PLAYING A TRICK ON UNCERTAINTY

If you have to decide between two alternatives without
knowing which one is more favourable, then you may
quite as well flip a coin - Right?
No, you can do better.

F. Thomas Bruss

Université Libre de Bruxelles

You want to sell your house. Your ad "Sell for the best offer above 800.000 Euro" has been running for weeks already in the newspaper. But now, next Sunday is the deadline.

Two potential buyers have announced their definite interest. Mr. X from Paris called, saying that he will make an offer exceeding 800.000 Euro but that he would like to see the house again next Saturday before finalizing his exact offer. And then there is Mrs.Y who called from London to say essentially the same, except that she can come next Sunday only. Both Mr. X and Mrs. Y insisted that they would need your irrevocable Yes or No on the very day of their visit.

You would have liked so much to find out more! If you only had been able to obtain an indication of what limit Mr. X and Mrs. Y were prepared to pay!
However, all you got on the phone was a short laugh and something like "Please let me see the house again." True business people, both of them! You also have made already your inquiries: Both are serious and reliable, and both have the necessary funds. But then, it seems hard to guess who of the two could be expected to be the more interested one.

It is time to analyse the exact circumstances of your situation. Clearly, you will have again the opportunity to point out the splendid features of your house. However, this will not change your dilemma: If you accept the offer of Mr.X you will lose the one of Mrs.Y, and if you want to wait for the offer of Mrs.Y you lose the one of Mr. X. This seems like gambling! You will lose the better of the two offers with probability $1/2$, won't you?

Another idea comes to your mind. Paris …., London ….., it is not likely that Mr. X and Mrs. Y know each other. Should you perhaps try to push up the price by telling each of them how much interested the other one is? Perhaps trying with Mr. X first? - But No, you dismiss this idea, a man like Mr .X would hardly be impressed by this - rather on the contrary. Trying with Mrs. Y perhaps? But then, the day she comes, Mr. X is already out of the game and can no longer serve as a means of pressure.

And again you arrive at the same conclusion as before: You may as well flip a coin in order to decide. Perhaps you should simply make the deal with Mr. X to have at least your Sunday free!

Game with two cards

Such situations in real life occur in many different variations. A special offer in the supermarket, a nice apartment, an attractive job offer, or even the woman or man for life: One must so often decide without knowing whether something better is still to come.
To clear the view for the problem, we summarize the essence in terms of a little game: You ask your son and your daughter to write, each one secretely, and not consulting each other either, one arbitrary number on his/her card. You point out that "arbitrary" means really as they want: Large, small, negative, decimal point, ..., everything is allowed. Then they place their cards, face down, on the table. You can now turn over the card of your son, inspect the number, and then decide whether you accept it. If you refuse it, then you receive automatically your daughter's card. Now both numbers are compared. If you have chosen the larger number, you win, otherwise you lose.

The difference between these numbers is now without importance, you just want to win. If the numbers happened to be the same, the game would be repeated, but this case is improbable. Further, if you think you may have some advantage from knowing your children well, you can imagine them being replaced by others. Alternatively, one person may also fill in both cards.
This really looks now like a purified game of chance with win probability 1/2.

But now the surprise: There is a strategy with which you can increase your win probability above 1/2. It is based on an idea of Professor Thomas Cover (Stanford University). Let X and Y be the two different numbers on the cards.

Strategy :

Think of an arbitrary number Z. Now uncover the first number, X,
and choose this number if it is larger than Z, otherwise choose Y.
Why should this strange strategy be better than choosing X or Y at random?

Here is the proof :

Recall X is the first number, Y the second. Let Min=min{X,Y} be the smaller and Max=max{X,Y} the larger one. There are exactly three possibilities:

(A) Both numbers X and Y are not larger than Z,

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Min            Max        Z
_______|__________|______|______________
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(B) Z lies between X and Y (possibly coinciding with the smaller one),

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        Min      Z        Max
_______|________|________|_______________
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(C) Both X and Y exceed Z.

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           Z        Min           Max
______|_________|_________|_________
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According to the strategy, you choose the number Y in case (A) and the number X in case (C). In these two cases you end up with a random choice between the larger and the smaller number, hence you win with probability 1/2. In case (B), however, you win with certainty, because if X is the larger of the two numbers you accept it, but if it is the smaller one, you refuse it. Thus your total win probability is now

$$w = a/2 + c/2 + b,$$

where a, b, and c denote the unknown probabilities of the events (A), (B) and (C) respectively. One of these events is bound to happen, of course, that is $a+b+c=1$, and hence

$$w = (a+b+c)/2 + b/2 = 1/2 + b/2.$$

Thus the win probability exceeds equal chance by $b/2 > 0$, because (B) can occur.

[Remark: To make this precise it suffices to choose Z according to any density which is strictly positive everywhere on the real line.]

How can you apply this strategy most skillfully? Evidently you should try to make the case (B) as probable as possible. This means you should choose Z such that it has the largest possible probability of falling between X and Y. Since these two numbers are unknown, no general recommendation can be made. In concrete cases, however, one may quite well have some ideas.

Optimal choice of a threshold.

Our house-selling problem is such a concrete case. On the first card is the offer of Mr. X and you will not know the number on the other card, the offer of Mrs. Y, when you must say Yes or No to Mr. X. The first difference to the card game with arbitrary numbers is that you know that both X and Y are above 800.000. The second difference is that the amount $|X-Y|$ is now of real interest to you.

