Petri Games are Monotonic but Difficult to Decide*

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Abstract. In this paper, we study two-player games played on infinite but monotonic game structures. We concentrate on coverability games, a natural subclass of reachability games in the context of monotonic game structures. On the negative side, we show that surprisingly, and contrary to the one-player case, coverability is undecidable on two-player monotonic game structures. On the positive side, we identify an interesting subclass of two-player monotonic game structures, for which coverability is decidable and for which we can effectively construct winning strategies. Furthermore, we show how to define two-player game structures that belong to that subclass with Petri nets. The results of this paper are compared to recent results obtained independently by Abdulla, Bouajjani and d'Orso on similar game structures where they identify another subclass of monotonic game structures with decidable results.

1 Introduction

Model-checking methods were originally proposed for finite-state systems. Nevertheless, much *recent* interest has concerned the application of model-checking methods to infinite-state systems. Several interesting classes of infinite state systems has been shown decidable. For example, Alur et al [AD94] showed that timed automata have a decidable *reachability problem.* Finkel et al in [FS01], and Abdulla et al in [ACJT96] have shown that infinite, but *monotonic*, transition systems (also called well-structured transition systems) have a decidable *coverability problem.* For instance, Petri nets and lossy channels systems define monotonic transition systems.

Timed automata, Petri nets, and lossy channels systems are usually used to model *reactive systems* embedded in an environment. But those formalisms define transition systems that are semantics models for *closed systems.* In closed systems, we do not distinguish between the reactive system and its environment. So the properties that we can verify on transitions systems are properties in which we can not distinguish between the role of the reactive system and the

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role of the environment. If we want to distinguish the role of the reactive system and the environment in which it is embedded, we can use games played on state spaces.

Usual transition systems can be considered as one-player game on which only closed-system verification problems can be formulated. The *control* and *modular verification* problems of systems can be studied as two-player games played on state spaces, where *one player* represents the reactive system and the *other player* represents the environment. If the state space on which the game is played is infinite then we have to solve infinite-state games. Infinite-state games has not yet been studied as intensively as traditional verification problems on infinite-state transition systems. Nevertheless, recently there have been several interesting works in that direction. Here are some examples. In [MAJ95], Maler et al study how to solve games defined by timed automata. In [Wal96], Walukiewicz studies how to solve infinite games defined by push down automata. In [dAHM01], Henzinger et al study symbolic algorithms to solve general infinite-state games.

In this paper, we study two-player games played on infinite but *monotonic game structures* (for a well-quasi ordering). We concentrate on *coverability games*, a natural subclass of reachability games in the context of monotonic game structures (coverability, contrary to reachability, have shown to be decidable for all well-structured transition systems). On the negative side, we show that surprisingly, and contrary to the one-player case (well-structured transition systems), coverability is *undecidable* on two-player monotonic game structures. On the positive side, we identify an interesting *subclass* of two-player monotonic game structures, for which coverability is *decidable* and for which we can effectively construct *winning strategies*. Furthermore, we show how to define two-player game structures that belong to that subclass with *Petri nets*. The results of this paper are compared to recent results obtained independently by Abdulla, Bouajjani and d'Orso in [ABd03] on similar game structures where they identify *another* subclass of monotonic game structures with decidable results.

Structure of the paper In Section 2, we recall preliminaries. In Section 3, we define monotonic game structures and coverability games. We show that coverability games are undecidable on general monotonic game structures. In Section 4, we identify a subclass of monotonic games structures for which player 1 coverability games are decidable. We show how Petri nets can be used to define monotonic game structures that fall in that class. For those decidable coverability games, we show how to effectively construct winning strategies. In Section 5, we compare our work with the independent work of Abdulla et al on monotonic games.

2 Preliminaries

In this section, we recall some standard definitions and well-known results for games, well-quasi orderings, and two-counter machines.

Games A (two-player) *game structure* G is a tuple $\langle C, C_1, C_2, \rightarrow\rangle$ where C is a (potentially infinite) *set of configurations* partitioned into the set of player 1 configurations C_1 and the set of player 2 configurations C_2 (that is $C_1 \cap C_2 = \emptyset$ and $C = C_1 \cup C_2$), and $\rightarrow \subseteq (C_1 \times C_2) \cup (C_2 \times C_1)$ is the *transition relation.* In the following, we note $c \rightarrow c'$ when $(c, c') \in \rightarrow$. A *play* P in the game structure G from a configuration c is either an infinite sequence of configurations $c_0 c_1 \ldots c_n \ldots$ such that $c_0 = c$ and $c_i \rightarrow c_{i+1}$ for all $i \geq 0$, or a finite maximal sequence of configurations $c_0 c_1 \ldots c_n$ such that $c_0 = c$ and for all i, $0 \leq i < n$, we have that $c_i \rightarrow c_{i+1}$, and there does not exist $c \in C$ such that $c_n \rightarrow c$. We write $\ell(P)$ to denote the length of the play P, which is equal to the number of configurations in P, if P is finite, and is equal to $+\infty$ if P is infinite. Let G be a game structure and c be a configuration of G, we note $P(G, c)$ for the set of all plays in G starting from configuration c. A *winning condition* W for a game structure G and a configuration c is a subset $W \subseteq P(G, c)$, that is: a subset of plays starting in c. A game is a triple (G, c, W) where G is a game structure, c is a configuration of G, and W is the subset of plays starting in c. During a play, players apply strategies. Let C^* denotes finite sequences of configurations from set of configurations C. A *strategy* for player 1 (1-strategy for short) is a partial function $\mathcal{S} : C^* \rightarrow C_2$ such that if we have $\mathcal{S}(c_1 c_2 \ldots c_n) = c'$, then $c_n \rightarrow c'$. We define a 2-*strategy* symmetrically. A strategy is *memory free* if it is such that for any $c_1 c_2 \ldots c_n \in C^*$, we have that $\mathcal{S}(c_1 c_2 \ldots c_n) = \mathcal{S}(c_n)$, that is the strategy only depends on the current configuration and not on the history of the play. The *outcome of a strategy* is defined as follows. Let $\mathcal{S}$ be a 1-strategy, the outcome of $\mathcal{S}$ in configuration c is the set of all plays $P = c_0 c_1 \ldots c_n \ldots \in P(G, c)$, noted $\mathsf{Outcome}(G, c, \mathcal{S})$, such that: for any i, $0 \leq i \leq \ell(P)$, we have that if $c_i \in C_1$ and $\mathcal{S}(c_0 c_1 \ldots c_i)$ is defined then $c_{i+1} = \mathcal{S}(c_0 c_1 \ldots c_i)$. So, the outcome of a 1-strategy S from a configuration c is the set of plays starting in c that are generated when player 1 plays with strategy $\mathcal{S}$. We say that player 1 *has a winning strategy* for the game (G, c, W) if $\mathsf{Outcome}(G, c, \mathcal{S}) \subseteq W$ for some $\mathcal{S}$. We call $\mathcal{S}$ a *winning strategy.* In this paper, we will concentrate on particular games called reachability games. A *reachability game* is defined by a triple (G, c, F) where G is a game structure with set of configurations C, c is a configuration of G, and $F \subseteq C$ is a subset of configurations, called the winning configurations. The triple (G, c, F) defines the game (G, c, W) where W is the set of plays starting in c that contain at least one configuration of F. The 1-*reachability* problem is defined as follows: given a reachability game defined by a triple (G, c, F), does player 1 have a wining strategy for this game. The 2-*reachability* problem is defined symmetrically. Given a set of configurations $S \subseteq C$ of a game G, we define the following sets: $\mathsf{CPre}_{1,G}(S)$ is the set of configurations where player 1 has a one step strategy to reach S, that is $\mathsf{CPre}_{1,G}(S)$ equals

