

Optimal design of the experiment of the century

Richard Gill

Utrecht/Eurandom

`http://www.math.uu.nl/people/gill`

+ Wim van Dam, Peter Grünwald

Quantum vs. classical (=local realist) *theory*:

QM vs. LR

EPR: Einstein, Podolsky, Rosen (1935):

QM is incomplete, if true (assuming LR)

Bell (1964):

LR is false, if QM true

Same, but now with proposed *experiment*:

CHSH: Clauser, Horne, Shimony, Holt (1969)

GHZ: Greenberger, Horne, Zeilinger (1989)

Hardy: Hardy (1993)

Bell: Bell (1964)

Mermin: Mermin (1985)

Acín: Acín, Durt, Gisin, Latorre (2002)

CHSH

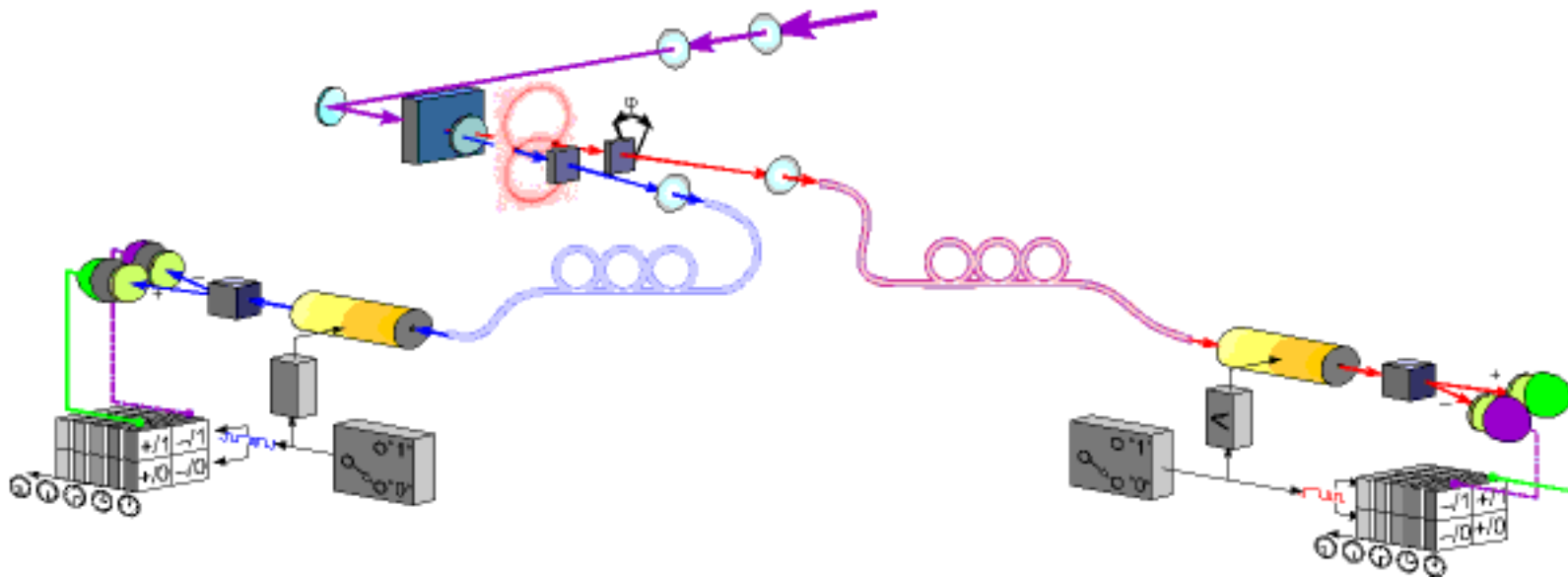


Innsbruck, 1998

<http://www.quantum.at>

---> people -> Gregor Weihs -> Bell experiment

CHSH

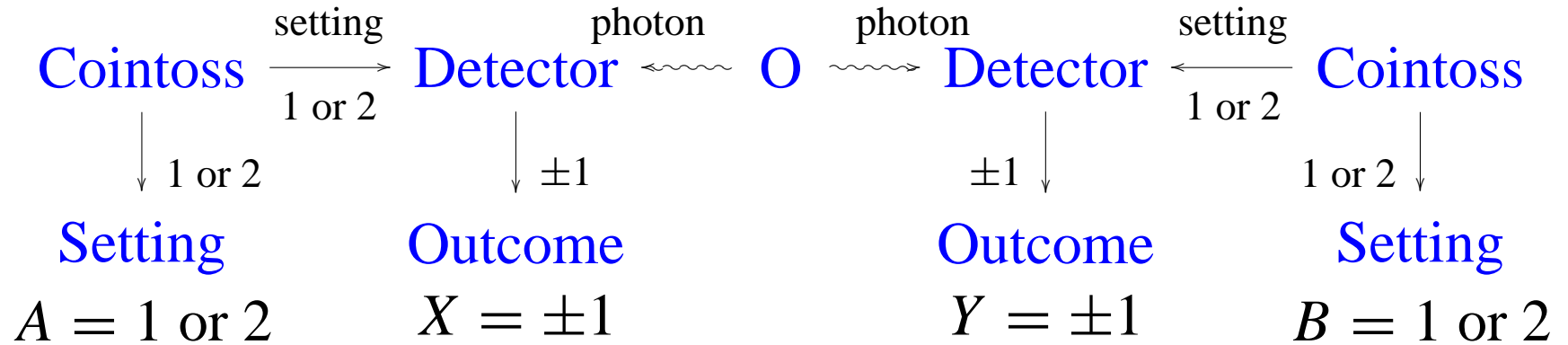


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CHSH



$Q_{ab}^{\psi \dots}$: probability distribution of (X, Y) given $A = a, B = b$

$$Q_{ab}^{\psi \dots}(x, y) = Q_{ab}^{\psi \dots}\{X = x, Y = y\}$$

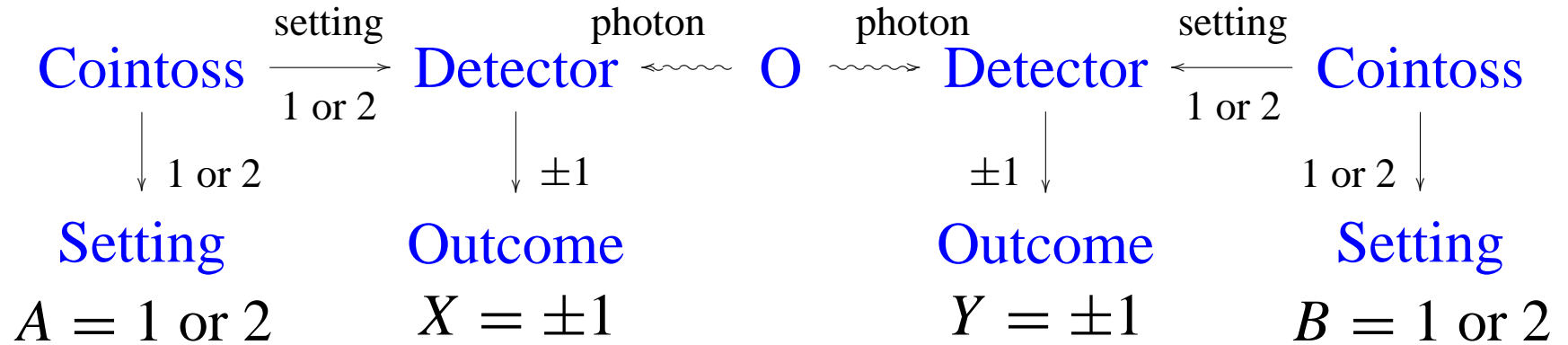
σ : probability distribution of (A, B)

