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CSTs (Covering Sharing Trees): *Compact Data Structures for Parameterized Verification*

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Abstract. The *control state reachability problem* is decidable for well-structured infinite-state systems like unbounded Petri Nets, Vector Addition Systems, Lossy Petri Nets, and Broadcast Protocols. An abstract algorithm that solves the problem is given in [AČJT96, FS01]. The algorithm computes the closure of the *predecessor* operator w.r.t. a given upward-closed set of target states. When applied to this class of verification problems, traditional (infinite-state) symbolic model checkers suffer from the *state explosion problem* even for very small examples. We provide a new data structure to represent in a compact way collections of upwards closed sets over numerical domains. This way, we turn the abstract algorithm of [AČJT96,FS01] into a practical method. We also combine of the backward reachability algorithm with a *new heuristic rule* based on *structural properties* of Petri Nets. We evaluate the method on several infinite state Petri Nets and parameterized systems taken from the literature [ABC+95,EM00,Fin93,MC99], and we compare the results with other finite and infinite-state verification tools.

Keywords Infinite-state Verification, Constraints, Data Structures, Heuristics, Static Analysis, Backward Reachability

1 Introduction

In the last years many efforts have been made to extend the theoretical results and practical methods developed for verification of finite-state systems [BCB+90] to systems with *infinite* state space (see e.g. [AČJT96, BM99,BW98,EFM99,FS01]). This class of systems comprises well-known examples like Vector Addition Systems [Min67], extensions of Petri Nets [Cia94,Ter94], Integral Relational Automata [Čer94], and more recent examples like Broadcast Protocols [EN98] and Lossy Petri Nets [BM99]. The *control state reachability* problem is decidable for all previous systems [AČJT96, BM99,EFM99,Fin90,FS01]. The abstract algorithm of [AČJT96,FS01] computes the closure of the *predecessor* operator w.r.t. a given upward-closed set of states. The algorithm can be used to check, e.g., invariant properties like *mutual exclusion* and *coverability for markings of Petri Nets*.

As in the finite-state case, the success of symbolic model checking for this class of problems depends on the data structures used as implicit representation of sets of states. Over numerical domains, upward-closed sets can be represented via a sub-class of integer arithmetic constraints (see e.g. [AČJT96,DEP99]). In this setting, the state-space generated by the algorithm of [AČJT96, FS01] can be represented as a large (often huge) *disjunction* of arithmetic constraints. In [DEP99], the authors tested constraint-based model checking methods over integers (based on a solver for Presburger arithmetic [BGP97]) and reals (based on polyhedra [HHW97]) on verification problems that can be expressed via upward-closed sets. Though the examples in [DEP99] would be considered of negligible size in finite-state model checking (e.g. 6 transitions and 10 variables, cf. [BCB+90]), the methods taken into consideration suffer from the *symbolic state explosion* problem. Specifically, the length of the arithmetic formulas used to represent the intermediate results obtained during the fixpoint computation 'explodes'. Some of the experiments required (when terminating) execution times in the order of days. Based on these observations, it seems natural to look for BDD-like data structures to represent 'compactly' the generalization of boolean formulas we are interested in.

In this paper we propose a new symbolic representation for upward-closed sets starting from the *sharing trees* of Zampuniéris and Le Charlier [ZL94], proposed originally to represent compactly large finite collections of tuples. Sharing trees are acyclic graphs used to represent large sets of tuples, e.g., of integer numbers. The intuition behind the choice of this data structure is the following. An upward-closed set U is determined by its finite set of minimal points (tuples of integers). Thus, we can represent the set U via a sharing tree whose paths correspond to its minimal points, or to a finite set of generators (if there are not only minimal elements). The semantics of the graph structure is the infinite set of integer tuples greater or equal to the tuples represented by the sharing tree. The data structure with the new intepretation is called *Covering Sharing Tree*. This way, we managed to turn the abstract algorithm of [AČJT96, FS01] into a 'practical method' working on the examples studied in [DEP99, Ter94] in acceptable time cost.

Technically, our contributions are as follows.

Covering Sharing trees We introduce a logic (the logic $\mathcal{U}$) where collections of upward-closed sets can be represented as disjunctive formulas. $\mathcal{U}$-formulas are used for verification problems of infinite-state systems as boolean formulas are used for the finite-state case. We show that sharing trees can be used to obtain compact representations of $\mathcal{U}$-formulas (there exist a $\mathcal{U}$-formula that can be represented using a sharing tree whose size is logarithmic in the size of the formula). We show how basic operations on $\mathcal{U}$-formulas (e.g. conjunction and disjunction) can be implemented symbolically on the corresponding sharing trees. Sharing trees can be viewed as the BDDs for $\mathcal{U}$-formulas. In this paper we investigate the theoretical complexity of all operations on CSTs needed in symbolic model checking. Some of the CSTs operations have exponential worst case complexity. For this operations we will propose algorithms and heuristics that seem to behave well in practice.

Simulation-based Heuristics In practical cases, covering sharing trees may still become very large, e.g., due to the presence of redundant generators. As we prove in the paper, removing all redundancies from a covering sharing tree representing a $\mathcal{U}$-formula cannot always be done in polynomial time (in the size of the covering sharing tree). For this reason, we propose approximations with polynomial time algorithms that can be used to eliminate redundancies. The algorithms are based on *simulation relations* defined between nodes of covering sharing trees. The same techniques can be used to give sufficient conditions for the subsumption test of convering sharing trees representing $\mathcal{U}$-formulas (i.e. for the termination test of the algorithm of [AČJT96]). The complete test is co-NP hard in the size of the covering sharing trees.

Structural Analysis and Backward Reachability In order to attack *symbolic state explosion* for Petri Nets of large size, we combine our CST-based technology with heuristics based on the *Structural Theory* of Petri Nets. More precisely, the heuristic rule we propose in this paper is based on a *fully automatic* static analysis [STC98], whose results can be used during the backward reachability algorithm to significantly cut the search space. As the other general purpose heuristics presented in the first part of the paper (e.g. simulation relations), our *structural heuristic* works in polynomial time on the graph structure of CSTs. When combined with our CSTs-symbolic representation, the heuristic rule allow us to scale up the dimension of the case-studies of one order of magnitude.

Experimental Evaluation As *practical contribution*, we present a set of benchmarks we obtained with an optimized implementation of the CST-library, integrated with the structural heuristic. We have applied the resulting model checking algorithm to a large set of examples of parameterized verification problems that can be solved using decision procedures for coverability of Petri Nets (e.g. mutual exclusion for parametric models like the Mesh and Multipoll examples of [ABC$^+$95, MC99], and semi-liveness for the PNCSA protocol of [Fin93]). We have also applied our method to verify safety properties of *finite-state* systems (e.g. some of the above mentioned examples for fixed values of the parameter). For these examples, we have compared our results with the results obtained with the specialized tool GreatSPN [CFGR95] for computing the reachability set of Petri Nets. As foreseen by Bultan in [Bul00], in most of the cases proving a parameterized property turns out to be more efficient than proving its finite-state instances.

Plan of the Paper. In Section 2 we introduce some preliminary notions we will use throughout the paper. In Section 3 we define the logic $\mathcal{U}$and in Section 4 we define the operations on formula of the logic $\mathcal{U}$in disjonctive form. In Section 5 we introduce the notion of *sharing tree*. In Section 6 we introduce the symbolic representation of $\mathcal{U}$-formulas via *Covering Sharing Trees*. In Section 7 we define the symbolic operations for *Covering Sharing Trees*. In Section 8 we define simulation relations for nodes of covering sharing trees and discuss their application in the operations for $\mathcal{U}$-formulas. In Section 9 we illustrate how these technology can be used for verification of *parameterized verification problems*. In Section 9.2 we define a CST-based symbolic model checking procedure for parameterized systems modeled via Petri Nets. In Section 10 we define new structural heuristics based on the Structural Theory of Petri Nets. In Sections 11 we present an experimental evaluation of our techniques. In Sections 12 we address some conclusions and future research directions and discuss related works.

This paper includes and extends with proofs and new experiments the material presented in [DR00,DRV01].

2 Preliminaries

In this paper we will work on sets of tuples of integer values. We will use $\mathbb{Z}$ to denote integer numbers, $\mathbb{N}$ to denote non-negative integer numbers, and D^k to denotes sets of tuples with k components taken from D. Furthermore, we will use $\mathbf{t}$ to denote a tuple $\langle t_1, \ldots, t_k \rangle$ in D^k, i.e. such that $t_i \in D$ for $i : 1, \ldots, k$.

The *componentwise ordering* $\preccurlyeq$ over tuples in $\mathbb{Z}^k$ is defined as follows: $\mathbf{t} \preccurlyeq \mathbf{t}'$ if and only if $t_i \leq t'_i$ for $i : 1, \ldots, k$. As an example, $\langle 1,0,0 \rangle \preccurlyeq \langle 3,2,1 \rangle$, whereas $\langle 1,0,0 \rangle \not\preccurlyeq \langle 0,1,0 \rangle$ and $\langle 0,1,0 \rangle \not\preccurlyeq \langle 1,0,0 \rangle$ (i.e. $\langle 1,0,0 \rangle$ and $\langle 0,1,0 \rangle$ are not comparable).

If $\mathbb{Z}$ is extended with special values like $-\infty$ and $+\infty$, then the ordering is naturally extended by taking $-\infty \leq c$ and $c \leq +\infty$ for any $c \in \mathbb{Z}$, and $-\infty \leq +\infty$. As an example, $\langle -\infty, 10, 0 \rangle \preccurlyeq \langle 0, +\infty, 0 \rangle$.

Given a tuple $\mathbf{t} \in \mathbb{Z}^k$, we define $cone(\mathbf{t})$ as the *upward-closed set* (or *cone*) generated by $\mathbf{t}$, namely,

$$cone(\mathbf{t}) = \{\mathbf{t}' \in \mathbb{Z}^k \mid \mathbf{t} \preccurlyeq \mathbf{t}'\}$$

We extend this definition to a set S in the natural way. A set S of tuples is *upward-closed* if and only if $S = cone(S)$.

When restricted to *non negative* integer values, the ordering $\preccurlyeq$ turns out to be a *well-quasi ordering*, i.e., for any infinite sequence $\mathbf{t_1 t_2} \ldots$ of tuples in $\mathbb{N}^k$ there always exists to indices $i < j$ such that $\mathbf{t_i} \preccurlyeq \mathbf{t_j}$. A consequence of this property is that *upward-closed sets* of tuples in $\mathbb{N}^k$ are always *generated* by a *finite sets of minimal tuples* (finite basis). This is also true if we take as domain for the components of our tuples $\mathbb{N} \cup \{+\infty\}$. Note that this property does not hold for $\mathbb{Z}^k$.

3 The Logic of Upward-Closed Sets

In this section we introduce the logic $\mathcal{U}$ that we use to define collections of upward-closed sets. The idea is to start from a subset of linear integer arithmetic constraints having the basic form $x \geq c$ for some constant c, and to enrich them with *disjunction*, *conjunction* and a special operation to *translate* a given coordinate from its current value x_i to $x_i + d$ for some displacement $d \in Int$. The latter kind of formula will be useful to concisely represent *transition* relations for systems defined over integer data paths (e.g. operations like increment and decrement for *counter* machines). Formally, let $V = \{x_1, \ldots, x_k\}$ be a finite set of variables. The set of $\mathcal{U}$-formulas is defined then by the following grammar.

$$\Phi \quad ::= \quad x_i \geq c \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Phi[x_i + d/x_i],$$

where $c \in \mathbb{Z} \cup \{-\infty\}$, $d \in \mathbb{Z}$. $\mathcal{U}$-formulas are interpreted over $\mathbb{Z}^k$. In the following we will use the tuple $\mathbf{t} = \langle t_1, \ldots, t_k \rangle \in \mathbb{Z}^k$ to denote the valuation of the variables $\langle x_1, \ldots, x_k \rangle$ in which x_i is mapped to t_i for $i : 1, \ldots, k$. Satisfaction of a formula with respect to a valuation $\mathbf{t}$ is defined as follows.

Definition 1 (Satisfiability Relation). Given a $\mathcal{U}$-formulas Φ, and a valuation $\mathbf{t}$, we define the satisfiability relation $\models$ as follows:

- let $\Phi = x_i \geq c$, then $\mathbf{t} \models \Phi$ iff $t_i \geq c$;
- let $\Phi = \Phi_1 \wedge \Phi_2$, then $\mathbf{t} \models \Phi$ iff $\mathbf{t} \models \Phi_1$ and $\mathbf{t} \models \Phi_2$;
- let $\Phi = \Phi_1 \vee \Phi_2$, then $\mathbf{t} \models \Phi$ iff $\mathbf{t} \models \Phi_1$ or $\mathbf{t} \models \Phi_2$;
- let $\Phi = \Psi[x_i + c/x_i]$, then $\mathbf{t} \models \Phi$ iff $\mathbf{t}' \models \Psi$ and $\mathbf{t}'$ is such that $t'_i = t_i + c$, whereas $t'_j = t_j$ for any $j \neq i$.

The *denotation* of a $\mathcal{U}$-formula is defined as follows.

Definition 2 (Denotation of $\mathcal{U}$-formulas). Given a $\mathcal{U}$-formula Φ, its denotation $[\![\Phi]\!]$ is defined as the set of all valuations $\mathbf{t}$ such that $\mathbf{t} \models \Phi$.

We also introduce the following relation between $\mathcal{U}$-formulas.

Definition 3 (Subsumption Relation). A formula Φ_1 is *subsumed* by a formula Φ_2, written $\Phi_1 \models \Phi_2$, if and only if $[\![\Phi_1]\!] \subseteq [\![\Phi_2]\!]$.

We say that two $\mathcal{U}$-formulas Φ_1 and Φ_2 are equivalent, written $\Phi_1 \equiv \Phi_2$, if their denotations coincide. As an example, let $k = 3$ and consider the formula $\Phi = ((x_1 \geq 3 \vee x_2 \geq 6) \wedge x_3 \geq 3)[x_1 + 2/x_1]$. The denotation $[\![\Phi]\!]$ contains the tuples $\langle 1,0,4 \rangle$ and $\langle -4,7,4 \rangle$, whereas it does not contain the tuple $\langle 0,1,4 \rangle$.

Note that, if we restrict the domain of interpretation of our formulas to $\mathbb{N}^k$, the class of $\mathcal{U}$-formulas can be used to denote upward-closed sets of tuples of *non negative* integer values, i.e., all sets $I \subseteq \mathbb{N}^k$ such that if $\mathbf{t} \in I$ then $cone(\mathbf{t}) \subseteq I$.