An offer of 900.000 Euro or more would be nice, but is not very likely. On the other hand, if you accepted Mr. X's offer of 801.000 Euro, say, then you would not suffer much regret if Mrs. Y would offer 802.000 Euro. There is no point in trying to hedge against a loss of too modest a magnitude. Therefore it may be best to choose Z clearly above 800.000 Euro, but then again not too large. If you asked me what I would do: I would toss a die and, for each eye of my result, add 5.000 Euro to the amount 800.500 Euro. So, for instance, if I obtained 3, I would choose Z=815.500. But, by all means, there is nothing special about this suggestion and you may be much happier with your own idea.

Why toss a die? Why not simply fix Z=820.000, say, if we feel this should be more or less in the right order of magnitude? Apart from our probability argument there is another reason: In such a game-theoretical situation, it is often better to be unpredictable. If we act in a predictable way, the other player may adjust his behaviour. Hence the introduction of a random component.

What is our strategy worth? - Definitely more than the random choice, as we have seen. We cannot really quantify the advantage compared with a random choice, but some additional 10.000 Euro or so may quite well be in it (in expectation.)

Let us go a step further and look again into our two-card game. Now you and I play the game. Suppose you are the one who writes the two numbers on the two cards and it is I who has to choose one. As before, I win if I choose the larger one. Suppose you would like to decrease my win probability. What should you do?

The answer is simple. You just choose two numbers which are very close to each other. Take 6.123455 and 6.123456, say. My advantage of using a Z-strategy is now hardly worth talking about because my chosen Z will have little chance of falling between these two numbers.

In real-world problems, things are often different, however. Real-world strategies are developed by one party and typically not communicated to the other party. What difference does this make?
To find this out, I made, several years ago, a test with Vesalius College business students. Everybody in the audience received two cards to write down his or her numbers, and then I passed from one to the other to make my choice. I had not mentioned Z-strategies before.

My score was 32 successes for 41 or 42 participants. With a random choice we would expect some 21, and some three or four more with a bit of luck. 32 however should not be explained by pure luck alone, as they knew. Even the best students were puzzled. It is difficult to see what one does not expect. But you, dear reader, you probably guess correctly: I had applied a Z-strategy, even a particularly naive one. I had chosen Z = 0.

Why was this so successful? – I think it was because I could prepare the field for the strategy: My remark "The numbers may also be negative had seemingly succeeded in being sufficiently casual. The fact is that numerous students made use of negative numbers, and all those who had written down just *one* negative number made me automatically a winner.

This experiment shows that strategic thinking has no simple rules Some people preach that the key to success in strategic behaviour is always narrowing down the adversary's field of action. However, this is not true. If we believe that our adversary does not expect our strategy it can be unwise to narrow down the set of his or her options. The less one can do, the more one thinks about each step. Indeed, in our experiment, by allowing negative numbers we did not narrow down but actually enlargened the set of options for the students. This probably helped to distract from paying attention.

A few words about Mathematics

You have just learned to know a little problem in a field of Mathematics, which, compared to other fields, is still in its infancy: strategic thinking, as a part of probability theory. Even at this introductory level several questions are still open. For instance, does there exist a strategy for the two card game which is generally more efficient than a Z-strategy? - The proof of existence or non-existence would already be a true progress. However, at the current state of knowledge, I see no sufficiently safe foundation to attack such a proof, not even to make this question sufficiently precise.

Is it not truely surprising that nobody can optimize, in a strict mathematical meaning, the sale of a house to two potential buyers? Given that we see so many impressive things Mathematics has achieved in our world, we would agree that it is, wouldn't we? Take examples of other optimization problems, as seen in modern airplane engineering, for instance. Compared to such problems, our little problem seems ridiculously simple. However, this is not true. Airplane engineers have a huge time advantage: They can work, routinely, with many methods the mathematical foundations of which have been well establishesd for two to three centuries. This is not the case for our little problem.

Such contrasts do exist in many fields of Mathematics.

Is this a symptom of the eternal youth of a discipline?

Yes, I think it is.

Footnote: : This is the author's English translation of his article *Der Ungewissheit ein Schnippchen schlagen* published in Spektrum der Wissenschaft (= German Edition of Scientific American). The latter was based on the author's *Unerwartete Strategien* published in Mitteilungen der Deutschen Mathema-tikervereinigung; (German Mathematical Society.) The idea was instigated by Cover's Problem. (References below.)

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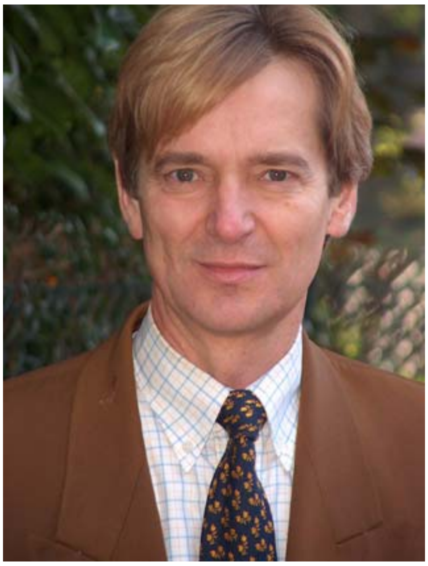

About the author : F. Thomas Bruss studied Mathematics in Saarbruecken (FRG), Cambridge, and Sheffield (UK). He started his career as assistent and first assistant in Namur (B) from where he moved on to the United States, where he was Visiting Associate Professor at UC Santa Barbara, U of Arizona, and UCLA, successively. In 1990 he was appointed Professor at Vesalius College of the Vrije Universiteit Brussels. Since 1993 he is Professor of Mathematics and Statistics at the Université Libre de Bruxelles. His research is in probability : limit theorems, branching processes, probabilistic models, and optimal stopping. He is fellow of the Institute of Mathematical Statistics and Feodor-Lynen-fellow of the von-Humboldt Foundation.

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