Losing the +1

or directed path-width games are monotone

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Abstract. We show the extra cop required for monotonicity in the game defined in [1] is not necessary.

1 Introduction

In [1] Barát defined a cops and robber game for directed path-width. In addition he defined a mixed-search game and was able to show:

Theorem 1.1. *k cops have a mixed search on a digraph $\mathcal{G}$ if and only if k cops have a monotone mixed search on $\mathcal{G}$.*

Where a monotone mixed search is one which clears each edge only once. He then observed that for a digraph $\mathcal{G}$, he could generate a digraph $\mathcal{G}^d$ such that:

Theorem 1.2. 1) *If k cops have a winning strategy on $\mathcal{G}$ (in the directed path-width game) then k cops have a mixed search on $\mathcal{G}^d$.*

2) *If k cops have a monotone mixed search on $\mathcal{G}^d$ then $k+1$ cops have a monotone winning strategy on $\mathcal{G}$.*

The combination of these theorems implies:

Corollary 1.3. *If k cops have a winning strategy on $\mathcal{G}$ in the directed path-width game, then $k+1$ cops have a monotone winning strategy on $\mathcal{G}$.*

He conjectured that the $k+1$ in the above theorem could be reduced to k with a more technical argument. Here we provide such an argument.

2 Games

In this section we recap the definitions of the games used by Barát.

Definition 2.1. A *mixed search* in a directed graph $\mathcal{G}$ is a sequence of pairs

$$(A_0, Z_0), \ldots, (A_n, Z_n)$$

(intuitively Z_i is the set of vertices occupied by the cops immediately before the $(i+1)$st step and A_i is the set of clear edges) such that

(I) For $0 \leq i \leq n$, $A_i \subseteq E(\mathcal{G})$, $Z_i \subseteq V(\mathcal{G})$

(II) For $0 \leq i \leq n$, if $v \in V(\mathcal{G})$ is such that there exists $(u, v) \notin E(\mathcal{G})$ and $(v, w) \in E(\mathcal{G})$ then $v \in Z_i$

(III) $A_0 = \emptyset$, $A_n = E(\mathcal{G})$

(IV) (Possible moves) for $1 \leq i \leq n$, either:

   (a) (placing new cops) $Z_i \supseteq Z_{i-1}$, and $A_i = A_{i-1}$, or
   (b) (removing cops) $Z_i \subset Z_{i-1}$ and A_i is the new set of non-contaminated edges. That is, $f \in A_i$ if every directed path containing an edge not in A_{i-1} *before* f *in order*, has an internal vertex in Z_i.
   (c) (*tail* searching e) $Z_i = Z_{i-1}$ and $A_i = A_{i-1} \cup \{e\}$ for some edge $e = (u, v) \notin A_{i-1}$ with $u \in Z_{i-1}$
   (d) (sliding) $Z_i = (Z_{i-1} \setminus \{w\}) \cup \{v\}$ for some $w \in Z_{i-1}$ and $v \notin Z_{i-1}$ and there is an edge $e = (v, w) \in E(\mathcal{G})$ such that every other edge with head w belongs to A_{i-1}, and $A_i = A_{i-1} \cup \{e\}$
   (e) (extension) $Z_i = Z_{i-1}$ and $A_i = A_{i-1} \cup \{e\}$ for some edge $e = (v, w) \notin A_{i-1}$, where $w \notin Z_i$ and every (possibly 0) edge with head v belongs to A_{i-1}

Definition 2.2. A *node search* of $\mathcal{G}$ is a mixed search of $\mathcal{G}$ where sliding and extension are not allowed.

Technically for this definition to coincide with the search game defined for directed path-width, we require at least one cop. A graph with no edges can be mix searched with 0 cops but the formal node search game requires one cop. This case is easily handled, so from now we only consider graphs with at least one edge, and therefore requiring at least one cop to mix (node) search.

Definition 2.3. A mixed (node) search $(A_0, Z_0), \ldots, (A_n, Z_n)$ is *monotone* if

$$A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n.$$

3 Proof

We now turn to a more technical version of Theorem 1.2.

Theorem 3.1. *Let $\mathcal{G}$ be a digraph with at least one edge. Let $\mathcal{G}^*$ be the digraph obtained by duplicating all edges of $\mathcal{G}$ and then for each vertex v adding a vertex v' and edges $e_v^o = (v, v')$ and $e_v^i = (v', v)$. Then*

1) *If k cops have a node search on $\mathcal{G}$ then k cops have a mixed search on $\mathcal{G}^*$*
2) *If k cops have a montone mixed search on $\mathcal{G}^*$ then k cops have a monotone node search on $\mathcal{G}$.*

Proof. 1): Given a node search of $\mathcal{G}$ each duplicate edge in $\mathcal{G}^*$ can be cleared immediately after the original (using the same rule). For each v the edges e_v^o and e_v^i can be cleared when a cop visits v – e_v^o by tail search and then e_v^i by extension. The only vertices that may not be visited by a cop during a node search are the sinks of $\mathcal{G}$. As $\mathcal{G}$ has at least one edge, there is at least one cop, and one cop suffices to clear all the edges corresponding to the sinks after all other edges have been cleared.

2): We observe that it suffices to show that a monotone mixed search of $\mathcal{G}^*$ need only use extension or sliding on the edges e_v^i. First we note (as in [1]) that the first edge cleared in a duplicated pair cannot be cleared by sliding, and whenever one edge is cleared by tail searching or extension, the other can be cleared immediately after using the same rule. Secondly we observe that for any $v \in V(\mathcal{G})$, e_v^o can always be cleared before e_v^i:

Lemma 3.2. *If $(X_0, \ldots X_r)$ is a progressive raid of width k such that for some v, e_v^i is cleared before e_v^o then there exists a progressive raid of width at most k such that e_v^o is cleared before e_v^i.*

Proof. Suppose $e_v^i \in X_m \setminus X_{m-1}$ and $e_v^o \in X_n \setminus X_{n-1}$. Then for all $m \leq j < n$, let $X'_j = (X_j \setminus \{e_v^i\}) \cup \{e_v^o\}$. Then we claim$(X_0, \ldots, X_{m-1}, X'_m, \ldots, X'_{n-1}, X_n, \ldots)$ is a progressive raid of width at most k. It is straightforward to see that it is a progressive raid as we have merely switched e_v^i and e_v^o. To show that the width has not increased, we observe that $\delta(X'_j) = (\delta(X_j) \setminus \{v'\}) \cup \{v\}$, for all $m \leq j < n$, as v' is no longer a dangerous vertex, but v will now become one (if it was not already one). □

From this it follows that e_v^o is never cleared by extension, and if it was cleared by sliding a cop from v' it can be cleared by placing the cop on v instead of v' and tail searching.

Now suppose a duplicated edge $e = (v, w) \in E(\mathcal{G}^*)$ is cleared by extension. By the definition of extension, e_v^i must be cleared before e, and e_v^i can only be cleared after a cop occupies v (and clears e_v^o). Now, if a cop is on v when e is cleared, e can be cleared by tail searching, so assume a cop is not on v. However, at the first point when the cop was removed from v after e_v^o was cleared, all edges with head v were clear (as otherwise e_v^o would become recontaminated). So e can be cleared by a tail search immediately before the cop is removed as it will not become recontaminated when the cop leaves. □

The case when $\mathcal{G}$ has no edges is trivial, giving us:

Corollary 3.3. *If there is a capture of an invisible robber in $\mathcal{G}$ with k cops, then there is a monotone capture with k cops.*

References

1. János Barát. Directed path-width and monotonicity in digraph searching. To appear in *Graphs and Combinatorics*.