We introduce next the notion of *disjunctive form*.

Definition 4 (Disjunctive form). A $\mathcal{U}$-formula in disjunctive form has the following form:

$$\bigvee_{i:1,\ldots,n} (x_1 \geq c_{i,1} \wedge \ldots \wedge x_{k_i} \geq c_{i,k_i}).$$

By adding, when necessary, formulas of the form $x_i \geq -\infty$, it is easy to see that any $\mathcal{U}$-formula can be reduced to an *equivalent disjunctive formula*. In fact, we can always push a substitution down to atomic formulas, i.e.,

$$(\Phi_1 \bullet \Phi_2)[x_i + c/x_i] \equiv \Phi_1[x_i + c/x_i] \bullet \Phi_2[x_i + c/x_i]$$

for $\bullet \in \{\wedge, \vee\}$. Furthermore, we have that

$$(x_i \geq d)[x_i + c/x_i] \equiv x_i \geq d - c,$$

whereas

$$(x_i \geq d)[x_j + c/x_j] \equiv x_i \geq d \text{ for } i \neq j.$$

As an example, consider again the formula Φ defined as

$$((x_1 \geq 3 \vee x_2 \geq 6) \wedge x_3 \geq 3)[x_1 + 2/x_1]$$

It can be equivalently written as $\Phi_1 \vee \Phi_2$, where

$$\Phi_1 \;=\; x_1 \geq 1 \;\wedge\; x_2 \geq -\infty \;\wedge\; x_3 \geq 3$$

and

$$\Phi_2 \;=\; x_1 \geq -\infty \;\wedge\; x_2 \geq 6 \;\wedge\; x_3 \geq 3$$

To ease the notation, in the sequel of the paper we will use Φ, Ψ, etc. to denote *arbitrary* $\mathcal{U}$-formulas, and ϕ, ψ, etc. to denote formulas in disjunctive form. We now define the notion of *generator* of a disjunctive formula.

Definition 5 (Generators). The set of *generators* of a *disjunctive* formula φ are defined as $gen(\varphi)$ is the set of tuples $\langle c_1, \ldots, c_k \rangle$ such that $x_1 \geq c_1 \wedge \ldots \wedge x_k \geq c_k$ is one of the disjuncts in φ.

We define the minimal elements as follows.

Definition 6 (Minimal Elements). The *minimal elements* $min(\varphi)$ of a *disjunctive* formula φ are the tuples of $gen(\varphi)$ that are *minimal* with respect to the component-wise ordering $\preccurlyeq$.

Finally, we say that a disjunctive formula is in *normal form* whenever $gen(\varphi) = min(\varphi)$.

As an example, consider the formula $\varphi = (x \geq 1 \wedge y \geq 2) \vee (x \geq 3 \wedge y \geq 1) \vee (x \geq 2 \wedge y \geq 0)$. Then, $gen(\varphi) = \{\langle 1, 2\rangle, \langle 3, 1\rangle, \langle 2, 0\rangle\}$, and $min(\varphi) = \{\langle 1, 2\rangle, \langle 2, 0\rangle\}$, i.e., φ is *not* in normal form. Note that, given a disjunctive formula φ, $[\![\varphi]\!]$ coincides with the union of the denotations $cone(t)$ computed over all $\mathbf{t} \in min(\varphi)$.

4 Operations on formulas in disjunctive form

4.1 Disjunction

Trivially, formulas in disjunctive form are closed under $\vee$. Furthermore, given two formulas in normal form φ_1 and φ_2 we can build a formula ψ in normal form such that $[\![\psi]\!] = [\![\varphi_1]\!] \cup [\![\varphi_2]\!]$ in time $|\varphi_1| \times |\varphi_2|$ (the time needed to remove the redundancies).

4.2 Conjunction

Furthermore, given the disjunctive formulas φ and ψ, the disjunctive formula for $\varphi \wedge \psi$ is defined as follows: $\bigvee_{\mathbf{t} \in gen(\varphi), \mathbf{t}' \in gen(\psi)} (x_1 \geq max(t_1, t'_1) \wedge \ldots \wedge x_k \geq max(t_k, t'_k))$, i.e., $gen(\varphi \wedge \psi) = \{\mathbf{s} \mid \exists \mathbf{t} \in gen(\varphi), \mathbf{t}' \in gen(\psi)$ and $s_i = max(t_i, t'_i)\}$. Note that, even if the operands φ and ψ are in normal form, then the resulting formula may not be in normal form.

4.3 Substitution

The formula $\varphi[x_i + c/x_i]$ is equivalent to the formula φ' obtained from φ by replacing every atom $x_i \geq d$ with $x_i \geq d - c$ ($-\infty - c = -\infty$ and $+\infty - c = +\infty$ for any $c \in \mathbb{Z}$), i.e., $gen(\varphi[x_i + c/x_i]) = \{\mathbf{t}' \mid t'_i + c = t_i,\ t'_j = t_j\ j \neq i,\ and\ \mathbf{t} \in gen(\varphi)\}$.

4.4 Satisfiability

Given a valuation $\mathbf{t}$, we first note that $\mathbf{t} \models \varphi$ iff there exists $\mathbf{t}' \in gen(\varphi)$ such that $\mathbf{t}' \preccurlyeq \mathbf{t}$. Thus, checking $\mathbf{t} \models \varphi$ can be done in time linear in the size of φ.

Note that the cost of checking if a valuation $\mathbf{t}$ satisfies an arbitrary $\mathcal{U}$-formula Φ is still linear in the size of the formula $\mathcal{U}$. We simply have to follow the definition of $\models$ given by induction on the structure of Φ.

4.5 Subsumption

Let φ and ψ be in disjunctive form. We can check $\varphi \models \psi$ in time quadratic in the size of the formulas. In fact, $\varphi \models \psi$ holds iff for all $\mathbf{t} \in gen(\varphi)$ there exists $\mathbf{t}' \in gen(\psi)$ such that $\mathbf{t}' \preccurlyeq \mathbf{t}$.

It is important to remark that the subsumption test is much harder for *arbitrary* $\mathcal{U}$-formulas, as stated in the following theorem.

Theorem 1. Given two arbitrary $\mathcal{U}$-formulas Φ and Ψ, checking that $\Phi \models \Psi$ is co-NP complete in the size of the formulas.

Proof. The complement of the problem is in NP. In fact, we can decide it by 'guessing' a valuation of length k and testing if satisfies Φ but not Ψ in time linear in the size of the formulas (see previous Section 4.4).

To prove co-NP hardness, we use a reduction from the *validity* problem (see e.g. [Joh90]) for propositional formulas in disjunctive normal form:

> *Given a propositional formula $F = D_1 \vee \ldots \vee D_n$ where each disjunct D_i is a conjunction of literals over the set of propositional variables $p_1, \ldots, p_m$, is F true for all valuations?*

This problem is co-NP complete [Joh90].

The reduction works as follows. Given a propositional formula F, we build a $\mathcal{U}$-formula $\Phi_{val(F)}$ (polynomial in the size of F) that represents all valuations that make F true. Then, we reduce validity for F to the subsumption problem $\Phi_{val(true)} \models \Phi_{val(F)}$. Without loss of generality, we assume that *all variables* occur in each disjunct of F either as p, $\neg p$ or $any(p)$. The latter means that p can take any value (note: it is equivalent to say that p *does not* occur in the disjunct). The formula *true* is equivalent to $any(p_1) \wedge \ldots \wedge any(p_m)$. To encode the valuation for p_i, we use two variables from V, namely v_i and w_i. We use the formula $\Phi_{val(p_i)} = (v_i \geq 1 \wedge w_i \geq 0)$

when p_i evaluates to *true*; $\Phi_{val(\neg p_i)} = (v_i \geq 0 \wedge w_i \geq 1)$ when p_i evaluates to $false$. Note that neither $\Phi_{val(p_i)} \models \Phi_{val(\neg p_i)}$ nor $\Phi_{val(\neg p_i)} \models \Phi_{val(p_i)}$. We use the disjunction $\Phi_{val(any(p_i))} = (v_i \geq 1 \wedge w_i \geq 0) \vee (v_i \geq 0 \wedge w_i \geq 1)$ when p_i can be evaluated either to *true* or $false$. The previous encoding allows us to keep the size of $\Phi_{val(F)}$ polynomial in the size of F.

Formally, the formula $\Phi_{val(F)}$ is defined by induction on F as follows:

- $\Phi_{val(D_1 \vee D_2)} = \Phi_{val(D_1)} \vee \Phi_{val(D_2)}$;
- $\Phi_{val(C_1 \wedge C_2)} = \Phi_{val(C_1)} \wedge \Phi_{val(C_2)}$;
- $\Phi_{val(p_i)} = (v_i \geq 1 \wedge w_i \geq 0)$;
- $\Phi_{val(\neg p_i)} = (v_i \geq 0 \wedge w_i \geq 1)$;
- $\Phi_{val(any(p_i))} = (v_1 \geq 0 \wedge w_1 \geq 1) \vee (v_1 \geq 1 \wedge w_1 \geq 0)$.

Based on this construction, we obtain that

F is valid if and only if $\Phi_{val(true)} \models \Phi_{val(F)}$.

Since the reduction is *polynomial*, we obtain that the subsumption problem for $\mathcal{U}$-formulas is co-NP complete. □

4.6 Minimization

Given two formulas in disjunctive form $\varphi = \varphi_1 \vee ... \vee \varphi_k$ and $\psi = \psi_1 \vee ... \vee \psi_n$, the reduction in normal form of φ according to ψ, written $\varphi \div \psi$, consists in removing from φ all φ_i such that there exists ψ_j with $\varphi_i \models \psi_j$. This can be done in time linear in $|\varphi| \times |\psi|$. The denotation of $\varphi \div \psi$ is defined as

$$[\![\varphi \div \psi]\!] = \{\mathbf{c} \in cone(\mathbf{t}) \mid \mathbf{t} \in [\![\varphi]\!],\ \mathbf{t} \notin [\![\psi]\!]\}$$

4.7 Reduction in normal form.

Given a disjunctive formula $\varphi = \varphi_1 \vee \ldots \vee \varphi_k$ we can reduce it in normal form by eliminating all formulas φ_i such that the corresponding generator is not a minimal element. The reduction can be done in time quadratic in the size of φ.

In the following sections we will introduce special data structures, called *sharing trees* [ZL94], and we will show how to apply them to handle large sets of *generators* of formulas in disjunctive form. Specifically, we will redefine all the previous operations in terms of sharing trees so as to being able to store compactly upward-closed sets of markings generated by large sets of integer vectors.

5 Sharing Trees

In this paper we specialize the original definition of [ZL94] as follows.

Definition 7 (k-Sharing Tree). A k-sharing tree is a rooted directed acyclic graph $(N, V, val, succ)$, where

- $N = \{root\} \cup N_1 \ldots \cup N_k \cup \{end\}$ is the finite set of *nodes*; N_i is the set of nodes of *layer* i and, by convention, $N_0 = \{root\}$ and $N_{k+1} = \{end\}$;
- $val : N \rightsquigarrow \mathbb{Z} \cup \{\top, \bot\}$ is a labeling function for the nodes;
- $succ : N \rightsquigarrow 2^N$ defines the successors of a node.
- Furthermore,
  1. $val(n) = \top$ if and only if $n = root$;
  2. $val(n) = \bot$ if and only if $n = end$;
  3. $succ(end) = \emptyset$;
  4. for $i : 0, \ldots, k$, for all $n \in N_i$, $succ(n) \subseteq N_{i+1}$ and $succ(n) \neq \emptyset$;
  5. forall $n \in N$, for all $n_1, n_2 \in succ(n)$, if $n_1 \neq n_2$ then $val(n_1) \neq val(n_2)$.
  6. for $i : 0, \ldots, k$, for all $n_1, n_2 \in N_i$ s.t. $n_1 \neq n_2$, if $val(n_1) = val(n_2)$ then $succ(n_1) \neq succ(n_2)$.

In other words, a k-sharing tree is an acyclic graph with root and terminal node such that:

- all nodes of layer i have successors in the layer $i+1$ (cond. 4);
- a node cannot have two successors with the same label (cond. 5);
- finally, two nodes with the same label in the same layer do not have the same set of successors (cond. 6).

Note that the *empty sharing trees* is the sharing tree with only root and end nodes.

Definition 8 (Pre-sharing Tree). We say that S is a *pre-sharing* tree if it respects conditions (1)-(4) whereas conditions (5) and (6) are not ensured.

In the rest of the paper we use $root^S$, N^S, and $succ^S$ to refer to the *root*, *set of nodes*, *successor relation*, respectively, of the sharing-tree S.

Definition 9 (Path). A path of a k-sharing tree is a sequence of nodes $\langle root, n_1, \ldots, n_k, end \rangle$ such that $n_{i+1} \in succ(n_i)$ $i : 1, \ldots, k$-1.

Paths will represent tuples of size k of integer numbers. Formally, we introduce the following definition.

Definition 10 (Flat Denotation). The flat denotation of a k-sharing tree S, written $elem(S)$, is the set of elements represented by the paths in S. Formally, the set $elem(S)$ consists of all tuples $\langle val(n_1), \ldots, val(n_k) \rangle$ such that $\langle root, n_1, \ldots, n_k, end \rangle$ is a path in S.

Condition 5 and 6 ensure the *maximal sharing* of *prefixes* and *suffixes* among the paths (elements) of a k-sharing tree.

Definition 11 (Size). We define the *size* of a sharing tree S as the number of its *nodes* plus the number of its *edges*.

Note that the number of tuples in $elem(S)$ can be exponentially larger than the size of S (see also [Zam97]).

Operation	Effect	Complexity	Algorithm
emptyness test	$is_empty(S)$ *iff* $elem(S) = \emptyset$	constant	constant
membership test	$member(\mathbf{t}, S)$ *iff* $\mathbf{t} \in elem(S)$	poly	linear
containment test	$contained(S,T)$ *iff* $elem(S) \subseteq elem(T)$	poly	poly
union	$elem(union(S,T)) = elem(S) \cup elem(T)$	poly	poly
intersection	$elem(intersection(S,T)) = elem(S) \cap elem(T)$	poly	poly
minus	$elem(minus(S,T)) = elem(S) \setminus elem(T)$	poly	poly

Fig. 1. Complexity of the operations on sharing trees.