$$\begin{gathered}\{c \in C_1 \mid \exists c' \in S : c \rightarrow c'\} \\ \cup \{c \in C_2 \mid \exists c' \in S : c \rightarrow c' \text{ and } \forall c' \in C : c \rightarrow c' \text{ implies } c' \in S\}.\end{gathered}$$

The operator $\mathsf{CPre}_{2,G}(S)$ is defined symmetrically. We define $\mathsf{CPre}^0_{1,G}(S)$ as S, for any $n \in \mathbb{N}$, $\mathsf{CPre}^{n+1}_{1,G}(S)$ as $\mathsf{CPre}_{1,G}(\mathsf{CPre}^n_{1,G}(S))$ and $\mathsf{CPre}^*_{1,G}(S) = \bigcup_{n \in \mathbb{N}} \mathsf{CPre}^n_{1,G}(S)$. It is well known, see for example [dAHM01], that the following theorem holds:

Theorem 1. *For any reachability game (G, c, F), for $i \in \{1, 2\}$, we have that player i has a winning strategy for the game (G, c, F) iff $c \in \mathsf{CPre}^*_{i,G}(F)$.*

Note that the operator $\mathsf{CPre}_{i,G}$ is monotonic w.r.t. to the inclusion relation between sets of configurations.

Well quasi-orderings A *well quasi ordering* $\preccurlyeq$ on the elements of a set S, wqo for short, is a *reflexive* and *transitive* relation such that for any infinite sequence $s_0 s_1 \ldots s_n \ldots$ of elements in S, there exist indices i and j, such that $i < j$ and $s_i \preccurlyeq s_j$. In the following, we note $s_i \prec s_j$ if $s_i \preccurlyeq s_j$ but $s_j \not\preccurlyeq s_i$. For example, it is well known that the quasi order $\sqsubseteq \subseteq \mathbb{N}^k \times \mathbb{N}^k$ defined as $\langle m_1, m_2, \ldots, m_k\rangle \sqsubseteq \langle m'_1, m'_2, \ldots, m'_k\rangle$ if $m_i \leq m'_i$ for any $1 \leq i \leq k$ is a wqo. In this paper, we will concentrate on wqo. Given a wqo $\preccurlyeq$ over the elements of S, a set $U \subseteq S$ is called an $\preccurlyeq$*-upward closed set* if for any $s_1 \in U$, for any $s_2 \in S$ such that $s_1 \preccurlyeq s_2$, we have that $s_2 \in U$. We say that B_U generates U for $\preccurlyeq$ iff $\{s \mid \exists s' : s' \in B_U \wedge s' \preccurlyeq s\} = U$, B_U is called a generator set for U. For any $\preccurlyeq$-upward closed set $U \subseteq S$, we note $\mathsf{Min}(U)$ the set of elements $\{s \in U \mid \neg\exists s' \in U : s \neq s' \wedge s' \preccurlyeq s\}$. It is easy to show that this set is finite for any $\preccurlyeq$-upward closed set of elements provided that $\preccurlyeq$ is a wqo. Furthermore, $\mathsf{Min}(U)$ generates U, and so, can be seen as a *finite* representation of the potentially infinite set U. We now recall two useful results from [Hig52]:

Lemma 1. *Let S be a set of elements, $\preccurlyeq \subseteq S \times S$ be a wqo, and $S_0 S_1 \ldots S_n \ldots$ be a infinite sequence of $\preccurlyeq$-upward closed subsets of S such that $S_i \subseteq S_{i+1}$ for any $i \geq 0$, then there exists $j \geq 0$ such that for any $k \geq j$, $S_j = S_k$.*

Lemma 2. *Let S be a set of elements, $\preccurlyeq \subseteq S \times S$ be a wqo, such that for any $s_1, s_2 \in S$ we can compute a finite subset of S that generates the set $\{s \in S \mid s_1 \preccurlyeq s$ and $s_2 \preccurlyeq s\}$. Let U_1 and U_2 two $\preccurlyeq$-upward closed sets. Given B_1 a generator set of U_1 and B_2 a generator set of U_2 we can compute a generator set of $U_1 \cup U_2$ and a generator set of $U_1 \cap U_2$.*

Two counter-machines A 2-*counter machine* M is a pair (Q, I) where Q is a finite set of states and I is a finite set of instructions. Each instruction is of one of the two following forms:

- $(q : c_i := c_i + 1,\ \mathit{goto}\ q')$ where $q, q' \in Q$ and $i \in \{1, 2\}$
- $(q : \mathit{if}\ \ c_i = 0\ \ \mathit{then}\ \mathit{goto}\ q'\ \ \mathit{else}\ \ c_i = c_i - 1\ \ \mathit{goto}\ q'')$ where $q, q', q'' \in Q$ and $i \in \{1, 2\}$.

A *configuration of* M is a triple (q, k_1, k_2) with $q \in Q$ and $k_1, k_2 \in \mathbb{N}$. We note C_M the set of configurations of M. We define a transition relation $\xrightarrow{M} \subseteq C_M \times C_M$ on configurations of M as follows: $(q, k_1, k_2) \xrightarrow{M} (q', k'_1, k'_2)$ iff either

- $(q : c_i := c_i + 1\ \mathit{goto}\ q') \in I$, $k'_i = k_i + 1$ and $k'_{3-i} = k_{3-i}$, or
- $(q : \mathit{if}\ \ c_i = 0\ \mathit{then}\ \mathit{goto}\ q'\ \mathit{else}\ c_i = c_i - 1\ \mathit{goto}\ q'') \in I$ and $k'_i = k_i = 0$ and $k'_{3-i} = k_{3-i}$ or

– $(q : if\ c_i = 0\ then\ goto\ q''\ else\ c_i = c_i - 1\ goto\ q') \in I$ and $k'_i = k_i - 1$ and $k'_{3-i} = k_{3-i}$.