$$\sigma(a, b) = \sigma\{A = a, B = b\}$$

$Q^{\sigma \psi \dots}$: probability distribution of (A, B, X, Y)

$$Q^{\sigma \psi \dots}(a, b, x, y) = \sigma(a, b) Q_{ab}^{\psi \dots}(x, y)$$

CHSH



CHSH specifies $\psi \dots$:

$$Q_{ab}^{\psi \dots} \{X = \pm 1\} = Q_{ab}^{\psi \dots} \{Y = \pm 1\} = \frac{1}{2}$$

$$Q_{12}^{\psi \dots} \{X = Y\} = \frac{1}{2}(1 + 1/\sqrt{2}) = 0.85$$

$$Q_{ab}^{\psi \dots} \{X = Y\} = \frac{1}{2}(1 - 1/\sqrt{2}) = 0.15 \quad (ab) \neq (12)$$

Weih's chose σ uniform:

$$\sigma : \quad A \perp B, \quad A, B \sim \text{Bernoulli}(\frac{1}{2})$$

CHSH

		Zählungen	Bob			
			1, +	1, -	2, +	2, -
Alice	1, +	77988	313	1728	1636	179
	1, -	74935	1978	351	294	1143
	2, -	75892	418	1683	269	1100
	2, +	73456	1578	361	1386	156

Koinzidenzen

Zeitverschiebung	3,8	ns	Quantentheoretische Vorhersage		
Koinzidenzfenster	4,0	ns			
Koinzidenzengesamt	14573		ideal	bei 94% -97% Kontrast	
$\rho(1,1)$	-0,70	$\pm 0,01$	-0,71	0,68	$\pm 0,02$
$\rho(2,1)$	-0,61	$\pm 0,01$	-0,71	0,68	$\pm 0,02$
$\rho(1,2)$	0,71	$\pm 0,01$	0,71	0,68	$\pm 0,02$
$\rho(2,2)$	-0,71	$\pm 0,01$	-0,71	0,68	$\pm 0,02$
S	2,73	$\pm 0,02$	2,82	2,72	$\pm 0,04$
Verletzung	29,8	stddev			

Tabelle 5.3: Auswertung von *longdist35*, aufgenommen am 22.4.1998

Local Realism

$$\begin{aligned} \exists X_1, X_2, Y_1, Y_2 : X &\equiv X_A, \quad Y \equiv Y_B \\ (A, B) &\perp (X_1, X_2, Y_1, Y_2) \sim \pi \end{aligned}$$

P_{ab}^π : probability distribution of (X, Y) given $A = a, B = b$

$$P_{ab}^\pi(x, y) = P_{ab}^\pi\{X = x, Y = y\}$$

σ : probability distribution of (A, B)

$$\sigma(a, b) = \sigma\{A = a, B = b\}$$

$P^{\sigma\pi}$: probability distribution of (A, B, X, Y)

$$P^{\sigma\pi}(a, b, x, y) = \sigma(a, b)P_{ab}^\pi(x, y)$$

Local Realism

$$\begin{aligned} \exists X_1, X_2, Y_1, Y_2 : X &\equiv X_A, \quad Y \equiv Y_B \\ (A, B) &\perp (X_1, X_2, Y_1, Y_2) \sim \pi \end{aligned}$$

P_{ab}^π : probability distribution of (X, Y) given $A = a, B = b$

π : probability distribution of (X_1, X_2, Y_1, Y_2) , margins π_{ab}

$$\begin{aligned} P_{ab}^\pi(x, y) &= P_{ab}^\pi\{X = x, Y = y\} \\ &= \pi\{X_a = x, Y_b = y\} = \pi_{ab}(x, y) \\ &= \sum_{x_1, x_2, y_1, y_2 : x_a=x, y_b=y} \pi(x_1, x_2, y_1, y_2) \end{aligned}$$

Bell's Theorem

Fix σ , define

$\mathcal{P}^\sigma = \{P^{\sigma\pi} : \pi \text{ arbitrary}\}$, $\mathcal{Q}^\sigma = \{Q^{\sigma\psi} : \psi \dots \text{arbitrary}\}$.