Given a node n of the i-th layer of a k-sharing tree S, the sub-(sharing)tree S_n rooted at n is the $k-i+1$-sharing tree obtained as follows. We first isolate the graph rooted at n that consists of all nodes reachable from n (this subgraph has $k-i+1$ layers and a terminal node). Then, we add a layer with the single root node and we set $succ(root) = \{n\}$. From the previous definition, $elem(S_n)$ consists of all tuples $\langle m, m_{i+1}, \ldots, m_k\rangle$ obtained from the subpaths $\langle n, n_{i+1}, \ldots, n_k\rangle$ of S such that $m = val(n)$ and $m_j = val(n_j)$ $j : i+1, \ldots, k$.

As shown in [ZL94], given a set of tuples F of size k, there exists a unique (modulo isomorphisms of graphs) sharing tree such that $elem(S) = F$. In the same paper the authors defined algorithms for the basic operations on the flat denotations of sharing trees. The table of Fig. 1 gives the functionality and the complexity (in the size of sharing trees) for the operations we will consider in the rest of the paper: union, intersection, emptiness, containment, and equality test. In [ZL94], the authors showed that the operations in Fig. 1 can also be applied to pre-sharing trees that satisfy condition 5.

6 Covering Sharing Trees

As explained in the previous section, sharing trees can be used to compactly represent *finite sets* of tuples. Our goal, however, is to symbolically represent *infinite, upward-closed sets* of integer tuples. For this purpose, we need to *lift* the denotation of a sharing tree to a *rich one* in which the tuples stored in its paths represent *generators* for $\mathcal{U}$-formulas in disjunctive form. In this section we will study the impact of the lifting from flat to rich denotations on the complexity of the symbolic operations on sharing trees. Let us first show how to represent $\mathcal{U}$-formulas in disjunctive form.

Definition 12 (Covering Sharing Tree (CST)). Let φ be a $\mathcal{U}$-formula in disjunctive form over $x_1, \ldots, x_k$. The *Covering Sharing Tree* S_φ that represents φ is a k-sharing tree such that $elem(S_\varphi) = gen(\varphi)$.

Definition 13 (Rich Denotation). The rich denotation of S_φ is defined as follow

$$cones(S_\varphi) = \{\mathbf{t}' \mid \mathbf{t} \preceq \mathbf{t}',\ \mathbf{t} \in elem(S_\varphi)\}.$$

By definition, we obtain that $[\![\varphi]\!] = cones(S_\varphi)$. The notion of pre-sharing tree and sub-sharing tree naturally extend to CSTs.

Definition 14 (Irredundant CST). We say that S_φ is irredundant if and only if there exists no $\mathbf{t} \in elem(S_\varphi)$ such that $\mathbf{t}' \preccurlyeq \mathbf{t}$ for $\mathbf{t}' \in elem(S_\varphi)$ distinct from $\mathbf{t}$.

The following proposition explains the advantages of using sharing trees for representing $\mathcal{U}$-formulas.

Proposition 1. There exists a disjunctive formula in normal form φ such that the corresponding CST S_φ has size logarithmic in the size of φ.

Proof. Consider the $\mathcal{U}$-formula

$$\Phi = \bigwedge_{i=1}^{m} (v_i \geq 1 \wedge w_i \geq 0) \vee (v_i \geq 0 \wedge w_i \geq 1).$$

Let $\mathbf{1}$ a vector (with $2m$ components) with only 1 as components, and let $\mathbf{t} + \mathbf{t}'$ be the vector $\mathbf{t}''$ in which $t''_i = t_i + t'_i$ for $i : 1, \ldots, m$. The corresponding disjunctive formulas φ (obtained by distributing $\wedge$) is $\bigvee_{\mathbf{c}+\mathbf{d}=\mathbf{1}} \bigwedge_{i=1}^{m} (v_i \geq c_i \wedge w_i \geq d_i)$ for two vectors $\mathbf{c}$ and $\mathbf{d}$ of nonnegative integers. This formula has $\mathcal{O}(2^m)$ disjuncts. Also note that φ is in normal form. In contrast, as shown in Fig. 2, the CST S_φ has size logarithmic in φ and polynomial in Φ (each layer has two nodes and at most four arcs). □

7 CSTs-based Operations for $\mathcal{U}$-formulas

In this section we show how operations on the disjunctive formulas φ and ψ can be defined symbolically on the representations given via the CSTs S_φ and S_ψ. We use the term *symbolically* because the algorithms that we propose work directly on the graphs S_φ and S_ψ and not by *enumerating* the tuples that they represent. In the sequel we will show that the complexity of the operations strongly depends on the form of the CSTs we consider. Specifically, we will distinguish between operations that require all arguments to be *irredundant* (IRR) CSTs, and operations that relax this conditions and accept *generic* (GEN) CSTs in input. The table in Fig. 3 summarizes the complexity results presented in this section. Let us start from the simplest operations.

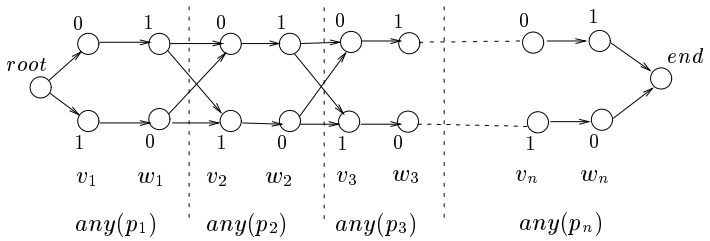

Fig. 2. Sharing tree for an exponential formula in DNF.

Operation	Effect	Type Args	Complexity	Algorithm
Emptyness	$CST_Empty(S)$ iff $cones(S) = \emptyset$	GEN	constant	constant
Satisfiability	$CST_Sat(\mathbf{t}, S)$ iff $cone(\mathbf{t}) \subseteq cones(S)$	GEN	poly	linear
Substitution	$CST_Substitute(S, c, i) = T$ s.t. $cones(T) = cones(S)[t_i + c/t_i]$	GEN	poly	linear
Subsumption	$CST_Subsumption(S, T)$ iff $cones(T) \subseteq cones(S)$	IRR $\subseteq$ IRR	co-NP	exp
		GEN $\subseteq$ GEN	co-NP	exp
Equivalence	$CST_Equiv(S, T)$ iff $cones(S) = cones(T)$	IRR $\equiv$ IRR	poly	linear
		GEN $\equiv$ GEN	co-NP	exp
Disjunction	$CST_Union(S, T) = U$ s.t. $cones(U) = cones(S) \cup cones(T)$	GEN $\cup$ GEN = GEN	poly	poly
		IRR $\cup$ IRR = IRR	co-NP	exp
Conjunction	$CST_Intersection(S, T) = I$ s.t. $cones(I) = cones(S) \cap cones(T)$	GEN $\cap$ GEN = GEN	co-NP	exp
		IRR $\cap$ IRR = IRR	co-NP	exp
Minimization	$CST_Minimization(S, T) = D$ s.t. $elem(D) = \{\mathbf{m} \in elem(S) \mid \mathbf{m} \notin cones(T)\}$	GEN $\div$ GEN = GEN	co-NP	exp
		IRR $\div$ IRR = IRR	co-NP	exp

Fig. 3. Complexity of the operations on CSTs.

7.1 Emptyness Test

First of all, let us note that checking emptyness has constant time. In fact, it is sufficient to test if the CST contains only the *root* and *end* nodes.

7.2 Satisfiability Test

Checking that $\mathbf{t} \models \varphi$ on the CST S_φ has cost *linear* in the size of φ, i.e., possibly logarithmic in the size of φ. In fact, the following theorem holds.

Theorem 2. Let S be a CST and $\mathbf{t}$ be a vector of length k. We can check if there exists $\mathbf{t}' \in elem(S)$ such that $\mathbf{t}' \preccurlyeq \mathbf{t}$, namely $\mathbf{t}$ is subsumed by S, in time linear in the number of edges of S.

Proof. We exhibit an algorithm that has time complexity that is linear in the size of S. The algorithm is based on a *layer-by-layer* comparison of the labels of the nodes of S with the components of the vector: at a given layer, we mark all nodes that belong to a path (tuple) that is candidate to subsume the vector. Specifically, we first mark $root_S$. Then, if a node m at the i-th layer of S is marked, we mark all successors n of m such that $t_{i+1} \geq val(n)$. At the end of the procedure, the vector $\mathbf{t}$ is subsumed by $cones(S)$ if and only if the node end_S is marked (note: we assume that $t_0 = \top$ and $t_{k+1} = \bot$). □

7.3 Substitution

The substitution operation applied on a formula φ represented by a CST S_φ can be computed in linear time in the lenght of S_φ.

Theorem 3. *Given the CST S_φ, there exists a linear-time algorithm that computes the CST $S_{\varphi[x_i+c/x_i]}$ such that $elem(S_{\varphi[x_i+c/x_i]}) = gen(\varphi[x_i + c/x_i])$.*

Proof. Given the CST S_φ, we build the CST $S_{\varphi[x_i+c/x_i]}$ as follows. $S_{\varphi[x_i+c/x_i]}$ has the same components as S_φ except from the valuation function. Let val be the evaluation mapping of S_φ and val' be the evaluation mapping of $S_{\varphi[x_i+c/x_i]}$. Then, $val'(n) = val(n) - c$ for every node $n \in N_i$, and $val'(m) = val(m)$ for $m \in N_j$ and $j \neq i$ (we use the same set of nodes for the two CSTs). □

7.4 Subsumption

Let us first remark the following. Even in an *irredundant* CST, the constraints (atomic $\mathcal{U}$-formula) associated to a given variable (i.e. to the nodes of the corresponding layer) are not required to represent disjoint sets of tuples. In other words, we may look at a CST S as a sort of non-deterministic automata accepting the tuples of his *rich denotation*. Non-determinism is the reason why we often obtain a compact representation. The price we have to

pay is, however, that checking subsumption between two CSTs presents a similar complexity as *trace inclusion* for non deterministic, acyclic automata. We first analyze the complexity of the problem for *irredundant* CSTs.

The case $IRR \subseteq IRR$. In this case we have the following result.

Theorem 4. The subsumption problem for two irredundant CSTs is co-NP complete in the size of the CSTs.

Proof. Let S_1 and S_2 be irredundant CSTs. The problem is checking whether or not $cones(S_1) \subseteq cones(S_2)$. We can decide the complement of the problem by guessing a tuple of length k and testing (following from Theorem 2, in linear time) that it is subsumed by S_1 and not by S_2.

To prove hardness, we use a reduction from *validity* as in Theorem 1. More precisely, given a propositional formula in DNF $F = D_1 \vee \ldots \vee D_n$ we define the CST $S_{val(F)}$ that encodes all valuations that satisfy F. We reduce then validity to the subsumption test between $S_{val(true)}$ and $S_{val(F)}$.

Following the proof of Theorem 1, we first build the two $\mathcal{U}$-formulas $\Phi_{val(true)}$ and $\Phi_{val(F)}$ that encode the valuations that make *true* and F true, respectively. Consider the $\mathcal{U}$-formulas $\Phi_{val(D_i)}$ we used in the proof of Theorem 1 to represent all evaluations that satisfy a disjunct D_i of a *propositional formula* F in DNF. The transformation of $\Phi_{val(D_i)}$ to a disjunctive formula gives a formula φ_i exponential in the size of $\Phi_{val(D_i)}$. However, the representation of φ_i using a sharing tree S_{φ_i} is polynomial in the size of the *original* formula $\Phi_{val(D_i)}$, i.e., logarithmic in φ_i. In other words, we can build S_{φ_i} 'directly' from $\Phi_{val(D_i)}$ (the formal proof is by induction on the structure of the formula). Following from the previous observations, we know that the CST $T_{val(true)}$ (with root r_t and terminal node e_t) that encodes $\Phi_{val(true)}$, and the CST S_i (with root r_i and terminal node e_i) that encodes $\Phi_{val(D_i)}$ have size polynomial in F. It remains now to define the 'irredundant' CST $S_{val(F)}$. Note that, as extensively discussed in Section 7.6, the *union* operation of Fig. 1 does not guarantee the result to be an irredundant CST. Thus, instead of using *union* we proceed as follows. By construction, every CST S_i is irredundant. Based on this fact, we first *merge* all S_i's into a single CST $S_{val(F)}$ with root r and terminal node e such that $succ(r) = \{r_1, \ldots, r_n\}$, $succ(e_1) = \{e\}, \ldots, succ(e_n) = \{e\}$. To ensure that for all pairs of distinct tuples (paths) $\mathbf{t}, \mathbf{t}'$ of $S_{val(F)}$ $\mathbf{t} \not\preccurlyeq \mathbf{t}'$ and $\mathbf{t}' \not\preccurlyeq \mathbf{t}$, we set

- $val(r_i) = i$;
- and $val(e_i) = 2n + 1 - i$ for $i : 1, \ldots, n$.

i.e., from 'left-to-right': increasing values for the r_i's; decreasing values for the e_i's. Clearly, the resulting CST is *irredundant* and has polynomial size in the size of F. Similarly, we define $S_{val(true)}$ as the CST with root r' and terminal node e' obtained from $T_{val(true)}$ setting $succ(r'_t) = \{r_t\}$ and $succ(e'_t) = \{e_t\}$. Finally, since the values of the nodes r'_t and e'_t should not influence the comparison of the paths of $S_{val(true)}$ and $S_{val(F)}$, we set $val(r_t) = val(e_t) = 2n + 1$ (a value strictly greater than the max label in $S_{val(F)}$). We give an example of the reduction in Fig. 4. Based on this construction, we obtain that

$$F \text{ is valid iff } cones(S_{val(true)}) \subseteq cones(S_{val(F)}).$$

Since the reduction is *polynomial*, we obtain that the subsumption problem for CSTs is co-NP complete (as for arbitrary $\mathcal{U}$-formulas). □

Following from the previous result, the cost of checking subsumption may be exponential in the number of edges of the input CSTs. The result follows from the fact that $\mathcal{U}$-formulas in disjunctive form can be represented compactly via CSTs.

The case $GEN \subseteq GEN$. From Theorem 4, we obtain the following corollary.

Corollary 1. *The subsumption problem for two CSTs is co-NP complete in the size of the CSTs.*

Proof. Given S_1 and S_2, we want to check $cones(S_1) \subseteq cones(S_2)$.

We can decide the complement of the problem by guessing a tuple of length k and testing (following from Theorem 2, in linear time) that it is subsumed by S_1 and not by S_2.