A 2-counter machine is *deterministic* if for each $q \in Q$ there exists at most one configuration of the form $(q : ...)$. The *2-counter machine reachability problem* is defined as follows: given a deterministic 2-counter machine $M = (Q, I)$ and two states $q_i, q_f \in Q$, is there a sequence $(q_0, k_0, k'_0) \xrightarrow{M} (q_1, k_1, k'_1) \xrightarrow{M} (q_2, k_2, k'_2) \xrightarrow{M} ... \xrightarrow{M} (q_n, k_n, k'_n)$ of transitions such that $q_0 = q_i$, $q_n = q_f$, $k_0 = 0$, and $k'_0 = 0$? It is well known, see for example [Min72], that:

Theorem 2. *The reachability problem is undecidable for deterministic 2-counter machines.*

3 Coverability Games

In this section, we define game structures that are monotonic w.r.t. a wqo. We show that Petri nets naturally define such game structures when we distinguish transitions that are controlled by one player, say the system, and transitions that are controlled by the other player, say the environment. We define natural problems for those monotonic game structures: the player 1 and player 2 coverability games. Unfortunately, and surprisingly, we show that contrary to the one player case (well-structured transition systems), coverability games are undecidable here.

3.1 Monotonic game structures

In this paper, we study reachability game problems defined on game structures that are monotonic w.r.t. a wqo on the set of configurations of each player in the following precise sense:

Definition 1 (Monotonicity). A *game structure* $\langle C, C_1, C_2, \rightarrow \rangle$ is *monotonic* for a well quasi order $\preccurlyeq \subseteq (C_1 \times C_1) \cup (C_2 \times C_2)$ if the following condition is verified: for any $c_1, c_2 \in C$, if $c_1 \rightarrow c_2$, then for all $c_3 \in C$, such that $c_1 \preccurlyeq c_3$, there exists $c_4 \in C$ with $c_3 \rightarrow c_4$ and $c_2 \preccurlyeq c_4$.

Given a game structure $G = \langle C, C_1, C_2, \rightarrow \rangle$ and a wqo $\preccurlyeq \subseteq (C_1 \times C_1) \cup (C_2 \times C_2)$ such that G is monotonic for $\preccurlyeq$, then we write $G = \langle C, C_1, C_2, \rightarrow, \preccurlyeq \rangle$ to underline that G is a monotonic game for $\preccurlyeq$. In [ABd03], the authors have established the following undecidability result about monotonic game structures:

Theorem 3. *The* 1 *reachability and* 2 *reachability game problems are undecidable on monotonic games.*

The result holds even if the set F that defines the set of configurations to reach is a finite set. We have proved this result independently in [Sam03] for a subclass of monotonic game structures defined by Petri nets, that we call Petri game structures. We define them in the next subsection.

3.2 Petri game structures

A *Petri net* is a pair $\langle \mathcal{P}, \mathcal{T} \rangle$ where $\mathcal{P}$ is a finite set of *places* and $\mathcal{T}$ is a finite set of *transitions*. A *transition* t is a pair $\langle \mathcal{I}, \mathcal{O} \rangle$ where $\mathcal{I}, \mathcal{O} : \mathcal{P} \to \mathbb{N}$. A *marking* of $\langle \mathcal{P}, \mathcal{T} \rangle$ is a function $m : \mathcal{P} \to \mathbb{N}$. A transition $t = \langle \mathcal{I}, \mathcal{O} \rangle$ is *firable* from a marking m if $m(p) \geq \mathcal{I}(p)$ for all $p \in \mathcal{P}$. We note $m \rightarrowtail^t m'$ if t is firable from m and $m'(p) = m(p) + \mathcal{O}(p) - \mathcal{I}(p)$ for all $p \in \mathcal{P}$. We note $m_1 \sqsubseteq m_2$ iff for any $p \in \mathcal{P}$, $m_1(p) \leq m_2(p)$.

Petri nets are usually used to model closed systems. In the next definition, we propose to use Petri nets to define two-player game structures by simply partitioning the set of transitions of the Petri net in two: one subset of the transitions are owned by player 1 (say the system) and the other subset of transitions are owned by player 2 (say the environment). With this in mind, Petri nets naturally define two-player game structures. This is formally expressed in the next definition:

Definition 2 (Petri game structures). Given a Petri net $\langle \mathcal{P}, \mathcal{T} \rangle$, a partition of $\mathcal{T}$ into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$, we define the following game structure $G = \langle C, C_1, C_2, \rightarrow \rangle$, called a *Petri game structure*, where $C_1 = \{(m,1) \mid m : \mathcal{P} \to \mathbb{N}\}$, $C_2 = \{(m,2) \mid m : \mathcal{P} \to \mathbb{N}\}$, and $\rightarrow = \{((m,1),(m',2)) \mid m \rightarrowtail^t m' \wedge t \in \mathcal{T}_1\} \cup \{((m,2),(m',1)) \mid m \rightarrowtail^t m' \wedge t \in \mathcal{T}_2\}$.

Let $G = \langle C, C_1, C_2, \rightarrow \rangle$ be a Petri game structure, we define $\overline{\sqsubseteq} \subseteq (C_1 \times C_1) \cup (C_2 \times C_2)$ as follows: for any $(m_1,1),(m_2,1) \in C_1$, $(m_1,1) \overline{\sqsubseteq} (m_2,1)$ iff $m_1 \sqsubseteq m_2$, and for any $(m_1,2),(m_2,2) \in C_1$, $(m_1,2) \overline{\sqsubseteq} (m_2,2)$ iff $m_1 \sqsubseteq m_2$. It is clear that $\overline{\sqsubseteq}$ is a wqo. Furthermore, we have that:

Lemma 3. *Any Petri game structure $G = \langle C, C_1, C_2, \rightarrow \rangle$ is monotonic for the wqo $\overline{\sqsubseteq}$.*

3.3 Coverability games and undecidability

As we are interested in monotonic game structures, it is natural to consider a particular kind of reachability games where the set of configurations to reach is an upward-closed set of configurations. We call those games *coverability games*.

Definition 3. A *coverability game* is defined by a reachability game (G, c, F) where G is a monotonic game structure for a given wqo $\preccurlyeq$ on the configurations of G, c is a configuration of G, and F is an upward closed set of configurations for the wqo $\preccurlyeq$.