\mathcal{P}^σ and \mathcal{Q}^σ are compact and convex, $\mathcal{P}^\sigma \subseteq \mathcal{Q}^\sigma$

$\mathcal{Q}^\sigma \setminus \mathcal{P}^\sigma \neq \emptyset$

CHSH proof: specific choice of $Q_{ab}^\psi(x, y)$

Hardy proof: specific choice of $Q_{ab}^\psi(x, y)$

GHZ proof: three parties, specific choice of $Q_{abc}^\psi(x, y, z)$

In *experiment* must also choose $\sigma(a, b)$, or $\sigma(a, b, c)$:
fair or biased coins, uncorrelated or correlated

Bell's Theorem

CHSH:

$$\pi\{X_1 = Y_2\} - \pi\{X_1 = Y_1\} - \pi\{X_2 = Y_1\} - \pi\{X_2 = Y_2\} \leq 0$$

hence

$$P_{12}^{\pi}\{X = Y\} - P_{11}^{\pi}\{X = Y\} - P_{21}^{\pi}\{X = Y\} - P_{22}^{\pi}\{X = Y\} \leq 0$$

but under QM we can have

$$Q_{12}^{\psi}\{X = Y\} - Q_{11}^{\psi}\{X = Y\} - Q_{21}^{\psi}\{X = Y\} - Q_{22}^{\psi}\{X = Y\} = \sqrt{2} - 1 > 0$$

Bell's Theorem

Hardy:

$$\pi_{11}(--) = 0 \text{ and } \pi_{12}(+-) = 0 \text{ and } \pi_{21}(-+) = 0 \text{ implies} \\ \pi_{22}(--) = 0$$

hence

$$P_{11}^{\pi}(--) = 0 \text{ and } P_{12}^{\pi}(+-) = 0 \text{ and } P_{21}^{\pi}(-+) = 0 \text{ implies} \\ P_{22}^{\pi}(--) = 0$$

but under QM we can have

$$Q_{11}^{\psi}(--) = 0 \text{ and } Q_{12}^{\psi}(-+) = 0 \text{ and } Q_{21}^{\psi}(+-) = 0 \text{ and} \\ Q_{22}^{\psi}(--) = 0.09$$

Bell's Theorem

GHZ:

$$\pi\{X_2Y_1Z_1 = +\} = \pi\{X_1Y_2Z_1 = +\} = \pi\{X_1Y_1Z_2 = +\} = 1$$

$$\text{implies } \pi\{X_2Y_2Z_2 = +\} = 1$$

hence

$$P_{211}^\pi\{XYZ = +\} = P_{121}^\pi\{XYZ = +\} = P_{112}^\pi\{XYZ = +\} = 1$$

$$\text{implies } P_{222}^\pi\{XYZ = +\} = 1$$

but under QM we can have

$$Q_{211}^\psi\{XYZ = +\} = Q_{121}^\psi\{XYZ = +\} = Q_{112}^\psi\{XYZ = +\} = 1$$

$$\text{and } Q_{222}^\psi\{XYZ = +\} = 0$$

Quantum Mechanics (von Neumann)

ρ , \hat{X} operators on Hilbert space \mathfrak{H} , $\psi \in \mathfrak{H}$; $f : \mathbb{R} \rightarrow \mathbb{R}$:

ρ is a *state* if $\rho \geq 0$, $\text{trace}(\rho) = 1$

$\rho = \int \Pi_{[\psi]} P(d\psi)$ (mixture of pure states)