Hardness follows from Theorem 4, in fact, the case $IRR \subseteq IRR$ can be reduced to the case $GEN \subseteq GEN$. □

Algorithm for Subsumption The subsumption test can be perfomed symbolically via the following exponential algorithm. (The algoritm computing the simplification operation for two CSTs will be presented in section 7.8.)

function $CST_Subsumption(S, S' : CSTs) : boolean$
  $S'' = CST_Minimization(S', S)$
  return $CST_Empty(S'')$

7.5 Equivalence Test and Reduction in Normal Form

Let us now consider the problem of testing *equivalence* and then reducing a CST in normal form. Given two CSTs S and S', we first study the complexity of the test $cones(S) = cones(S')$. Notice that *equivalence* is easier than *subsumption*. This will be evident when considering comparisons of irredundant CSTs.

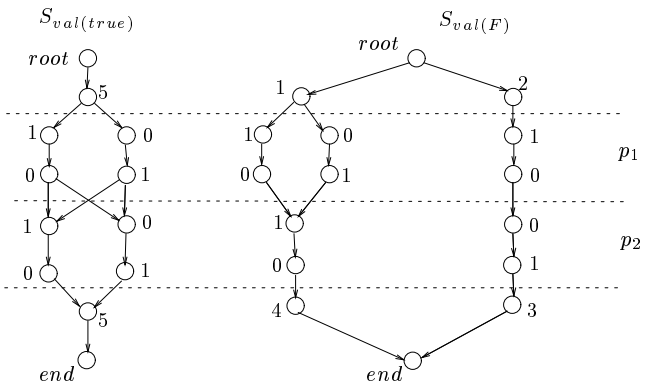

Fig. 4. The validity of $F = p_2 \vee (p_1 \wedge \neg p_2)$ is reduced to $cones(S_{val(true)}) \subseteq cones(S_{val(F)})$.

The case $IRR \equiv IRR$. If the CSTs S and S' are in normal form, then they necessarily represent the same set of minimal points. In this case we have then the following result.

Theorem 5. *Let S and S' be two irredundant CSTs. Then, $cones(S) = cones(S')$ holds if and only if $elem(S) = elem(S')$. Thus, we can decide the equivalence of S and S' with respect to their rich denotations in* linear time.

Proof. Checking if $elem(S) = elem(S')$ only consists to test if S and S' are isomorph. To do this, we start from the root nodes of S and S' and mark their sons as equilalent iff they have the same value and the same number of sons. We then treat all the equivalent nodes from the first layer to the last one in the same way. S and S' are equivalent iff all the nodes are equivalent to exactly one another. □

The case $GEN \equiv GEN$. The complexity of the equivalence test becomes harder, whenever one of operands is not irredundant. (As for subsumption, note that $GEN \equiv GEN$ cannot be reduced to $IRR \equiv IRR$, whereas the vice versa holds.)

Theorem 6. *Let S and S' be two CSTs. Checking $cones(S) = cones(S')$ is a* coNP-complete *problem.*

Proof. We can decide the complement of the problem by guessing a tuple of length k and testing in polynomial time that it is subsumed by S, but not by S', or, vice versa, it is subsumed by S', but not by S.

To prove hardness, let us first note that $cones(S) \subseteq cones(S')$ if and only if $cones(S) \cup cones(S') = cones(S')$. Furthermore, notice that a simple way to compute a generic CST T such that $cones(T) = cones(S) \cup cones(S')$ is to take T as the *sharing tree* union of S and S', i.e., $T = union(S, S')$. T will probably contain many redundant tuples, but this will not be a problem for our purposes. In fact, using the previous observation, and by the property of the union operation between sharing trees, we get a polynomial reduction from CSTs-subsumption (for generic CSTs) to CSTs-equivalence (for generic CSTs). From Cor. 1, it follows that checking equivalence between generic CSTs is co-NP hard. □

The previous results also applies to the case in which only one of the two CST is irredundant.

Corollary 2. *Let S be a generic CST, and S' be an irredundant CST. Checking $cones(S) = cones(S')$ is a* coNP-complete *problem.*

Proof. We use the same proof as for Theorem 6. □

As a consequence, we have the following result.

Corollary 3. *Given a CST S, there is no* polynomial *algorithm that computes the irrendundant CST S' such that $cones(S) = cones(S')$.*

Proof. Per absurdum. Suppose we can use our algorithm as a polynomial-time oracle for reduction in normal form. Given the CSTs S and the irredundant CST S', we first use the *oracle* to compute in polynomial time the irredundant CST T such that $cones(S) = cones(T)$. Then, we use Theorem 5 to decide in polynomial time if $cones(T) = cones(S')$. This way we have built a *polynomial* time algorithm that decides the problem of Cor. 2. This gives us a contradiction. □

Algorithm for Normalization The following algorithm computes the normal form of a given CST. The method *memo!* is used to cache intermediate results; the method *memo?* retrieves cached information. Using these two methods, we apply the method *subsumedPaths* only one time for each pair of nodes. As a consequence, the method *subsumedPaths* has polynomial complexity. To adjust condition 5 and 6 (that may be violated after running the procedure), we need to apply the algorithms of Appendix A. Thus, *NormalForm* method is exponential is the worst case (the cost of adjusting condition 5).

function *SubsumedPaths*(n_S : *node*, n'_S : *node*, *bool* :

```
boolean) : node
    newbool = bool;
    if val^S(n_S) = ⊥ then
        if bool = true then
            Sol = new node such that val^Sub(Sol) = ⊥;
        else
            Sol = null;
    else
        Sol = new node such that val^Sub(Sol) = val^S(n_S);
        For each s_S ∈ succ^S(n_S) and s'_S ∈ succ^S(n'_S) do
            if s_S ≠ s'_S then
                newbool = true;
            if val^S(s_S) ≥ val^S(s'_S) then
                Son = memo?(s_S, s'_S);
                    if Son = notmemorized then
                        Son = SubsumedPaths(s_S, s'_S, newbool);
                if Son ≠ null then
                    succ^Sub(Sol) = succ^Sub(Sol) ∪ {Son};
        if succ^Sub(Sol) = null then
            Sol = null;
    memo!(n_S, n'_S, Sol, newbool);
    return Sol;

function NormalForm(S : CST) : CST
    root_Sub = SubsumedPaths(root_S, root_S, false);
    Apply algorithms of [ZL94] that ensure condition
    5 − 6 on Sub;
    return Minus(S, Sub);
```

For a CST **S**, the *NormalForm* method first computes the *CST* **Sub** containing all the paths of S subsumed by another one. This is done by making two parallel traversals of **S**. We test during the traversals that the values of the nodes encountered during the first one are greater or equal than the value of the nodes encountered during the second one. As all paths are subsumed by themselves, we also maintain a boolean information to know whether or not we compare a path with another one. In this way, we obtain a pre-CST that may not satisfy conditions 5 or 6 in the definition of sharing trees and algorithms presented in [ZL94] must be applied to obtain a CST. Finally, we use an algorithm of [Zam97] that removes all the paths of **S** that are also in **Sub** (see Appendix A).

7.6 Disjunction

Given the formulas φ and ψ, we would like to symbolically compute their disjunction working over the corresponding CSTs S_φ and S_ψ. Let us first consider the case in which the resulting CST needs not to be irredundant.

The case $GEN \cup GEN = GEN$. As already observed in the proof of Theorem 6, in this case we have an algorithm that works in polynomial time.

Theorem 7. *Let S and S' be two CSTs. Then, we can compute a CST T such that $cones(T) = cones(S) \cup cones(S')$ by resorting to the operation of union of sharing trees $T = union(S, S')$, polynomial in the size of S and S' (see Fig. 1).*

The case $IRR \cup IRR = IRR$. The problem becomes *harder* if we require the resulting CST to be *irredundant*.

Theorem 8. *Given two irredundant CSTs S and S', there is no* polynomial *algorithm that computes the irredundant CST T such that $cones(T) = cones(S) \cup cones(S')$.*

Proof. Per absurdum. Suppose we can use our algorithm as a polynomial-time oracle for computing the union of irredundant CSTs. Let us note that $cones(S) \subseteq cones(S')$ holds if and only if $cones(S) \cup cones(S') = cones(S')$. In order to decide the subsumption problem $cones(S) \subseteq cones(S')$, we can first invoke the oracle to compute the irredundant CST T such that $cones(T) = cones(S) \cup cones(S')$. Then, we use Theorem 5 to decide in polynomial time if $cones(T) = cones(S')$. This way we built a *polynomial* time algorithm to decide the *subsumption* problem for irredundant CSTs. This gives us the contradiction. □

In this this case, the most simple way to compute the disjunction is to first apply the polynomial algorithm and then reduce in normal form the resulting CST.

7.7 Conjunction

Given the formulas φ and ψ, we would like to symbolically compute their conjunction working over the corresponding CSTs S_φ and S_ψ, i.e., compute a CST T such that $cones(T) = cones(S_\varphi) \cap cones(S_\psi)$. Let us first study the case in which the operands are irredundant CSTs.

The case $IRR \cap IRR = IRR$. The following result shows that intersection is hard when the results is required to be irredundant.

Theorem 9. *Given two CSTs S and S', there is no* polynomial *algorithm that computes the irredundant CST T such that $cones(T) = cones(S) \cap cones(S')$.*

Proof. Per absurdum. Suppose we can use our algorithm as a polynomial-time oracle for computing the intersection of irredundant CSTs. Let us note that $cones(S) \subseteq cones(S')$ holds if and only if $cones(S) \cap cones(S') = cones(S)$. In order to decide the subsumption problem $cones(S) \subseteq cones(S')$, we can first invoke the oracle to compute the irredundant CST T such that $cones(T) = cones(S) \cap cones(S')$. Then, we use Theorem 5 to decide in polynomial time if $cones(T) = cones(S')$. This way we built a *polynomial* time algorithm to decide the *subsumption* problem for irredundant CSTs. This gives us the contradiction. □

The case $GEN \cap GEN = GEN$. The following result shows that interesection is hard, even if we do not require the resulting CSTs to be irredundant.

Theorem 10. *Given two irredundant CSTs S and S', there is no* polynomial *algorithm that computes the CST T such that $cones(T) = cones(S) \cap cones(S')$.*

Proof. Following [Joh90], we exhibit a polynomial *Turing* reduction from *validity*. Specifically, we build an OTM (Oracle Turing Machine) that uses *interesction* as an oracle to decide in polynomial time *validity*. To prove the validity of a propositional formula F in DNF, we first construct the CSTs $S_{val(F)}$ and $S_{val(true)}$, as shown in the proof of Theorem 4. This step can be done in *polynomial* time. By construction, we obtain then that F is *valid* if and only if $cones(S_{val(true)}) \subseteq cones(S_{val(F)})$, i.e., if and only if $cones(S_{val(true)}) \subseteq cones(S_{val(F)}) \cap cones(S_{val(true)})$.

We apply then the oracle to compute the CST T such that $cones(T) = cones(S_{val(true)}) \cap cones(S_{val(F)})$. Note that, by construction, the intersection of the cones induced by $S_{val(true)}$ with those of $S_{val(F)}$ will always returns cones whose minimal points have the labels of the minimal points of $S_{val(true)}$ in the extra layers (the first and the last layer) used in the construction of Theorem 4. By hypothesis, T may contain redundant tuples. So we cannot apply a direct equivalence test between T and $S_{val(true)}$ (as shown in Theorem 6, this test is exponential for generic operands). However, note that $S_{val(true)}$ is irredundant by construction. Thus, to check $cones(S_{val(true)}) \subseteq cones(T)$ we simply have to check that *at least* all generators of $S_{val(true)}$ occur in T, i.e., we have to check that $elem(S_{val(true)}) \subseteq elem(T)$. This is the containment test between the flat denotation of *sharing trees* that we know to be polynomial (see Fig. 1). Thus, the resulting Turing reduction is polynomial and co-NP harndess for our search problem follows from the co-NP completeness of *validity*. □

Algorithm for Conjunction Let us now describe the algoriothm we can use to compute interesction between CSTs. Given S and S', we compute S'' such that $cones(S) \cap cones(S') = cones(S'')$ as follows. We first define a pre-sharing tree P with the following components:

1. $N^P = \{root\} \cup N_1^P \cup \ldots \cup N_k^P \cup \{end\}$ with $N_i^P = \{(n,m) \mid n \in N_i^S, m \in N_i^{S'}\}$ (i.e. a node in S'' correspond to a pair consisting of a node of S and a node of S' (of the same layer));
2. $val^P((n,m)) = max(val^S(n), val^{S'}(m))$, and
3. for all $(n,m) \in N_1 \cup \ldots N_n$, we set $succ^P((n,m)) = \{(n',m') \mid n' \in succ^S(n), m' \in succ^{S'}(m)\}$, $succ^P(root) = N_1^P$, and for all $(n,m) \in N_k^P$ we set $succ^P((n,m)) = \{end\}$.

We obtain the CST S'' from P by enforcing conditions (5-6) of Section 5 with the algorithms in Appendix A, Fig. 25 and Fig. 26. The algorithm to adjust condition 5 may have exponential time complexity in the size of the CST, whereas the time complexity to adjust condition 6 is polynmial. Note that S'' can contain redundancies, even if S and S' are irredundant. To get an irredandant CST, we still must put S'' in normal form (exponential time operation).

7.8 Minimization

Given the CSTs S_φ and S_ψ associated to the formulas φ and ψ, the operations of simplification returns the CST $S_{\varphi \div \psi}$ such that $cones(S_{\varphi \div \psi}) = cones(\varphi \div \psi)$. As for the subsumption test, we cannot define a polynomial algorithm to build $S_{\varphi \div \psi}$.

Theorem 11. *Given the CSTs S_φ and S_ψ, there is no* polynomial *algorithm, that computes the CST T such that $cones(T) = cones(\varphi \div \psi)$.*

Proof. We exhibit a polynomial *Turing* reduction from *validity*. Specifically, we build an OTM (Oracle Turing Machine) that uses $\div$ as an oracle to decide in polynomial time *validity*. We first construct the CSTs $S_{val(F)}$ and $S_{val(true)}$, as shown in the proof of Theorem 4.