The 1-*coverability problem* is defined as follows: given a coverability game defined by a triple (G, c, F), does player 1 have a winning strategy for this game ? The 2-*coverability problem* is defined symmetrically. The following theorem states the undecidability of coverability games on monotonic game structures.

Theorem 4. *The* 1*-coverability and* 2*-coverability problems are undecidable for monotonic games.*

Proof. In the following, we reduce the 2-counter machine reachability problem to the 1-coverability problem. Given a 2-counter machine $M = (Q, I)$ and two states $q_i, q_f \in Q$ we construct a 1-coverability game such that there is a winning 1-strategy in that game if and only if (M, q_i, q_f) is a positive instance of the 2-counter reachability problem. Intuitively, player 1 will simulate the 2-counter machine M and player 2 will verify the simulation. Formally, we define the monotonic game structure $G = (C, C_1, C_2, \rightarrow, \preccurlyeq)$ as follows:

- the set of configurations is $C = Q \times \mathbb{N}^2 \times \{G, B, T_1, T_2\} \times \{1, 2\}$, which is partitioned into C_1 and C_2 as follows: for $j \in \{1, 2\}$, $C_j = Q \times \mathbb{N}^2 \times \{G, B, T_1, T_2\} \times \{j\}$;
- the transition relation is defined as $\rightarrow = \xrightarrow{Inc} \cup \xrightarrow{Test} \cup \xrightarrow{Ver}$ where:
  - $\xrightarrow{Inc}$ contains the pairs $\{((q, k_1, k_2, G, 1), (q', k_1', k_2', G, 2))$ such that there exists $i \in \{1, 2\}$ such that $(q : c_i := c_i + 1,\ goto\ q') \in I$, $k_i' = k_i + 1$, and $k_{3-i}' = k_{3-i}$;
  - $\xrightarrow{Test}$ contains the pairs $((q, k_1, k_2, G, 1), (q', k_1', k_2', G, 2))$ such that there exists $(q : if\ c_i = 0\ then\ goto\ q''\ else\ c_i = c_i - 1,\ goto\ q') \in I$, $k_i' = k_i - 1$, and $k_{3-i}' = k_{3-i}$, and the pairs $((q, k_1, k_2, G, 1), (q', k_1', k_2', T_i, 2))$ such that there exists $(q : if\ c_i = 0\ then\ goto\ q'\ else\ c_i = c_i - 1,\ goto\ q'') \in I$, $k_i' = k_i$, and $k_{3-i}' = k_{3-i}$.
  - $\xrightarrow{Ver}$ contains the pairs $((q, k_1, k_2, X, 1), (q, k_1, k_2, G, 2))$ such that $q \in Q$, $k_1, k_2 \in \mathbb{N}$, $X \in \{G, T_1, T_2\}$, and the pairs $((q, k_1, k_2, T_i, 2), (q, k_1, k_2, B, 1))$ such that $q \in Q$, $k_1, k_2 \in \mathbb{N}$, and $k_i > 0$.
- finally, $\preccurlyeq = \{((q, k_1, k_2, X, j), (q, k_1', k_2', X, j)) \mid k_1 \leq k_1', k_2 \leq k_2'\}$

It is easy to verify that $\preccurlyeq$ is a wqo over $(C_1 \times C_1) \cup (C_2 \times C_2)$ and that G is a monotonic game structure for that wqo. Let $F = \{(q_f, G, k_1, k_2, 1) \mid k_1, k_2 \in \mathbb{N}\}$. The 1-coverability game is defined by the monotonic game structure G, the upward closed set F and the initial configuration $(q_i, G, 0, 0, 1)$. Let $\mathcal{S}$ the memory free 1-strategy defined as follows:

- $\mathcal{S}(q, k_1, k_2, G, 1) = (q', k_1, k_2, T_i, 2)$ if $\exists (q : if\ c_i = 0\ then\ goto\ q''\ else\ c_i = c_i - 1,\ goto\ q') \in I$ with $i \in \{1, 2\}$ and $k_i = 0$;
- $\mathcal{S}(q, k_1, k_2, G, 1) = (q', k_1', k_2', G, 2)$ if $\exists (q : if\ c_i = 0\ then\ goto\ q'\ else\ c_i = c_i - 1,\ goto\ q'') \in I$ with $i \in \{1, 2\}$, $k_i' = k_i - 1$ and $k_{3-i}' = k_{3-i}$.

We consider a play in that game, we can suppose that the play stops as soon as the set F is reached. If player 1 does not play following the 1-strategy $\mathcal{S}$, a configuration of the set $\{(q, k_1, k_2, T_i, 2) \mid i \in \{1, 2\}, k_i > 0\}$ is reached and player 2 can reach a configuration $(q, k_1, k_2, B, 1)$ from which the game is blocked. So if player 1 does not play following the memory free 1-strategy, player 2 can win the play. If player 1 plays following the strategy $\mathcal{S}$, he simulates perfectly the run of the 2-counter machine and he wins the coverability game if and only if there is a sequence $(q_0, k_0, k_0') \xrightarrow{M} (q_1, k_1, k_1') \xrightarrow{M} (q_2, k_2, k_2') \xrightarrow{M} \ldots \xrightarrow{M} (q_n, k_n, k_n')$ of transitions such that $q_0 = q_i$, $q_n = q_f$, $k_0 = 0$, and $k_0' = 0$.

As a consequence of the reduction, the 1-coverability problem is undecidable. Since we can invert the roles of the players 1 and 2 we can prove in the same way that the 2-coverability problem is undecidable.

It is easy to show that the reduction above can be specialized for Petri game structures. This has be done in details in [Sam03], so the next stronger theorem also holds:

Theorem 5. *The* 1*-coverability and* 2*-coverability problems are undecidable for Petri game structures.*

4 B-game structures

In the previous section, we have shown that *even* coverability problems are undecidable on monotonic games. This is in opposition with the situation for well-structured transition systems that can be seen as one-player game structures. The authors of [ABd03] have studied downward closed game structures (a subclass of monotonic game structures) for which they obtain interesting decidability results. We study here another subclass of monotonic game structures with (other) decidability results. We call those game structures B-games. They are defined as follows:

Definition 4 (B-game structures). A *B-game structure* G is a monotonic game structure $\langle C, C_1, C_2, \rightarrow, \preccurlyeq \rangle$ with the following additional property: for any $c_1, c_2 \in C_2$, $c_3 \in C_1$ if $c_1 \rightarrow c_3$ and $c_2 \preccurlyeq c_1$ then there exists $c_4 \in C_1$ such that $c_2 \rightarrow c_4$ and $c_4 \preccurlyeq c_3$.