\hat{X} is an *observable* if $\hat{X}^* = \hat{X}$

$f(\hat{X})$ too is an observable

(replace eigenvalues by f of eigenvalues) e.g. $\mathbb{1}_A\{\hat{X}\}$

Experiment: *measure* observable \hat{X} on system in state ρ

Outcome is random, call it X

$$Q^{\rho, \hat{X}}\{X \in A\} = \text{trace}(\rho \mathbb{1}_A(\hat{X}))$$

$$E_Q(f(X)) = \text{trace}(\rho f(\hat{X}))$$

Quantum Mechanics

Commuting observables can be measured simultaneously

$$Q^{\rho, \hat{X}}\{X \in A, Y \in B\} = \text{trace}(\rho \mathbb{1}_A(\hat{X}) \mathbb{1}_B(\hat{Y}))$$

Product space $\mathfrak{H} \otimes \mathfrak{K}$

Joint state ρ on $\mathfrak{H} \otimes \mathfrak{K}$

Observables \hat{X} on \mathfrak{H} , \hat{Y} on \mathfrak{K} define commuting observables

$$\hat{X} \otimes 1, 1 \otimes \hat{Y} \text{ on } \mathfrak{H} \otimes \mathfrak{K}$$

Joint measurement of \hat{X} on \mathfrak{H} with \hat{Y} on \mathfrak{K} :

$$Q^{\rho, \hat{X}, \hat{Y}}\{X \in A, Y \in B\} = \text{trace}(\rho \mathbb{1}_A(\hat{X}) \otimes \mathbb{1}_B(\hat{Y}))$$

CHSH

$$\mathfrak{H} = \mathfrak{K} = \mathbb{C}^2$$

$|\odot\rangle, |\ominus\rangle$ orthonormal basis of \mathbb{C}^2

$$\psi = |\odot\rangle \otimes |\odot\rangle + |\ominus\rangle \otimes |\ominus\rangle$$

$$\rho = \Pi_{[\psi]}$$

\hat{X}_α : eigenvalues ± 1 ,

eigenvectors $\cos \alpha |\odot\rangle + \sin \alpha |\ominus\rangle$, $\sin \alpha |\odot\rangle - \cos \alpha |\ominus\rangle$

\hat{Y}_β : ...

$$Q^{\rho, \hat{X}_\alpha, \hat{Y}_\beta} \{X = \pm 1\} = \frac{1}{2} = Q^{\rho, \hat{X}_\alpha, \hat{Y}_\beta} \{Y = \pm 1\}$$

$$Q^{\rho, \hat{X}_\alpha, \hat{Y}_\beta} \{X = Y\} = \cos^2(\alpha - \beta)$$

CHSH

$$\begin{aligned} & \alpha_2 = \pi/4 \\ & \alpha_1 = 0 \\ & \beta_2 = -\pi/8 \\ & \beta_1 = -\pi/8 - \pi/4 \end{aligned}$$

$$\alpha_1 - \beta_2 = \pi/8$$

$$\alpha_a - \beta_b = \pi/2 \pm \pi/8 \quad \text{for } ab \neq 12$$

$$\cos^2(\alpha_1 - \beta_2) = 0.85$$

$$\cos^2(\alpha_a - \beta_b) = 0.15 \quad \text{for } ab \neq 12$$

Given experiment, which σ is best? Which experiment is best?

Fix experiment: $Q_{ab}^\psi(x, y)$

Experimenter chooses σ

Repeat N times (large), collect data $\sim Q^{\sigma\psi}$

Local realist must explain with π

Bayesian data analyst:

$$-\log(\Pr\{\mathcal{P}^\sigma | \text{data}\})/N \approx \inf_\pi D(Q^{\sigma\psi} : P^{\sigma\pi})$$

Frequentist data analyst:

p -value of best test statistic of $H_0 : \mathcal{P}^\sigma$ vs. $H_1 : \mathcal{Q}^\sigma$

$$-\log(p(\text{data}))/N \approx \inf_\pi D(Q^{\sigma\psi} : P^{\sigma\pi})$$

Computer scientist: Increased description length for data $\sim Q^{\sigma\psi}$
when using best coding for \mathcal{P}^σ