We apply then the oracle to compute the CST T such that $cones(T) = cones(S_{val(true) \div F})$. Finally we note that F is valid if and only if T is the empty CST. Checking emptyness can be done in constant time. Thus, the resulting Turing reduction is polynomial and co-NP harndess for our search problem follows from the co-NP completeness of *validity*. □

Algorithm for Minimization The following algorithm computes the simplification operation for two *CST* $\mathbf{S}$ and $\mathbf{S'}$. It is very similar to the algorithm computing the normal form, its worst case time complexity is exponential.

```
function SubsumedPaths(n_S : node, n_T : node) : node
  if val^S(n_S) = ⊥ then
    Sol = new node such that val^Sub(Sol) = ⊥;
  else
    Sol = new node such that val^Sub(Sol) = val^S(n_S);
    For each s_S ∈ succ^S(n_S) and s_T ∈ succ^T(n_T) do
      if val^S(s_S) ≥ val^T(s_T) then
        Son = memo?(s_S, s_T);
          if Son = notmemorized then
            Son = SubsumedPaths(s_S, s_T);
        if Son ≠ null then
          succ^Sub(Sol) = succ^Sub(Sol) ∪ {Son};
    if succ^Sub(Sol) = null then
      Sol = null;
  memo!(n_S, n_T, Sol);
  return Sol;

function CST_Minimization(S : CST, T : CST) : CST
```

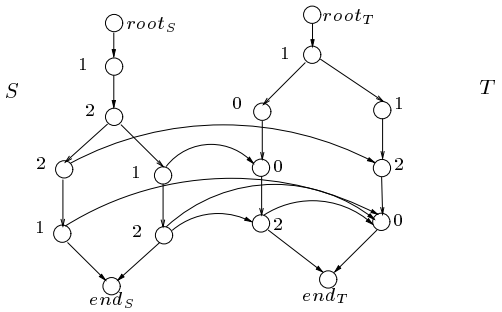

Fig. 5. The forward simulation is incomplete with respect to subsumption.

$root_{\mathbf{Sub}} = SubsumedPaths(root_{\mathbf{S}}, root_{\mathbf{T}})$;
Apply algorithms of [ZL94] *that ensure condition* $5-6$ *on* **Sub**;
return $Minus(\mathbf{S}, \mathbf{Sub})$;

8 Simulation Relations and Related Heuristics

In the previous section we have proved that the subsumption problem for two $\mathcal{U}$-formulas represented as CSTs , the computations of generators of the normal form of a $\mathcal{U}$-formula represented as a CST and the simplification of a $\mathcal{U}$-formula according to another represented both as CSTs are co-NP hard. In this section we will introduce *approximations* of the subsumption relation that can be tested more efficiently. The main idea is to exploit the *graph-structure* underlying CSTs, and in particular the possibility of defining *relations* between nodes of CSTs.

Technically, we will try to find sufficient condition for subsumption and equivalence by using *simulation relations*. Let S_1 and S_2 be two CSTs. Note that, if $elem(S_1) \subseteq elem(S_2)$ then $cones(S_1) \subseteq cones(S_2)$. Besides giving a sufficient condition for checking subsumption, the previous fact suggests a possible strategy to reduce the cost of the 'complete' test. We first compute $T = minus(S_1, S_2)$ (polynomial in the size of S_1, S_2) and then test $cones(T) \subseteq cones(S_2)$ on the (possibly) smaller CST T.

In the rest of this section we will give more interesting polynomial-time *sufficient conditions* for the subsumption test, based on a notion of *simulation* bewteen nodes of CSTs. We will see that this notion of simulation is also useful to reduce CSTs and "approximate" the reduction in normal form and the simplification.

Given two nodes n and m of a CST S we are looking for a relation $\curvearrowright^F$ such that: $n \curvearrowright^F m$ 'implies' $cones(S_n) \subseteq cones(S_m)$.

Definition 15 (Forward Simulation). Given two CSTs S and T, let n be a node of the i-th layer of S, and m be a node of the i-th layer of T. We say that n is *forward simulated by* m, written $n \curvearrowright^F m$, if $val^S(n) \geq val^T(m)$ and for all $s \in succ^S(n)$ there exists $t \in succ^T(m)$ such that $s \curvearrowright^F t$.

Note that, if $S = T$ then the simulation relation is *reflexive* and *transitive*.

Let $father^S(n) = \{\ m \mid n \in succ^S(m)\ \}$ be the set of fathers of a node n at layer i of a CST S ($fathers^S(n) \subseteq N_{i-1}$). We define the backward simulation as follows:

Definition 16 (Backward simulation). Given two CSTs S and T, let n be a node of the i-th layer of S, and m be a node of the i-th layer of T. We say that n is *backward simulated by* m, written $n \curvearrowright^B m$, if $val^S(n) \geq val^T(m)$ and for all $s \in fathers^S(n)$ there exists $t \in fathers^T(m)$ such that $s \curvearrowright^B t$.

The following result (taken from [HHK95]) shows that the previous simulations can be computed efficiently.

Theorem 12 (From [HHK95]). The forward and backward simulation relations between the nodes of the CST S and the nodes of the CST T can be computed in $O(m \cdot n)$ where m is the sum of the number of nodes in S and in T, and n is the sum of the number of edges in S and in T.

8.1 Properties of the simulation

The following propositions relate subsumption and the simulation $\curvearrowright^F$.

Lemma 1. Given the CSTs S and T, let S_n and T_m be the sub-CSTs rooted at nodes n and m, respectively. If $n \curvearrowright^F m$ then $cones(S_n) \subseteq cones(S_m)$.

Proof. We will show that for every node n of S in layer $k-l$, for $0 \leq l \leq k$ the property is verified. We reason by induction on the value of l.

Base case: $l = 0$. Let us consider $n \in N^S_{k-l}$ and $m \in N^T_{k-l}$ such that $n \curvearrowright^F m$. By definition of CSTs, we have that $succ^S(n) = \{end\}$, $succ^T(m) = \{end\}$, $elem(S_{end}) = \emptyset$, and $elem(T_{end}) = \emptyset$. Thus, $elem(S_n) = \langle val^S(n) \rangle$ and $elem(T_m) = \langle val^T(m) \rangle$, and, by definition of $\curvearrowright^F$, we have that $val^S(n) \geq val^T(m)$.

Induction step: $l > 0$. By induction hypothesis, for all nodes $n' \in N^S_{k-i}$and $m' \in N^T_{k-i}$, with $0 \leq i < l$,

we know that if $n' \frown^F m'$ then $cones(S_{n'}) \subseteq cones(S_{m'})$. Let us now consider $n \in N^S_{k-l}$ and $m \in N^T_{k-l}$ with $n \frown^F m$. By definition of $\frown^F$, we know that for $s \in succ^S(n)$ there exists $t \in succ^T(m)$ such that $s \frown^F t$. We also know that $val^S(n) \geq val^T(m)$. So $elem(S_n) = \{\langle val^S(n)\rangle \times elem(S_{n'}) \mid n' \in succ^S(n)\}$ and $elem(T_m) = \{\langle val^T(m)\rangle \times elem(T_{m'}) \mid n' \in succ^T(m)\}$, and thus we have $cones(S_n) \subseteq cones(T_m)$. □

The converse does not hold (in accord with the co-NP hardness result for subsumption). As a counterexample, take the two trees in Fig. 5. The curly arrows represent the simulation relation between nodes of S and T. Note that none of the nodes of layer 2 in T simulates the single node of S at the same layer. However, the denotation of S are contained in that of T. In fact, $\langle 1,1,2,0\rangle \preccurlyeq \langle 1,2,2,1\rangle$ and $\langle 1,0,0,2\rangle \preccurlyeq \langle 1,2,1,2\rangle$.

As Lemma 1 can be reformulated for backward simulation, the following theorem holds.

Theorem 13. Let $root_S, end_S$ and $root_T, end_T$ be the root and terminal nodes of S and T, respectively. If $root_S \frown^F root_T$ then $cones(S) \subseteq cones(T)$. Symmetrically, if $end_S \frown^B end_T$ then $cones(S) \subseteq cones(T)$.

The previous theorem gives us sufficient conditions for testing subsumption.

8.2 Use of simulations to remove redundancies

As for the subsumption test, the simulations we introduced in Section 8 can be used to 'approximate' the exact normalization procedure and the exact simplification procedure. For this purpose, we introduce a rule that allows us to exploit the information given from the simulation relations in order to 'locally' remove edges of a CST.

Definition 17 (Edge Removal). Given a CST S with nodes N^S and successors $succ^S$, let us assume that for $n, m \in N^S$ there exist $s \in succ^S(n)$ and $t \in succ^S(m)$ $(s \neq t \vee n \neq m)$ such that $s \frown^F t \wedge n \frown^B m$. Then, we define $remove(S, \langle n, s\rangle)$ as the pre-CST with successor relation $succ'$ obtained from S by setting $succ'(n) = succ(n) \setminus \{s\}$.

The following proposition states the 'correctness' of the previous rule. Its correctness follows directly from the definition of *remove*.

Proposition 2. S and $remove(S, \langle n, s\rangle)$ have the same denotations, i.e., $cones(S) = cones(remove(S, \langle n, s\rangle))$;

A possible strategy to apply Def. 17 consists of the 'on-the-fly' removal of edges during the computation of the simulation relations. Specifically, during a bottom-up traversal of the CST (assume the backward simulation is already computed), we first apply Rule 17 to every pairs of edges between nodes of layers N_i and N_{i+1}. As removing edges between nodes of these layers doesn't change the structure of the CST in the set of layers $S_1 = \{ N_j$ such that $j \leq i\}$ and the set of layers $S_2 = \{N_j$ such that $j \geq i+1$ $\}$, the backward simulation doesn't change for the nodes of layers of S_1 and the forward simulation doesn't change for the nodes of layers of S_2. We then compute the forward simulation relation for the nodes of the layer N_i, look at layers N_{i-1} and N_i, and so on. The rule to remove edges is applied exhaustively at each step.

Note that after the bottom-up traversal, the forward simulation is computed on the nodes of the resulting pre-CST but the backward simulation could have changed. An example is given in Fig. 7, where B and F represent backward and forward simulation, respectively. Since $3 \frown^B 2$ and $6 \frown^F 5$, we can remove the arc $3 \rightarrow 6$ during the bottom-up traversal, doing so we will add $\langle 6, 0\rangle$ to $\frown^B$. By doing a top to bottom traversal, some new edges should be removed again. As a consequence, we get better and better results iterating bottom-up and top to bottom traversals. Note also that the pre-CST $remove(S, \langle n, s\rangle)$ may violate the condition 6 of the definition of CSTs (Section 6). For instance, removing the node with label 6 from the CST in Fig. 6 makes the two nodes with label 2 (belonging to vectors that are not in the subsumption relation) have the same set of successors. As already mentioned in the course of the paper, condition 6 can be restored using the algorithm in Appendix A, Fig. 26, polynomial in the size of the CST.

To 'approximate' the simplification procedure, we can define a definition similar to definition 17 where the nodes s, n and the nodes t, m are in two differents CSTs.

A simplified version of Rule 17 that requires only a local test for every node of a CST is given as follows.

Definition 18 (Local Edge Removal). Given a CST S with nodes N^S, successors $succ^S$ and fathers $father^S$, let assume that for $n, m \in N^S$ there exist $s \in succ^S(n)$ and $t \in succ^S(m)$ $(s \neq t \vee n \neq m)$ such that $val^S(s) \geq val^S(t)$,$succ^S(s) \subseteq succ^S(t)$ and $father^S(n) \subseteq father^S(m)$. Then, we define $local_remove(S, \langle n, s\rangle)$ as the pre-CST with successor relation $succ'$ obtained from S by setting $succ'(n) = succ^S(n) \setminus \{s\}$.

Though less effective than Rule 17, Rule 18 can be paired with it in order to simplify the computation of the simulation.

9 Application to Parameterized Verification

The theory of well-structured systems [AČJT96, FS01] gives us decision procedures to verify a large class of safety properties of *parameterized* systems modeled as Petri Nets [AČJT96, FS01], Lossy Vector Addition Systems [BM99], and Broadcast Protocols [EFM99]. The decision procedures are based on *backward reachability*

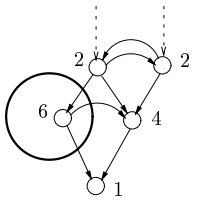

Fig. 6. Violation of cond. 6 after removing two edges and one node.

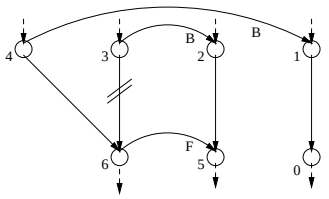

Fig. 7. Removing edge $3 \rightarrow 6$ changes the backward simulation.

algorithms like the one proposed in [AČJT96], whose termination (for Petri Nets and their extensions) is guaranteed by Dickson's lemma. It is important to recall that forward approaches like Karp-Miller's coverability tree are not robust when applied to extensions of Petri Nets like Broadcast protocols [EFM99].

Differently from the finite-state case, in parameterized verification *symbolic* representations are ineluctable in order to make the approach *effective*: we *need* to finitely represent infinite, upward-closed collections of states. In the backward approach of [AČJT96,FS01] we need to represent infinite, *upward-closed* sets of *markings*, when we restrict our attention to Petri Nets. Two examples of symbolic representations for *upward-closed sets* of marking are collections of minimal points [AČJT96], and *linear arithmetic constraints* [DEP99]. The complexity of the algorithm of [AČJT96] is non-elementary. For this reason, naive implementations of the backward approach suffer from the *symbolic state explosion* problem: the number of minimal points or, equivalently, the size of the constraints become unmanageable after few iterations. Symbolic state explosion is the counterpart of the state explosion problem we known from finite-state verification.

In this section we will focus on application of CSTs to verification problems of Petri Nets. As explained in the first part of the paper, Covering Sharing Trees can be used to compactly store the minimal points of an upward-closed set of markings. Using this symbolic representation, in the preliminary results given in [DR00] we managed to prove properties of Petri Nets (of small size) that could not be managed from other infinite-state model checkers (working backwards) like HyTech [HHW97].

In order to attack *symbolic state explosion* for Petri Nets of large size, we will combine our CST-based technology with heuristics based on the *structural properties* of Petri Nets. More precisely, the heuristic rule we propose in this paper is based on a *fully automatic* static analysis[STC98], whose results can be used during the backward reachability algorithm to significantly cut the search space. As the other general purpose heuristics presented in the first part of the paper (e.g. simulation relations), our *structural heuristic* works in polynomial time on the graph structure of CSTs. When combined with our CSTs-symbolic representation, the heuristic rule allow us to scale up the dimension of the case-studies of one order of magnitude.

Before entering in more details, in the next section we will briefly recall the main ideas behind the connection between parameterized systems, Petri Nets, and backward reachability.

9.1 Petri Nets and Verification of Safety Properties

Following [GS92], asynchronous concurrent systems can be naturally represented as Petri Nets in which places and transitions are used to model *local states*, *internal actions* and communication via *rendez-vous* (see e.g. [Rei86,STC98] for an introduction to Petri Nets). At this level of abstraction, processes can be viewed as undistinguishable black *tokens*. A *marking* $\mathbf{m} = \langle m_1, \ldots, m_n \rangle$, a mapping from places to *non-negative* integers, can be viewed as an abstraction of a *global system state* in which we only keep track of the number of processes in every state. The number of processes in the system is determined by the *initial marking* $\mathbf{m_0}$. The *backward reachability* approach is based on the property of the relation $\preccurlyeq$ of being a *well-quasi ordering*. This property ensures the termination of backward reachability, whenever the starting point of the exploration is an upward-closed set of markings [Fin90, AČJT96, FS01].