For those B-game structures, we can prove the following lemma:

Lemma 4. *Let G be a B-game structure $\langle C, C_1, C_2, \rightarrow, \preccurlyeq \rangle$, let $U \subseteq C$ be any $\preccurlyeq$-upward-closed set of configurations of G, then $\mathsf{CPre}_{1,G}(U)$ is an $\preccurlyeq$-upward-closed set of configurations.*

Proof. Remember that $\mathsf{CPre}_{1,G}(U) = \{c \in C_1 \mid \exists c' \in U : c \rightarrow c'\} \cup \{c \in C_2 \mid \exists c' \in U : c \rightarrow c'$ and $\forall c' \in C : c \rightarrow c'$ implies $c' \in U\}$. Let $c \in \mathsf{CPre}_{1,G}(U)$ and c_0 a configuration such that $c \preccurlyeq c_0$. We will show that $c_0 \in \mathsf{CPre}_{1,G}(U)$. We study two cases. Case 1: $c \in C_1$. There exists $c' \in U$ such that $c \rightarrow c'$. Since G is monotonic and $c \preccurlyeq c_0$, there exists $c'_0 \in C$ with $c_0 \rightarrow c'_0$ and $c' \preccurlyeq c'_0$. $c'_0 \in U$ because U is upward closed and so $c_0 \in \mathsf{CPre}_{1,G}(U)$. Case 2: $c \in C_2$. There exists $c' \in U$ such that $c \rightarrow c'$ and for all $c' \in C$, $c \rightarrow c'$ implies $c' \in U$. Since G is monotonic there exists $c'_0 \in C$ with $c_0 \rightarrow c'_0$ and $c' \preccurlyeq c'_0$. Let c'_0 a configuration such that $c_0 \rightarrow c'_0$. Since G is a B-game and $c \preccurlyeq c_0$, there exists $c' \in C$ with $c \rightarrow c'$ and $c' \preccurlyeq c'_0$. $c \rightarrow c'$ implies $c' \in U$. U is upward closed, so $c'_0 \in U$ and $c_0 \in \mathsf{CPre}_{1,G}(U)$. The study of this 2 cases allows us to conclude that $\mathsf{CPre}_{1,G}(U)$ is upward closed.

The previous lemma together with lemma 1 allow us to state the result about B-game structures:

Lemma 5. *Let G be a B-game structure $\langle C, C_1, C_2, \rightarrow, \preccurlyeq \rangle$ and let $F \subseteq C$ be an $\preccurlyeq$-upward-closed set, the sequence $S_0 S_1 \dots S_n \dots$ of sets of configurations defined by $S_0 = F$, and for any $i \geq 1$, $S_i = \bigcup_{l=0}^{l=i} \mathsf{CPre}^l_{1,G}(F)$, is such that there exists $j \geq 0$ such that for any $k \geq j$, $S_k = S_{k+1}$.*

This last lemma means that the iteration of $\mathsf{CPre}_{1,G}$ operator starting from a $\preccurlyeq$-upward closed set stabilizes after a finite number of steps. So, if $\mathsf{CPre}_{1,G}(U)$ can be effectively computed for any upward-closed U and if $\preccurlyeq$ is a decidable wqo, then the 1-coverability problem is decidable. In the next subsection, we define B-Petri game structures for which we can effectively compute a finite generator for $\mathsf{CPre}_{1,G}(U)$ for any upward-closed set U.

4.1 B-Petri game structures

Intuitively, B-Petri game structures are Petri game structures where player 2 can only have input places that are bounded. We formally define B-Petri game structures in the following definition:

Definition 5 (B-Petri game structures). Given a tuple $\langle \mathcal{P}_b, \mathcal{P}_u, \mathcal{T}_1, \mathcal{T}_2, f \rangle$ where

- $\langle \mathcal{P}_b \cup \mathcal{P}_u, \mathcal{T}_1 \cup \mathcal{T}_2 \rangle$ is a Petri net such that $\mathcal{P}_b \cap \mathcal{P}_u = \emptyset$ and $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$;
- $f : \mathcal{P}_b \to \mathbb{N}$ is a function that bounds the number of tokens that can be simultaneously in places that belongs to $\mathcal{P}_b$;
- $\forall \langle \mathcal{I}, \mathcal{O} \rangle \in \mathcal{T}_2$ and $\forall p \in \mathcal{P}_u$ we have $\mathcal{I}(p) = 0$[1].

we define the following monotonic game structure $\langle C, C_1, C_2, \to, \widetilde{\sqsubseteq} \rangle$, where the configurations are defined as follows: for $i \in \{1, 2\}$, C_i is the set of pairs $\langle m, i \rangle$ where m is a marking such that $m(p) \leq f(p)$ for all $p \in \mathcal{P}_b$. And the transition relation is defined as: for any $\langle m_1, i_1 \rangle \in C_{i_1}$ and $\langle m_2, i_2 \rangle \in C_{i_2}$, we have $\langle m_1, i_1 \rangle \to \langle m_2, i_2 \rangle$ if $m_1 \rightarrowtail^t m_2$ for some $t \in \mathcal{T}_{i_1}$, and either $i_1 = 1$ and $i_2 = 2$, or $i_1 = 2$ and $i_2 = 1$. And $\widetilde{\sqsubseteq}$ is a wqo defined as follows: for any two configurations $\langle m_1, i_1 \rangle$ and $\langle m_2, i_2 \rangle$, we have $\langle m_1, i_1 \rangle \widetilde{\sqsubseteq} \langle m_2, i_2 \rangle$ if and only if $m_1(p) = m_2(p)$ for all $p \in \mathcal{P}_b$, $m_1(p) \leq m_2(p)$ for all $p \in \mathcal{P}_u$ and $i_1 = i_2$.

The next lemma states formally that any B-Petri game structure is a B-game structure.

Lemma 6. *Any B-Petri game structure is a B-game structure.*

Proof. B-Petri game structures are monotonic for the same reason as Petri game structures are monotonic. We show that they also satisfy the additional requirement expressed in definition 4. Let $\langle \mathcal{P}_b, \mathcal{P}_u, \mathcal{T}_1, \mathcal{T}_2, f \rangle$ be a tuple that defines the B-Petri game structure $\langle C, C_1, C_2, \to, \widetilde{\sqsubseteq} \rangle$, $\langle m_1, 2 \rangle$, $\langle m_2, 2 \rangle$ and $\langle m_1', 1 \rangle$ configurations such that $\langle m_1, 2 \rangle \to \langle m_1', 1 \rangle$ and $\langle m_2, 2 \rangle \widetilde{\sqsubseteq} \langle m_1, 2 \rangle$. We know that there exists $\langle \mathcal{I}, \mathcal{O} \rangle \in \mathcal{T}_2$ such that $m_1 \rightarrowtail^{\langle \mathcal{I}, \mathcal{O} \rangle} m_1'$. Since for any $p \in \mathcal{P}_u$, $\mathcal{I}(p) = 0$, and for any $p \in \mathcal{P}_b$, $m_1(p) = m_2(p)$, we have that $m_2 \rightarrowtail^{\langle \mathcal{I}, \mathcal{O} \rangle} m_2'$ with $\langle m_2', 2 \rangle \widetilde{\sqsubseteq} \langle m_1', 2 \rangle$. Therefore $\langle C, C_1, C_2, \to, \widetilde{\sqsubseteq} \rangle$ defines a B-game structure.