$D(Q:P) \geq 0$ relative entropy (Kullback-Leibler divergence)
of P relative to (from) Q

$$\begin{aligned} D(Q:P) &= E_Q \log \frac{dQ}{dP} \\ &= \sum_{abxy} Q(a, b, x, y) \log \frac{Q(a, b, x, y)}{P(a, b, x, y)} \\ &= \sum_{ab} \sigma(a, b) D(Q_{ab}^\psi : P_{ab}^\pi) \end{aligned}$$

Problem: how to compute

$$\sup_{\sigma} \inf_{\pi} \sum_{ab} \sigma(a, b) D(Q_{ab}^\psi : P_{ab}^\pi)$$

Missing Data

Fix σ, ψ, \dots

Computation of

$$\inf_{\pi} E_Q \log \frac{dQ}{dP^{\pi}}$$

is computation of the *maximum likelihood estimator* of π given observations on (A, B, X, Y) with cell frequencies equal to their expectations under $Q^{\sigma\psi}$

The problem is a *nonparametric missing data problem*:

observe (A, B, X, Y)

function of (A, B) and (X_1, X_2, Y_1, Y_2)

$(A, B) \perp (X_1, X_2, Y_1, Y_2)$

$(X_1, X_2, Y_1, Y_2) \sim \pi$ completely unknown

Use VD-IQM; EM too slow.

Game Theory (von Neumann)

Define $D(\sigma, \pi) = D(Q^{\sigma\psi} : P^{\sigma\pi})$

$$= \sum_{ab} \sigma(a, b) D(Q_{ab}^{\psi} : P_{ab}^{\pi}) = \sum_{ab} \sigma(a, b) D_{ab}(\pi)$$

$$\sup_{\sigma} \inf_{\pi} D(\sigma, \pi) \leq \inf_{\pi} \sup_{\sigma} D(\sigma, \pi)$$

The *game has a value* when equality holds

The *game has a saddlepoint* when $\exists \sigma^*, \pi^*$ such that

$$\sigma^* = ! \max_{\sigma} D(\sigma, \pi^*)$$

$$\pi^* = ! \min_{\pi} D(\sigma^*, \pi)$$

Game Theory

Proposition: Game has saddle point implies game has value

Typically no value, no saddle point

Von Neumann: typically, *randomized game* has saddle point

Game Theory

$D(\sigma, \pi)$ is linear in σ , convex in π

(nonparametric missing data problem)

If σ is arbitrary, randomizing over σ doesn't change the game!

Game randomized in π has saddle point

By convexity in π , saddle point $\tilde{\pi}^*$ is non-randomized π^*

By linearity in σ , saddle point π^* is an equalizer for $D_{ab}(\pi)$

If σ is arbitrary, we need just search for σ^*, π^* such that

$D_{ab}(\pi^*)$ is constant in (a, b)

π^* is npmle for π based on $Q^{\sigma^* \psi}(a, b, x, y)$

Results

CHSH: 0.0463 “bits per trial”; σ uniform is optimal

Hardy: 0.0280 (uncorrelated 0.0279, uniform 0.0278)

GHZ: 0.407 (uniform 0.203)

Mermin: 0.021 (uniform 0.016)

Bell: 0.016 (uncorrelated 0.015, uniform 0.013)

Acín trinary-outcome: 0.058 (maximally entangled),
0.072 (maximally resistant).

Gill trinary outcome: 0.0768, Popescu recursive 0.031

Conjecture: CHSH is best two-party binary-outcome experiment
GHZ best three-party binary-outcome

Numerical experiments: R aka GNU S + VD-IQM
Vertex direction iterative quadratic minimization
hidden variables in AIDS epidemiology

Detection loophole, memory loophole, random arrival times:
martingale methods from clinical cancer trials

Time, Finite Statistics and Bell's Fifth Position
quant-ph/0301059