 4\rho_r \Theta_1 \end{aligned}$$

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