In the context of B-Petri game structures, we show that $\mathsf{CPre}^1_G(U)$ is effectively constructible for any upward-closed set U. First, we need the following lemma:

[1] This condition ensures that player 2 can not test unbounded places.

Lemma 7. *Let $G = \langle C, C_1, C_2, \rightarrow, \widetilde{\sqsubseteq}\rangle$ be a B-game structure defined by a tuple $\langle \mathcal{P}_b \cup \mathcal{P}_u, \mathcal{T}_1 \cup \mathcal{T}_2, f\rangle$. Let $U_1 \subseteq C$ and $U_2 \subseteq C$ two upward closed sets. Given B_1 a generator set of U_1 and B_2 a generator set of U_2 we can compute a generator set for $U_1 \cup U_2$ and a generator set for $U_1 \cap U_2$.*

Proof. To apply lemma 2, we prove that given any two configurations (m, i) and (m', i'), we can compute a finite generator set for $\{c \mid (m, i)\widetilde{\sqsubseteq}c \wedge (m', i')\widetilde{\sqsubseteq}c\}$. If $i \neq i'$ or if $\exists p \in \mathcal{P}_b : m(p) \neq m'(p)$ then $\{c \mid (m, i)\widetilde{\sqsubseteq}c$ and $(m', i')\widetilde{\sqsubseteq}c\} = \emptyset$. If $i = i'$ and $\forall p \in \mathcal{P}_b : m(p) = m'(p)$ then $\{(m'', i)\}$ is a finite generator set of $\{c \mid c \geq (m, i)$ and $c \geq (m', i')\}$ with

– $\forall p \in \mathcal{P}_b \; m''(p) = m(p)$
– $\forall p \in \mathcal{P}_u \; m''(p) = max(m(p), m'(p))$

We have established that, in the context of B-Petri game structures, we can effectively compute union and intersection of upward-closed sets. We now show that for any upward-closed set U, we can effectively compute a finite generator for $\mathsf{CPre}_{1,G}(U)$.

Lemma 8. *Let $G = \langle C, C_1, C_2, \rightarrow, \widetilde{\sqsubseteq}\rangle$ be a B-game structure defined by the tuple $\langle \mathcal{P}_b \cup \mathcal{P}_u, \mathcal{T}_1 \cup \mathcal{T}_2, f\rangle$, then for any upward-closed set of markings U defined by a finite generator set B_U, a finite generator set for $\mathsf{CPre}_{1,G}(U)$ exists and is effectively contructible.*

Proof. From lemma 4, we know that $\mathsf{CPre}_{1,G}(U)$ is upward closed. From B_U, we will construct a finite generator set for $\mathsf{CPre}^1_G(U)$. By lemma 7, we know that given any $\widetilde{\sqsubseteq}$-upward closed set U_1 and U_2 with finite generator sets B_{U_1} and B_{U_2}, we are able to construct the finite generator $B_{U_1 \cup U_2}$ for $U_1 \cup U_2$ and $B_{U_1 \cap U_2}$ for $U_1 \cap U_2$. This can be extended for any finite unions and intersections. For $j \in \{1, 2\}$, let $B_j = B_U \cap C_j$, it is clear that B_j is a finite generator set of $U \cap C_j$. Note that $\mathsf{CPre}_{1,G}(U) = \mathsf{CPre}_{1,G}(U \cap C_1) \cup \mathsf{CPre}_{1,G}(U \cap C_2)$. Remember that $\mathsf{CPre}_{1,G}(U \cap C_1) = \{(m, 2) \mid \exists t \in \mathcal{T}_2 : m \xrightarrow{t} m'$ and $\forall t' \in \mathcal{T}_2 : m \xrightarrow{t'} m''$ implies $(m'', 1) \in U \cap C_1\}$. First we construct $MT_1 = \{(m, \langle \mathcal{I}, \mathcal{O}\rangle) \in B_1 \times \mathcal{T}_2 \mid \forall p \in \mathcal{P}_b : m(p) \geq \mathcal{O}(p)\}$. For each $(m, \langle \mathcal{I}, \mathcal{O}\rangle) \in MT_1$, we compute the marking $\mu(m, \langle \mathcal{I}, \mathcal{O}\rangle)$ such that $\forall p \in \mathcal{P} \; \mu(m, \langle \mathcal{I}, \mathcal{O}\rangle)(p) = max(\mathcal{I}(p), m(p) + \mathcal{I}(p) - \mathcal{O}(p))$ and the set of transitions $T(m, \langle \mathcal{I}, \mathcal{O}\rangle) = \{\langle \mathcal{I}', \mathcal{O}'\rangle \in \mathcal{T}_2 \mid \forall p \in \mathcal{P}_b : \mu(m, \langle \mathcal{I}, \mathcal{O}\rangle)(p) \geq \mathcal{I}'(p)\}$. Then for each $\langle \mathcal{I}', \mathcal{O}'\rangle \in \mathcal{T}_2$, we construct the following set of markings: $M(\langle \mathcal{I}', \mathcal{O}'\rangle) = \{m' \in B_f \mid \forall p \in \mathcal{P}_b : m'(p) \geq \mathcal{O}'(p)\}$. Finally we note Γ_1 the following set of configurations:

$$\bigcup_{(m,t) \in MT_1} [\{c \mid (\mu(m, t), 2)\widetilde{\sqsubseteq}c\} \cap (\bigcap_{t' \in T(m,t)} \bigcup_{m' \in M(t')} \{c \mid (\mu(m', t'), 2)\widetilde{\sqsubseteq}c\})]$$