As an example, consider the Petri Net of Fig. 8, a monitor for a parameterized system with two mutually exclusive critical sections (cs_1 and cs_2). Initially, all K processes are in p_1. To enter cs_1, a process tests

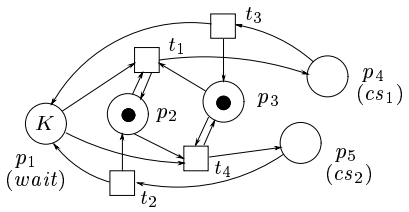

Fig. 8. An example of Petri Net with parametric initial marking $\langle K, 1, 1, 0, 0\rangle$, $K \geq 1$.

for the presence of processes in cs_2 using p_2, and locks cs_1 using p_3 (transition t_1), and vice versa. Processes leave the critical section using transitions t_3 and t_4. Note that the set U of violations to mutual exclusion is the *upward-closed* set generated by the *minimal violations* $\langle 0, 0, 0, 2, 0\rangle$, $\langle 0, 0, 0, 0, 2\rangle$, and $\langle 0, 0, 0, 1, 1\rangle$ (*at least* 2 tokens in p_4+p_5). To prove that the protocol guarantees *mutual exclusion* for *any* value of K, it is enough to show that no admissible initial marking is in the set of *predecessor* markings $pre^*(U)$ of U (pre is the operator that returns the set of markings that reach some marking in U by *firing* a transition). To compute $pre^*(U)$, we iterate the application of the predecessor operator pre until we reach a fixpoint. During the computation, every newly generated marking is stored only if it is *not subsumed* by an already visited one. The backward reachability graph of our example is given in Fig. 9. In Fig. 9 we have omitted all redundant markings (about 30). As mentioned in the introduction, the symbolic backward approach based on the enumeration of minimal points of sets of markings suffers from the symbolic state explosion problem. More sophisticated data structures are necessary to make the approach feasible in practice.

9.2 Symbolic Verification of Parameterized Systems

Following, e.g., [Rei86,STC98], a Petri Net can be represented via a *vector addition system*, i.e., a machine model that consists of n variables $x_1, \ldots, x_n$ (one for each place) ranging over non-negative integers, and m *guarded commands* (one for each transition) defined over the data variables. The guarded command associate to the i-th transition contains a guard $x_j \geq c_{i,j}$ and an assignment $x_j := x_j + d_{i,j}$ for every place p_j. The guard is such that

- $c_{i,j} = 0$ if $d_{i,j} \geq 0$: when reasoning backwards, it avoids to assign negative values to x_j;
- $c_{i,j} \geq d_{i,j}$ if $d_{i,j} < 0$: when reasoning fowards, it avoids to assign negative values to x_j.

Markings represent then the current state of the n-counters associated to the places. Having this view in mind, it should be very easy to see how to apply $\mathcal{U}$-formulas for representing *covering* problems of Petri Nets, and thus how to use CSTsto efficiently solve them.

Given a Petri Net N and an upward-closed set of markings U, based on the connection between Petri Nets and vector addition systems, we will show that we can define a $\mathcal{U}$-formula to represent the set of *predecessors* of markings in U with respect to the transitions in N. Actually, we already described in Section 3 how to represent an upward-closed set U as a $\mathcal{U}$-formula φ_U. It remains to construct the $\mathcal{U}$-formula that represents $pre(U)$.

Definition 19 (The $\mathcal{U}$-formula for $pre(U)$). Let $\varphi_{i,j} = x_j \geq c_{i,j}$ and $x_j := x_j + d_{i,j}$ be the guard and action associated to the i-th guarder command (transition) and to the j-th variable (place). The following $\mathcal{U}$-formula can be used to represent $pre(U)$:

$$pre(\varphi_U) \equiv \bigvee_{i=1}^{m} \bigvee_{j=1}^{k} (\varphi_i \ \wedge \ \gamma_{i,j}),$$

where

- $\varphi_U = \varphi_{U_1} \vee \ldots \vee \varphi_{U_k}$ with $\varphi_{U_j} = (x_1 \geq c_{1,j} \wedge \ldots \wedge x_n \geq c_{n,j})$
- $\varphi_i = \varphi_{i,1} \wedge \ldots \wedge \varphi_{i,n}$ represents the guards of the i-th guarded command for $i = 1, \ldots, m$
- $\gamma_{i,j} = \varphi_{U_j}[x_1 + d_{i,1}/x_1] \ldots [x_n + d_{i,n}/x_n]$ represents the effect of applying backwards the action of the i-the guarded command for $i = 1, \ldots, m$.
- $pre(\varphi_U)$ is in disjunctive form.

It is easy to verify that $[\![\varphi_{pre(U)}]\!] = pre(U)$. Note that, we assume here that the right-hand side is put in disjunctive form, i.e., the guard φ_i is distributed over the possible disjunctive formula γ_i. Based on this idea, in order to verify problems that can be expressed via upward-closed sets, we can build the following symbolic backward reachability algorithm that works using $\mathcal{U}$-formulas and their operations:

```
Function BS(φ_I,φ_U: U-formulas):Bool;
begin
  φ:=φ_U;
  repeat
    ψ := φ;
    φ := ψ ∨ pre(ψ); (pre(ψ) as in Def. 19)
  until φ |= ψ
  return (φ_I ∧ φ ≡ false);
end.
```

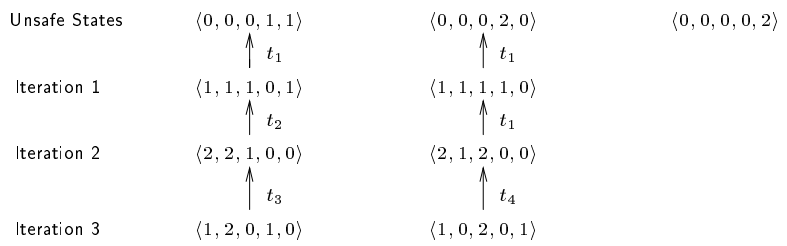

Fig. 9. Backward reachability graph.

By construction, the intermediate results are always in disjunctive form. At each step of the algorithm, we use the subsumption test to break the cycle in which we repeatedly apply *pre*.

Interestingly, this symbolic algorithm coincides with the backward search scheme implemented by the so-called *constraint-based* model checkers like HyTech [HHW97] and the tool based on the Omega Library of [BGP97]. These tools use *linear arithmetic constraints* as assertional languages to represent *infinite* collection of system states (i.e. assignments of reals or integers to a given number of data variables). Furthermore, they internally maintain formulas in disjunctive form (linear arithmetic constraints are defined as conjunctions of atomic formulas). Thus, they can be used as effective implementation platform for the backward search algorithm based on $\mathcal{U}$-formulas described above. In [DEP99], we tested this technology on some Petri Net example of small size. We report on these experiments in the following section.

9.3 Symbolic State Explosion

The *table manufacturing system* of Teruel [Ter94] is a production system modeled via a *weighted Petri Net* with 7 places and 6 transitions. The values of the weights on the transitions vary from 2 to 4. This Petri Net can be modeled using a VAS with 13 variables, 7 for the places and 6 extra variables, say $T_1, \ldots, T_6$, to keep track of the number of firings of each transition. As shown in [Ter94], all deadlock-free initial markings are exactly the predecessors of the set of states where the first transition is fired at least 3 times, and all the others are fired at least twice (i.e. $U = T_1 \geq 3 \ \wedge \ \bigwedge_{j>1} T_j \geq 2$). In [DEP99], different general purpose constraint-based checkers have been used to compute the set of predecessors via backward reachability. As shown in Fig. 10, the comparison was carried out using HyTech, the tool based on the Omega library of Bultan, Gerber and Pugh's [BGP97], and a specialized implementation proposed in [DEP99], in which collection of linear constraints are stored as a list of list of bit-vectors. As shown in the table of Fig. 10, the general purpose model checkers suffer from the symbolic state explosion, due to the way in which constraints are stored, i.e., maintaining them in disjunctive normal form.

The specialized data structures proposed in [DEP99] perform instead reasonably well. The data structure of [DEP99] are based on the following idea. The main idea is to represent a constraint like $x \geq c$ via a pair $\langle b, c\rangle$ in which b is a bit vector that uniquely represents x. For n variables, we need then n bits. A conjunction of constraints can be stored then as a *list of such pairs*. Finally, a collection of constraint is stored as a *list of list of pairs*. As an example, the $\mathcal{U}$-formula in disjunctive normal form over the three variables $\langle x_1, x_2, x_3\rangle$ $\varphi_1 \vee \varphi_2$, where $\varphi_1 = (x_1 \geq 2 \wedge x_2 \geq 1 \wedge x_3 \geq 4)$ and $\varphi_2 = (x_1 \geq 2 \wedge x_2 \geq 4 \wedge x_3 \geq 1)$ is represented as following.

φ_1 :	$\langle 100, 2\rangle$	$\langle 010, 1\rangle$	$\langle 001, 4\rangle$
φ_2 :	$\langle 100, 2\rangle$	$\langle 010, 4\rangle$	$\langle 100, 1\rangle$

As shown in [DEP99], bit-a-bit operations can be used to handle the resulting representation. Although when applied to $\mathcal{U}$-formulas, the use of bit-vectors to represent variables might seem a waste of space, this representation can be applied to capture more sophisticated classes of constraints (see [DEP99]). As an example, we can represent constraints like $x_1 + x_2 + x_3 \geq 5$ via the pair $\langle 111, 5\rangle$.

Unfortunately, this symbolic representation of $\mathcal{U}$-formulas suffers from another potential source of redundancy. Note in fact that, the $\mathcal{U}$-formula of our example can be written more compactly as $(x_1 \geq 2) \wedge ((x_2 \geq 1 \wedge x_3 \geq 4) \vee (x_2 \geq 4 \wedge x_3 \geq 1))$. Intuitively, what is missing in the data structure of [DEP99] is the possibility of *sharing* information among different disjuncts. This justifies the use of CSTs as heuristics to obtain compact data structures for this kind of verification problems.

9.4 CST-based Symbolic Backward Reachability

The first step towards a CST-based symbolic backward search algorithm consists in formulating the $\mathcal{U}$-based algorithm using the corresponding CST-operations. After this preliminary step, we will try to exploit the structure of CSTs in order to find optimizations one can use during backward search. As mentioned in the previous sections, it is often convenient to work on generic CSTs (i.e. in which we allow redundancies). As an example, in this

Examples	No.Rules	Unsafe States	Steps	BitVector[1]	HyTech[1]	Presb[2]
Man. System [Ter94]	6	$T_1 \geq 3, \wedge_{i>1} T_i \geq 2$	24	1090s	6925.27s	19h50m

[1] On a Sun Sparc 5.6. [2] On a Sun Ultra Sparc.

Fig. 10. Comparisons of Constraint-based verification tools.

case disjunction can be done in polynomial time. On the other hand, too many redundancies might cause an explosion of the space needed to store intermediate results. The trade off between complexity and compactness of the CSTs is a crucial aspect to take into account, e.g., in the computation of the CST-based symbolic predecessor operator. Another important point to consider is the complexity of the subsumption test used as a termination test for the fixpoint computation of pre^*. In the next sections we will discuss these two points and then present the resulting CST-based backward search algorithm. Let us start discussing the symbolic computation of pre.

9.5 A CST-based CST_Pre operation for Petri Nets

Given a CST S_φ representing a $\mathcal{U}$-formula in disjunctive formula φ (hence an upward-closed set of markings), we would like to symbolically compute $pre(\varphi)$ using operations on CSTs. In the following we will focus our attention of the backward effect of a single Petri Net transition t, represented via a guarded command in which the guard $\varphi_{t,j} = x_j \geq c_{i,j}$ and the action $x_j := x_j + d_{t,j}$ are associated to the j-th place. Let S_φ be a CST, then the operator $CST_Pre(t, S)$ is defined as follows:

$$CST_Pre(t, S_\varphi) = S_{\varphi_t} \wedge S_{\varphi[x_1+d_{i,1}/x_1]\ldots[x_k+d_{i,k}/x_k]}$$

where $\varphi_t = \varphi_{t,1} \wedge \ldots \wedge \varphi_{t,n}$ is the entire guard of the transition t. The CST $S_{\varphi[x_1+d_{i,1}/x_1]\ldots[x_k+d_{i,k}/x_k]}$ represents in a compact way the disjunction of constraints obtained after the backward application of the action of t. The final result is obtained by intersecting the resulting CST with the CST associated to the guard of t. Alhough the CST S_{φ_t} is a very special CST, i.e. it contains only one path, intersection makes this operation exponential in the size of S. Actually, this is also the cost of search problem of symbolically computing pre on CSTs. We will formally prove this claim in the following theorem.

Theorem 14. *Let S be an (irredundant) CST and t be a Petri Net transition. Then, there is no polynomial algorithm to compute the (irredundant) CST $pre(t, S)$ such that $cones(pre(t, S)) = pre(t, cones(S))$.*

Proof. Let us first consider the case in which the input CST S is irredundant and the result is required to be irredundant, and then extend the proof to generic CSTs.

The case $pre(IRR) = IRR$. As in Section 6, we exhibit a polynomial *Turing* reduction from *validity*. Specifically, we build an OTM (Oracle Turing Machine) that uses pre as an oracle to decide in polynomial time *validity*.