Since Γ_1 is defined by finite unions and intersections of upward closed sets of configurations for which we have a finite generator set we can compute a finite generator set of Γ_1. It is long but easy to verify that $\Gamma_1 = \mathsf{CPre}_{1,G}(U \cap C_1)$. $\mathsf{CPre}_{1,G}(U \cap C_2) = \{(m, 1) \mid \exists t \in \mathcal{T}_1 : m \xrightarrow{t} m'$ and $(m', 2) \in U\}$. First, we construct $MT_2 = \{(m, \langle \mathcal{I}, \mathcal{O}\rangle) \in B_2 \times \mathcal{T}_1 \mid \forall p \in \mathcal{P}_b : m(p) \geq \mathcal{O}(p)\}$. For each $(m, \langle \mathcal{I}, \mathcal{O} \rangle) \in MT_2$, we compute the marking $\mu(m, \langle \mathcal{I}, \mathcal{O} \rangle)$ such that $\forall p \in \mathcal{P}$, $\mu(m, \langle \mathcal{I}, \mathcal{O} \rangle)(p) = max(\mathcal{I}(p), m(p) + \mathcal{I}(p) - \mathcal{O}(p))$. Finally, we note Γ_2 the following set of configurations: $\Gamma_2 = \bigcup_{(m,t) \in MT_2} \{c \mid (\mu(m,t), 1) \widetilde{\sqsubseteq} c\}$. We can compute a finite generator set for Γ_2. It is easy to verify that $\Gamma_2 = \mathsf{CPre}_{1,G}(U \cap C_2)$.

We are now ready to establish our positive result:

Theorem 6. *The* 1*-coverability problem is decidable for B-Petri game structures.*

Proof. From lemma 8, we can construct for any $k \in \mathbb{N}$ a finite generator set for $\mathsf{CPre}^k_{1,G}(U)$ and from lemma 7, for any $k \in \mathbb{N}$ we can effectively construct a finite generator for $\bigcup_{n \in [0,k]} \mathsf{CPre}^n_{1,G}(U)$. From lemma 5, there exists $k \in \mathbb{N}$ such that $\bigcup_{n \in [0,k]} \mathsf{CPre}^n_{1,G}(U) = \mathsf{CPre}^*_{1,G}(U)$. Then, we can compute a finite generator for $\mathsf{CPre}^*_{1,G}(U)$ and determine if $c \in \mathsf{CPre}^*_{1,G}(U)$. Finally we deduce from theorem 1 that 1-coverability problem is decidable for B-Petri game structures.

Strategy synthesis We have shown the decidability of the 1 coverability problem. We now show that we can automatically construct winning strategies for those games. Let (G, c, U) be a B-Petri 1-coverability game defined by the B-Petri game structure G with the wqo $\widetilde{\sqsubseteq}$ as defined in definition 5, and U be an $\widetilde{\sqsubseteq}$-upward closed set of configurations defined by the finite generator set B_U. We assume that $c \in \mathsf{CPre}^*_{1,G}(U)$. Then we can construct a winning memory free 1-strategy $\mathcal{S}$ with the following algorithm. We consider the game (G, c, U) and a configuration c' we will compute $\mathcal{S}(c')$. If $c' \in U$ we do not need to define $\mathcal{S}(c')$. If $c' \in \mathsf{CPre}^*_{1,G}(U) \setminus U$, we compute $\mathsf{CPre}^n_{1,G}(U)$ for n where n is the smallest integer m such that $c' \in \mathsf{CPre}^m_{1,G}(U)$. We choose $\mathcal{S}(c')$ among $\{c'' \in \mathsf{CPre}^{m-1}_{1,G}(U) \mid c' \rightarrow c''\}$.

The synthesis algorithm above can be applied for any 1-coverability B-game (G, c, U) such that:

- the wqo is decidable;
- given a finite generator set for an upward closed set U we can construct a finite generator set for $\mathsf{CPre}_{1,G}(U)$
- for all configuration c, $\{c' \mid c \rightarrow c'\}$ is finite and computable.

On the other hand, we have the following negative result about B-games:

Theorem 7. *The* 2*-coverability problem is undecidable for B-game structures.*

Proof. We follow the same proof schema as in the proof of the theorem 4, that is we reduce the 2-counter machine reachability problem to the 2-coverability problem is for B-games. Given a 2-counter machine $M = (Q, I)$ and two states $q_i, q_f \in Q$ we construct a corresponding 2-coverability B-game such that there is a winning 2-strategy in that game if and only if (M, q_i, q_f) is a positive instance of the 2-counter reachability problem. Unlike the previous construction player 2 will simulate the 2-counter machine M and player 1 will verify the simulation. Since we have to construct a B-game player 2 can not decrement a counter and we will modify the game as follows: player 2 will be able to force player 1 to decrement a counter when needed.

Formally, we define the B-game structure $G = (C, C_1, C_2, \rightarrow, \preccurlyeq)$ as follows:

- $C = Q \times \mathbb{N}^2 \times \{G, B, T_1, T_2, D_1, D_2\} \times \{1, 2\}$
- for $j \in \{1,2\}$ $C_j = Q \times \mathbb{N}^2 \times \{G, B, T_1, T_2, D_1, D_2\} \times \{j\}$
- the transition relation is defined as $\rightarrow = \xrightarrow{Inc} \cup \xrightarrow{Test} \cup \xrightarrow{Ver}$ where $\xrightarrow{Inc} = \{((q, k_1, k_2, G, 2), (q', k_1', k_2', G, 1)) \mid \exists i \in \{1,2\}\ such\ that\ (q : c_i := c_i + 1,\ goto\ q') \in I, k_i' = k_i + 1\ and\ k_{3-i}' = k_{3-i}\}$
  $\xrightarrow{Test} = \{((q, k_1, k_2, G, 2), (q', k_1, k_2, T_i, 1)), ((q, k_1, k_2, G, 2), (q'', k_1, k_2, D_i, 1)) \mid \exists i \in \{1,2\}\ such\ that\ (q : if\ c_i = 0\ then\ goto\ q'\ else\ c_i = c_i - 1,\ goto\ q'') \in I, k_i' = k_i\ and\ k_{3-i}' = k_{3-i}\}$
  $\xrightarrow{Ver} = \{((q, k_1, k_2, X, 1), (q, k_1, k_2, G, 2)), \mid q \in Q\ k_1, k_2 \in \mathbb{N}, X \in \{G, T_1, T_2\}$
  $\cup\{((q, k_1, k_2, D_i, 1), (q, k_1', k_2', G, 2)), \mid q \in Q\ k_i' = k_i - 1\ k_{3-i}' = k_{3-i}\}$
  $\cup\{((q, k_1, k_2, T_i, 1), (q, k_1, k_2, B, 2)), \mid q \in Q\ k_1, k_2 \in \mathbb{N}, k_i > 0\}$
- and finally, $\preccurlyeq = \{((q, k_1, k_2, X, j), (q, k_1', k_2', X, j)) \mid k_1 \leq k_1', k_2 \leq k_2'\}$

It is easy to verify that G is a B-game structure: G is monotonic and if $(c_1, c_2) \in \xrightarrow{Inc} \cup \xrightarrow{Test}$ and $c_1' \preccurlyeq c_1$ there exists c_2' such that $c_2' \preccurlyeq c_2$ and $(c_1', c_2') \in \xrightarrow{Inc} \cup \xrightarrow{Test}$.