1. Let $F = D_1 \vee \ldots \vee D_n$ be a propositional formula in DNF defined over k symbols. We first construct the CST $T_{val(D_i)}$ for $i : 1, \ldots, n$ and $T_{val(true)}$, i.e. the CSTs without extra layers used in the proof of Theorem 4. This step can be done in *polynomial* time.
2. As a second step, we add two extra layers on the top of every newly computed CST $T_{val(D_i)}$; let us call T_i the so obtained CST. The labels in the two extra layers are such that all resulting T_i's have generators that are not comparable to each other. Specifically, we simply add the prefix with labels $\langle i, n-i+1\rangle$ to every tuple in T_i. Note that, for each T_i, we need only two nodes and polynomially many edges distributed in the two extra layers: the node in the first layer has label i, the node in the second layer has label $n-i+1$; and all paths going out from the root are forced to pass through the two nodes.
3. Now, we build the CST T_F by first merging all the T_i's in a single pre-sharing tree, and then reducing it in a CST by sharing all their *suffixes* using the polynomial time algorithm proposed in [Zam97] (see Appendix A). From now on, we will interpretated the resulting CST T_F as representing markings for the variables $x, y, x_1, \ldots, x_k$, x, y being the variables associate to the two newly introduced extra layers, and $x_1, \ldots, x_k$ being the variables associated to the propositional symbols in F. Note that n is the largest value occurring in the constraints associated to x, y is T_F.
4. We define now a Petri Net transition t having the following guard:

   $$x \geq n \wedge y \geq n \wedge x_1 \geq 0 \wedge \ldots \wedge x_k \geq 0.$$

   and the following action

   $$x' = x \wedge y' = y \wedge x'_1 = x_1 \wedge \ldots \wedge x'_k = x_k.$$

   This transition first removes and then put back n tokens from x and y. Thus at least n tokens must be in x and y in order for t to be fired. The transition has no effects on the other variables.
5. Notice that the result of computing pre over the constraints associated to x and y in T_F is always the constraint $x \geq n \wedge y \geq n$ (by construction, the constants in T_F associated to x and y range in $1, \ldots, n$). As a consequence, if we apply the oracle to t and

T_F, we will obtain an irredundant CST in which: (1) there is only one path associated to the variables x and y with labels $\langle n, n\rangle$; (2) every evaluation for the propositional symbol of F occurs at most once. Point (2) is a consequence of (1) and of the fact that the oracle returns an irredundant CST.

6. We build now the CST $T_{val(true)}$ associated to all possible evaluations of the propositional symbols in F, and then we add the prefix $\langle n, n\rangle$ to the tuples in $T_{val(true)}$ as for the T_i's. Let S_{true} be the resulting CST. By construction, we obtain then that F is *valid* if and only if $cones(pre(t, T_F)) = cones(S_{true})$. Finally, we note that, since both CST are irredundant, we can compute their equivalence in polynomial time (see Theorem 3).

The resulting Turing reduction is polynomial, thus co-NP harndess for our search problem follows from the co-NP completeness of *validity*.

The case $pre(GEN) = GEN$. We can extend the proof of co-NP hardness when *pre* works on *generic* CSTs as follows. Instead of labelling the extra layers of the CST T_i using the prefix $\langle i, n-i+1\rangle$, we always use the prefix $\langle i, i\rangle$. As a consequence, we will get a CST T_F that may contain redundant tuples. If now we apply the oracle (working on any CST) we will obtain a CST $pre(t, T_F)$ that may also contain redundant tuples. Now, we build S_{true} exactly as is the case $pre(IRR) = IRR$ By construction, however, we notice that F is *valid* if and only if $cones(pre(t, T_F))$ contains all generators of $cones(S_{true})$. Following from Fig. 1, we can check this property by using containment of the flat denotations $elem(S_{true}) \subseteq elem(T_F)$ in polynomial time. The resulting Turing reduction is again polynomial. Thus, co-NP harndess for our search problem follows from the co-NP completeness of *validity*. □

As shown in the previous theorem, the problems of computing *pre* arise from the presence of guards in a Petri Net transition. Note, in fact, that if counters range over negative values that the computation becomes trivially polynomial (it amounts to a substitution operation). Nevertheless, we will show later that in practice computing *pre* is never a bottleneck for the CST-based search engine. Intuitively, the reason is that in practice only few labels collapse to the same value. In most of the cases the application of *pre* amounts to a substitution operation.

9.6 CST-driven Optimizations of pre

Following the definition of *pre*, we directly get the following symbolic algorithm to compute the pre-image of a set of markings:

$$CST_Pre(S) = \cup_{t \in T}\ CST_Pre(t, S)$$

Proc $CST_Pre^*(\mathbf{S}_U : CST)$
  $\mathbf{S} := \mathbf{S}_U$;
  $\mathbf{T} := CST_Empty$;
  while $not(CST_Subsumption(\mathbf{T}, \mathbf{S}))$ **do**
    $\mathbf{T} := CST_Union(\mathbf{T}, \mathbf{S})$; $\mathbf{R} := CST_Empty$;
    for each *transition* t **do**
      $\mathbf{N} := CST_Pre(t, \mathbf{S})$;
      $\mathbf{R} := CST_Union(\mathbf{N}, \mathbf{R})$;
    $\mathbf{S} := \mathbf{R}$;
  return $\mathbf{T}$;

Fig. 11. Basic algoritm computing pre^* on a CST

where T is the set of transitions of our Petri Net. In this section, we will focus on some optimization of this algorithm. In the following, we will assume that a transition t consists of

- a guard $x_i \geq d_{t,i}$;
- and an action $x_i' = x_i + c_{t,i}$

for each variable x_i associated to the i-th place, and stored in the i-th layer of a CST. We will also extend the mapping $cones(\cdot)$ to denote the rich denotation of a pre-CST (i.e. in which condition 5 or 6 may not hold). First, we can note that it is easy to caracterize the set of predecessors of a upward-closed set with one minimal marking, as described by the following proposition.

Proposition 3. *Let* $\mathbf{m}$ *be a marking and* t *be a transition.* $pre(t, cone(\mathbf{m})) = cone(\mathbf{m}')$ *such that* $\mathbf{m}'(i) = max(g_{t,i}, \mathbf{m}(i) - c_{t,i})$

Follopwing Proposition 3, we can compute the minimal marking of the predessors set for a transition t via independent computations on each components. As the denotation of a CST is the union of the upward-closed set described by its paths, we can define an algorithm based on this idea. More precisely, given a CST S, it is very simple to compute a pre-CST that has the same rich denotation as $CST_Pre(t, S)$: we set the value of a node n in the i-th layer to the following value:

$$val'(n) = max(d_{t,i}, val(n) - c_{t,i})$$

The max operation correspond to the conjunction between the guard and the result of applying backward the action of t on x_i. We use the algorithm given in [Zam97] (see Appendix A) to adjust the two condition 5 and 6. Although the algorithm for ensuring condition 5 is exponential in the worst case, this method seems to perform well in practice. A basic algorithm computing pre^* on a CST S is given in Fig. 11. More precisely, while a fixpoint is not reached, the algorithm computes the *pre* operation on $\mathbf{S}$, the CST descibing the set of minimal markings computed at the last iteration. Applying this algorithm in practice, we compute large CSTs and after few iterations, $\mathbf{S}$ can become difficult to manipulate. In fact, not all the paths are needed in $\mathbf{S}$, as explain by the following proposition.

Proposition 4. *Let* $\mathbf{m}$ *and* $\mathbf{m}'$ *two markings and* t *a transition. If* $\mathbf{m} \preceq \mathbf{m}'$*, we have* $pre(t, cone(\mathbf{m}')) \subseteq pre(t, cone(\mathbf{m}))$ *and* $pre^*(cone(\mathbf{m}')) \subseteq pre^*(cone(\mathbf{m}))$

In this way, we know we can remove all the path of $\mathbf{S}$ subsumed by a path of $\mathbf{T}$. Unfortunately, it has been shown in section 7 that removing the paths of $\mathbf{S}$ subsumed by a path of $\mathbf{T}$ can not be done in polynomial time. But as the relation between the paths of S and the paths of $CST_Pre(t, S)$ is defined by the guard and the effect of t, we can define efficient algorithms ,i.e. using advantage of the CST structure, that permits to remove paths from $CST_Pre(t, S)$ that are subsumed by paths of S. For example, having a transition t defined by the guard $x_2 \geq 1 \wedge x_4 \geq 1$ and the effect defined by the constraints $x'_2 = x_2 - 1$, $x'_4 = x_4 - 1$, $x'_1 = x_1 + 1$ and $x'_5 = x_5 + 1$, we can find a marking defining an upward-closed set such that all the predecessors for t are included in the set.For example, applying t on the upward-closed set with the minimal marking $\langle 0, 2, 1, 4, 0\rangle$, we obtain its upward-closed subset with the minimal marking $\langle 0, 3, 1, 5, 0\rangle$. In this way, the resulting marking doesn't give new information and can be removed from $\mathbf{S}$ during the computation of pre^*. We can formalize the idea with the following proposition.

Proposition 5. *Having a marking* $\mathbf{m}$ *and a transition* t*. If for each* x_i *such that* $c_{t,i} > 0$*, we have* $d_{t,i}$ *such that* $\mathbf{m}(x_i) \leq d_{t,i}$*, then we have* $cone(pre(t, \mathbf{m})) \subseteq cone(\mathbf{m})$

When we compute $CST_Pre(t, \mathbf{S})$, it is interesting to efficiently detect and remove some paths of $\mathbf{S}$ satisfying proposition 5 before the computation of CST_Pre. The following proposition gives a way to detect that all the paths of $pre(t, \mathbf{S})$ are subsumed by paths of $\mathbf{S}$.

Proposition 6. *Let* t *be a transition of a Petri net* P *and* $\mathbf{S}$ *be a CST. If for all layers* i *such that* $c_{t,j} > 0$ *and for all* $n \in N_i^{\mathbf{S}}$ *such that* $val^{\mathbf{S}}(n) \leq d_{t,i}$*, then* $cones(CST_Pre(t, \mathbf{S})) \subseteq cones(\mathbf{S})$.

It is obvious that checking condition of proposition 6 can be done in linear time. This permits to know efficiently if it is interesting to compute the CST_Pre operation for a transition t on a CST $\mathbf{S}$.

We now focus on the case where some paths of $CST_Pre(t, \mathbf{S})$ are not subsumed by paths of $\mathbf{S}$. In this case, efficient algorithms are need to remove a maximum of paths satifying proposition 5. The following algorithm try to remove edges $e = \langle n, m\rangle$ of $\mathbf{S}$ such that all the paths passing by e define upward-closed set including their set of predecessors.

PrunePre($\mathbf{S} : CST, t : transition$)
  $Sub^{\uparrow}(n) = false, Sub^{\downarrow}(n) = false, \forall n \in N^{\mathbf{S}}$
  ComputeSub$^{\downarrow}(\mathbf{S}, t)$
  ComputeSub$^{\uparrow}(\mathbf{S}, t)$
  Remove all *edges* $e = \langle n, m\rangle$ *such that*
  $Sub^{\downarrow}(n) = false \wedge Sub^{\uparrow}(m) = false$

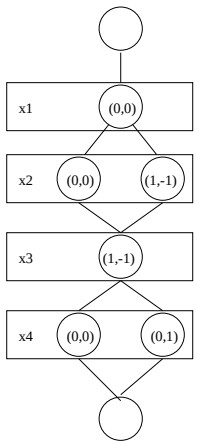

Fig. 12. example of transition tree

where **ComputeSub**$^{\downarrow}$ is defined as follow (we assume that $newval(n)$ is the new value of n when computing the pre-CST corresponding to $CST_Pre(t, \mathbf{S})$):

ComputeSub$^{\downarrow}$($\mathbf{S} : CST, t : transition$)
For all layer l **from** the first **to** the last **do**
  For all node n of the l-th layer **do**
    if $newval(n) \neq val^{\mathbf{S}}(n) \wedge c_{t,l} > 0$ **then**
      $Sub^{\downarrow}(n) = true$
    For all $n' \in succ^{\mathbf{S}}(n)$ **do**
      if $Sub^{\downarrow}(n) = true$ **then**
        $Sub^{\downarrow}(n') = true$

The function **computeSub**$^{\uparrow}$ is defined from **ComputeSub**$^{\downarrow}$ by traversing the CST from the last layer to the first one, replacing the relation $succ^S$ by $father^S$ and $Sub^{\downarrow}$ by $Sub^{\uparrow}$. The two following propositions are easy to prove.

Proposition 7. *Given a transition* t*, a CST* $\mathbf{S}$ *and the pre-CST* $\mathbf{S}'$ *obtained by applying* **PrunePre** *on* $\mathbf{S}$ *with* t *and then applying algorithm of Fig. 26 that ensures condition 6, we have* $CST_Pre(cones(\mathbf{S})) = CST_Pre(cones(\mathbf{S}'))$.

Proposition 8. *The complexity of* **PrunePre** *is linear in the size of the input CST.*

Proof. Stems from the linear complexity of the breath-first search.

Another way to compute the CST_Pre operation is to follow the idea given in [Zam97] by defining a sharing tree, called transition tree, describing the set of transitions. The value of the nodes of the transition tree is a couple where the first component describe a guard of transition (i.e. a constant $c_{t,i}$), the second component describe an effect of a transition (i.e. a constant $d_{t,i}$) and each path correspond to a transition. Fig. 12 shows the transition tree for the transitions t_1 with guard

```
proc CST_Pre*(S_U : CST)
  S := S_U;
  T := CST_Empty;
  while not(CST_Subsumption(T, S)) do
      T := CST_Union(T, S); R := CST_Empty;
      for each transition t do
          N := CST_Pre(S, t);
          CST_remove_redundancies(N, R, T);
          CST_Heuristic_Minimization(N);
          R := CST_Union(N, R);
      S := R;
  return T;
```

Fig. 13. The CST-based symbolic model checking algorithm.

$x_3 \geq 1$ and effect $x_3' = x_3 - 1$, t_2 with guard $x_3 \geq 1$ and effec t $x_3' = x_3 - 1 \wedge x_4' = x_4 + 1$, t_3 with guard $x_2 \geq 1 \wedge x_3 \geq 1$ and effect $x_2' = x_2 - 1 \wedge x_3' = x3 - 1$ and transition t_4 with guard $x_2 \geq 1 \wedge x_3 \geq 1$ and effect $x_2' = x_2 - 1 \wedge x_3' = x3 - 1 \wedge x_4' = x_4 + 1$. By computing the operation corresponding to the product of automata between the transition tree and a CST $\mathbf{S}$,we compute $CST_Pre(\mathbf{S})$ in one step (i.e. accroding to all the transitions). For a node $\langle n, m\rangle$ of the product where n is a node of the starting CST and m is a node of the transition tree, its value is computed as before according to the value of n and the guard and effect defined by m. This product can be done in polynomial time but the resulting pre-CST could violate condition 5-6. The algorithm of Fig. 25 and Fig. 26 must be then applied to obtain a CST.

Note that to compute efficiently the CST_Pre operation, we can use an algorithm similar to **PrunePre**.

9.7 The CST-based Backward Search Engine

We are ready now to formulate our CST-backward reachability algorithm. The algorithm in pseudo code is given in Fig. 13. The algorithm works as follows. First of all, we store intermediate results in two variables. The variable $\mathbf{S}$ stores the current *frontier* of the breadth-first performed by the algorithm. The variable $\mathbf{T}$ stores the set of visited generators.