Let $F = \{(q_f, k_1, k_2, G, 2) \mid k_1, k_2 \in \mathbb{N}\}$. The 2-coverability game is defined by the monotonic game structure G, the upward closed set F and the initial configuration $(q_i, G, 0, 0, 2)$. Let $\mathcal{S}$ the 2-strategy defined as follows:

- $\mathcal{S}(q, k_1, k_2, G, 1) = (q', k_1, k_2, T_i, 2)$ if $\exists (q : if\ c_i = 0\ then\ goto\ q'\ else\ c_i = c_i - 1,\ goto\ q'') \in I$ with $i \in \{1,2\}$ and $k_i = 0$
- $\mathcal{S}(q, k_1, k_2, G, 1) = (q', k_1, k_2, D_i, 2)$ if $\exists (q : if\ c_i = 0\ then\ goto\ q''\ else\ c_i = c_i - 1,\ goto\ q') \in I$ with $i \in \{1,2\}$, $k_i > 0$.

We consider a play in that game, we can suppose that the play stops as soon as the set F is reached. If player 2 does not play following the 2-strategy $\mathcal{S}$ we have two cases:

- a configuration of the set $\{(q, k_1, k_2, T_i, 1) \mid i \in \{1,2\}, k_i > 0\}$ is reached and player 1 can reach a configuration $(q, k_1, k_2, B, 1)$ from which the game is blocked.
- a configuration of the set $\{(q, k_1, k_2, D_i, 1) \mid i \in \{1,2\}, k_i = 0\}$ is reached and the game is blocked

So if player 2 does not play following the 2-strategy $\mathcal{S}$, player 1 can win the play. If player 2 plays following the strategy s, he simulates perfectly the run of the 2-counter machine and he wins the play if and only if there is a sequence $(q_0, k_0, k_0') \xrightarrow{M} (q_1, k_1, k_1') \xrightarrow{M} (q_2, k_2, k_2') \xrightarrow{M} ... \xrightarrow{M} (q_n, k_n, k_n')$ of transitions such that $q_0 = q_i$, $q_n = q_f$, $k_0 = 0$, and $k_0' = 0$. So the 2-coverability problem is undecidable for the B-games.

BB-game structures Note that from the results above, it is clear that we can obtain a subclass of monotonic game structures where both the 1-coverability and the 2-coverability problems are decidable: it is sufficient to impose to player 1 the restriction imposed to player 2 in definition 4. We call those monotonic game structures, the *BB-game structures* and define them as follows:

Definition 6 (BB-game structures). A *BB-game structure* G is a monotonic game structure $\langle C, C_1, C_2, \rightarrow, \preccurlyeq\rangle$ with the following two additional properties:

- for any $c_1, c_2 \in C_1$, $c_3 \in C_2$ if $c_1 \rightarrow c_3$ and $c_2 \preccurlyeq c_1$ then there exists $c_4 \in C_2$ such that $c_2 \rightarrow c_4$ and $c_4 \preccurlyeq c_3$.
- for any $c_1, c_2 \in C_2$, $c_3 \in C_1$ if $c_1 \rightarrow c_3$ and $c_2 \preccurlyeq c_1$ then there exists $c_4 \in C_1$ such that $c_2 \rightarrow c_4$ and $c_4 \preccurlyeq c_3$.

We can easily adapt our proofs above to obtain:

Theorem 8. *The* 1*-coverability and the* 2*-coverability problems are decidable for BB-game structures.*

5 Comparison with [ABd03]

The authors of [ABd03] have independently study monotonic game structures. In their paper, they concentrate on reachability problems instead of coverability problems. We compare here the negative and positive results obtained in the two papers.

Negative results We have decided to concentrate here on coverability game problems. There are at least two reasons for that. First, we know that coverability (contrary to reachability) is decidable on any well-structured transition systems (those can naturally be seen as one player game structures). So, we had some hope to obtain decidable for coverability games on monotonic two-player game structures. Second, the coverability games are intuitively the simplest games that we can study on monotonic game structures. Let us illustrate the second point. We can show that the undecidability of natural problems on monotonic game structures are implied by the undecidability of coverability on those game structures. As an example, let us consider the *place boundedness games* played on Petri game structures. A *place boundedness game* is defined by a Petri net $\langle \mathcal{P}, \mathcal{T}\rangle$, a place $p \in \mathcal{P}$, and a configuration c in the corresponding Petri game structure G. The player-1 place boundedness problem (player-2 place boundedness problem is defined symmetrically) asks if, given a Petri net $\langle \mathcal{P}, \mathcal{T}\rangle$, a place $p \in \mathcal{P}$, and a configuration c, player 1 has a strategy $\mathcal{S}$ such that there exists a bound $b \in \mathbb{N}$ such that for any play $P \in \mathsf{OutCome}(G, c, \mathcal{S})$, in any configuration of P, the marking of place p is less or equal to b. This problem is undecidable and this is a corollary of the undecidability for coverability games:

Theorem 9. *The player-*1 *and player-*2 *place boundedness problems are undecidable on Petri game structures.*

The authors of [ABd03] have obtained other interesting undecidability results that are not implied by ours, concerning parity-games on monotonic and downward-closed game structures (a subclass of monotonic game structures whose definition is recalled below).

Positive results The positive results that we obtain in this paper are orthogonal to the positive results obtained in [ABd03]. In fact, the positive results obtained in that paper are for a subclass of monotonic game structures called *downward-closed game structures*. Those game structures are defined as follows for player 2 (they are defined symmetrically for player 1):

Definition 7 (2 Downward-closed game structures). A monotonic game structure $G = \langle C, C_1, C_2, \rightarrow, \preccurlyeq \rangle$ is a 2 *downward-closed game structure* if the following property holds: for each $c_1, c_2 \in C_2$, and $c_3 \in C_1$, whenever $c_1 \rightarrow c_3$, and $c_1 \preccurlyeq c_2$, then $c_2 \rightarrow c_3$.

It is easy to establish that:

Theorem 10. *There are monotonic game structures that belong to the class of B-game structures but not to the class of* 2 *downward-closed game structures. And, there are monotonic game structures that belong to the class of* 2 *downward-closed game structures but not to the class of B-game structures.*

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