Before entering the main loop, we test subsumption bewteen $\mathbf{S}$ and $\mathbf{T}$. For this purpose, the following heuristics seems to work well in practice.

1. We first compute the forward and backward simulation relations between the nodes of $\mathbf{S}$ and the nodes of $\mathbf{T}$.
2. If the root of $\mathbf{S}$ is forward simulated by the root of $\mathbf{T}$ or if the end node of $\mathbf{S}$ is backward simulated by the end node of $\mathbf{T}$, then we know that all the generators of $\mathbf{S}$ are subsumed by some generators of $\mathbf{T}$ (see [DR00]), thus the fixpoint is reached.
3. If the test fails, we perform a depth-first, top-down visit of the CST $\mathbf{S}$ in order to compare its tuples with those of $\mathbf{T}$. During the depth-first visit, we use however the information previously computed via the forward simulation as follows. Each time we reach a node n that is forward simulated by a node of $\mathbf{T}$, we stop the exploration: all the elements in the subtree rooted at n will be subsumed by elements of $\mathbf{T}$.

In the main loop, we compute the new frontier $\mathbf{N}$ transition-by-transition via the symbolic operator $CST_Pre(\mathbf{S}, t)$. As a further heuristics to keep the size of $\mathbf{N}$ small, after computing $CST_Pre(\mathbf{S}, t)$, we apply simulation-based heuristics to remove *redundant* generators. The function $CST_remove_redundancies(\mathbf{N}, \mathbf{R}, \mathbf{T})$ uses simulations relation between nodes of $\mathbf{N}$ and nodes of $\mathbf{N}$ and $\mathbf{T}$. The function $CST_Heuristic_Minimization(\mathbf{N})$ uses simulations relation of nodes of $\mathbf{N}$.

9.8 Preliminary Experiments

Let us go back to the example of the Manufacturing System discussed in Section 9.3. The use of CST as assertional language to represent intermediate results dramatically speeds up the analysis of the verification problem proposed in [DEP99]. The computation based on the CST-search engine of Fig. 13 terminates after 39 seconds on a Sun Ultra Sparc (see Fig. 14). In a first experiment we have not removed the redundancies from S_{pre^i}, whereas in a second experiment we have applied the reduction based on the forward simulation (every 5 steps) (see Fig 14). Simulation-based reduction (every 5 steps) allows us to reduce the set of generators of a factor of ten (note: removing *all* redundancies yields 450 elements).

In the next section we will show how to further speed up backward analysis by considering heuristics inspired to the structural theory of Petri Nets.

10 Structural Heuristics

In the backward reachability approach every place of a Petri Net is considered as *unbounded*. In many practical cases, however, some places of the net might be bounded for any possible value of the parameters in the initial configuration. The *Structural Theory* of Petri Nets [STC98] can help us to distinguish between bounded and unbounded places and thus to restrict the search space we have to explore backwards. Let us recall the main results of this theory.

Let N be a Petri Net with n places, m transitions, and *token flow matrix* $\mathbf{C}$. We recall that $\mathbf{C}$ describes how tokens are reallocated by transitions: rows correspond to places, columns to transitions, and an element $c_{i,j}$ to the difference between the number of tokens added and removed for place i by the j-th transition. In the

Example	NV	NR	NS	LR	FSR	ET	NE	Nodes	Ratio
Man. Sys.	13	6	24	no	_	39s	7563	4420	4%
Man. Sys.	13	6	24	yes	5	68s	727	1772	12%

Fig. 14. Flags: NV=No. Variable; NR=No. Rules; ET=Execution Time; LR=Use of local reduction; FSR=Frequency in the use of sim. reduction (__=not used); NS=No Steps; NE=No. of Elem. of the result sharing tree; Ratio=Nodes/(NV*NE).

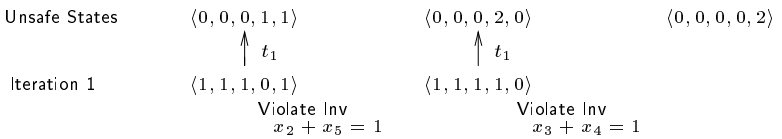

Fig. 15. Backward search space pruned using structural invariants.

following, we will use $\cdot$ to denote the vector product

$$\mathbf{a}^T \cdot \mathbf{b} = a_1 b_1 + \ldots a_n b_n$$

where $\mathbf{a}^T$ indicates the transpose of vector $\mathbf{a}$.

Place invariants [STC98] are among the possible information we can compute via a static analysis of N. A place invariant (also called P-semiflow) is a vector $\mathbf{p} = \langle p_1, \ldots, p_n \rangle$ with non-negative components that is a solution of the linear system

$$\mathbf{x}^T \cdot \mathbf{C} = 0, \ \mathbf{x} \geq \mathbf{0},$$

where $\mathbf{x}$ is a vector of variables of dimension n. Given an initial marking $\mathbf{m_0}$, and a place invariant $\mathbf{p}$, the set

$$\mathcal{O}(\mathbf{m_0}, \mathbf{p}) = \{\mathbf{m} \mid \mathbf{p}^T \cdot \mathbf{m} = \mathbf{p}^T \cdot \mathbf{m_0}\}$$

over-approximates the reachability set of the Petri Net. This property follows from the definition of place invariant, and from the *state equation*

$$\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \sigma$$

that characterizes a generic marking $\mathbf{m}$ reachable from $\mathbf{m_0}$ via the sequence of transitions represented by the *firing vector* σ (see [STC98]): if $\sigma = \langle c_1, \ldots, c_m \rangle$ then the j-th transition has been fired c_j times to reach $\mathbf{m}$ from $\mathbf{m_0}$. As a consequence, the equation

$$\mathbf{p}^T \cdot \mathbf{x} = \mathbf{p}^T \cdot \mathbf{m_0}$$

for some place invariant $\mathbf{p}$ gives us a *structural invariant* we can use to analyze the net.

Let us consider the example of Fig. 8. The state equation is given as follows

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix} = \begin{pmatrix} K \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

where $m_1, \ldots, m_5$ and $c_1, \ldots, c_5$ are required to be greater or equal than zero. Place invariants can be computed then by solving the following linear system

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

where $y_1, \ldots, y_5$ are required to be greater or equal than zero. After some simplifications, we get the following linear system

$$\begin{gathered} x_1 + x_2 = x_5 \\ x_1 + x_3 = x_4 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{gathered}$$

This gives us, for instance, the following *place invariants*: $\mathbf{p_1} = \langle 0, 1, 0, 0, 1 \rangle$, $\mathbf{p_2} = \langle 0, 0, 1, 1, 0 \rangle$, and $\mathbf{p_3} = \langle 1, 0, 0, 1, 1 \rangle$. Actually, these place invariants represent a basis to generate (via linear combinations) all other place invariants. Algorithms to compute generators are mentioned, e.g., in [STC98]. When applied to the state equation and to the parametric initial marking, we obtain then the following structural invariants:

$$\begin{gathered} x_2 + x_5 = 1 \\ x_3 + x_4 = 1 \\ x_1 + x_4 + x_5 = K. \end{gathered}$$

Notice that the first two invariants, namely $x_2 + x_5 = 1$ and $x_3 + x_4 = 1$, imply that the places x_2, x_3, x_4 and x_5 always contain at most one token, i.e., they imply $x_i \leq 1$ for $i : 2, \ldots, 5$. Unfortunately, these information are not sufficient to prove our mutual exclusion property expressed via the constraint $x_4 + x_5 \leq 1$: a configuration in which both x_4 and x_5 have one token satisfies the structural invariant $x_i \leq 1$ for $i : 2, \ldots, 5$ but not the mutual exclusion property. The reason of the weakness of the invariants is the fact that they do not take into account the guards of the transitions, an information that is not represented in the matrix $\mathbf{C}$. Despite of this problem, we can still try to exploit the information coming from the first two invariants.

A possible way to use the structural analysis would be to make what is usually called a *program specialization*, i.e., we could replace the subnet involving places linked by structural invariants (e.g. p_2, p_5 for $x_2 + x_5 = 1$) with a control part (a finite-state net). This way however, we run the risk of obtaining nets of exponential size. As an alternative, we propose to use the structural invariants directly within our CSTs-based search scheme as heuristics to *prune the search space*.

10.1 Pruning the Backward Search Space

Let U be an upward-closed set of markings denoting *unsafe states*, and let $U' = U \cap \mathcal{O}(\mathbf{m_0}, \mathbf{p})$ for some place invariant $\mathbf{p}$. We first note that if $U' = \emptyset$, then we can immediately infer that the net is *safe* with respect to the violations represented via U. As mentioned in the previous section, invariants are often not sufficient to directly verify the property, but they can be used to prune the backward search space. Let us consider again our example and the backward reachability graph of Fig. 15. After the first iteration, we obtain the generators $\langle 1, 1, 1, 0, 1\rangle$ and $\langle 1, 1, 1, 1, 0\rangle$. They are not subsumed by pre-existing generators. The first generator defines a set of markings that has no intersection with the set of markings defined by the invariant $x_2 + x_5 = 1$, while the second generator defines a set of markings that have no intersection with the set of markings defined by the invariant $x_3 + x_4 = 1$. So we know that no markings defined by those two generators will be reached from any instance of the parametric initial marking (the invariants do not depend on the value of K). In fact, the markings satisfying the invariant over-approximate the set of reachable markings. As a consequence, we can stop the backward search after the first iteration instead of having to consider 3 iterations as in the plain search presented in Fig. 9.

Let us now study how we can incorporate this idea in our CST-based backward search. We first note that, in general, the intersection between an upward-closed set U and the set of markings $\mathcal{O}(\mathbf{m_0}, \mathbf{p})$ induced by a place invariants $\mathbf{p}$ is not upward-closed. Thus, even if we work using a symbolic representation of invariants using linear constraints (e.g. $x_2 + x_5 = 1 \wedge x_3 + x_4 = 1$), we cannot conjunct the invariant with the intermediate results of the algorithm for pre^* without running the risk of breaking the property of well-quasi ordering of the resulting constraints. For instance, the constraint $x_2 + x_5 = 1 \wedge x_2 \geq 1 x_5 \geq 0$ is equivalent to $x_2 = 1 \wedge x_5 = 0$, whose denotation is not upward-closed.

What we can do, however, is to remove, from the CSTs computing after each step of the symbolic reachability algorithm, *all the generators* that define sets of markings that have no intersection with the set $\mathcal{O}(\mathbf{m_0}, \mathbf{p})$. In fact, we have the following property.

```
Proc CST_Pre*(S_U : CST)
  S := S_U;
  T := CST_Empty;
  while not(CST_Subsumption(T, S)) do
      T := CST_Union(T, S); R := CST_Empty;
      for each transition t do
          N := CST_Pre(S, t);
          CST_structural_reduction(N);
          CST_remove_redundancies(N, R, T);
          CST_Heuristic_Minimization(N);
          R := CST_Union(N, R);
      S := R;
  return T;
```

Fig. 16. The CST-based symbolic model checking algorithm enriched with pruning.

Theorem 15. *Given a Petri Net N with initial parametric marking $\mathbf{m_0}$, a place invariant $\mathbf{p}$, and an upward-closed set of markings U represented by a CST $\mathbf{S}$, suppose $cone(\mathbf{m}) \cap \mathcal{O}(\mathbf{m_0}, \mathbf{p}) = \emptyset$ for some $\mathbf{m} \in elem(\mathbf{S})$. Furthermore, let $\mathbf{S'}$ be the CST such that $elem(\mathbf{S'}) = elem(\mathbf{S}) \setminus \{\mathbf{m}\}$, and $\mathbf{m'_0}$ be any instance of $\mathbf{m_0}$. Then,*

$$\textit{If } \mathbf{m'_0} \in pre^*(cones(\mathbf{S})) \textit{ then } \mathbf{m'_0} \in pre^*(cones(\mathbf{S'})).$$

Proof. Per absurdum. Suppose there exists $\mathbf{m_0}'$ such that $\mathbf{m'_0} \in pre^*(cones(\mathbf{S}))$ and $\mathbf{m'_0} \notin pre^*(cones(\mathbf{S'}))$. This implies that from $\mathbf{m'_0}$ we can reach a marking $\mathbf{m}$ generated by a $\mathbf{m'}$ such that $\mathbf{m'}$ such that $cone(\mathbf{m}) \cap \mathcal{O}(\mathbf{m_0}, \mathbf{p}) = \emptyset$. Thus, $\mathbf{m}$ is itself not member of $\mathcal{O}(\mathbf{m_0}, \mathbf{p}) = \emptyset$. But this contradict the hypothesis that $\mathbf{p}$ is a place invariant with respect to $\mathbf{m_0}$, and thus to $\mathbf{m_0}'$. □

Thus, during the computation of $pre^*(U)$ we can prune the search space by removing *all* generators $\mathbf{m}$, such that $cone(\mathbf{m})$ has an empty intersection with the set of markings defined by a structural invariant. We call such generators *useless*. As shown in the following theorem, this process can be done in polynomial time.

Theorem 16. *Given a Petri Net N with initial marking $\mathbf{m_0}$, and a place invariant $\mathbf{p}$, such that $M = \mathbf{p}^T \cdot \mathbf{m_0}$ is an integer value (no parameters occur in M. Then, there exists an algorithm to compute the CST S' such that $\mathbf{m} \in elem(S')$ if and only if $\mathbf{m} \in elem(S)$ and $cone(\mathbf{m}) \cap \mathcal{O}(\mathbf{m_0}, \mathbf{p}) \neq \emptyset$, with complexity polynomial in the size of S, $\mathbf{p}$, and in the value M.*

Proof. Let N^S, $root^S$, end^S, etc., the components of S. Furthermore, let $\mathbf{p} = \langle p_1, \ldots, p_k\rangle$, $M = \mathbf{p}^T \cdot \mathbf{m_0}$.

The algorithm works as follows. We first constructs (in quadratic time) a pre-sharing tree T such that $\mathbf{m} \in elem(T)$ if and only if $\mathbf{m} \in elem(S)$ and $cone(\mathbf{m}) \cap \mathcal{O}(\mathbf{m_0}, \mathbf{p}) \neq \emptyset$.

T is defined as follows. Its set of nodes N^T consists of $root^T$, end^T, and the nodes of the i-th layer are $\langle s, v\rangle$ for $s \in N^S_i$, and each value $v \in \{0, \ldots, M\}$. We set

$val(\langle s, v\rangle) = val(s)$. A node $\langle s, v\rangle$ is in the i-th layer of T if and only if s is in the i-th layer of S. The successor relation $succ^T$ is defined as (recall that $\mathbf{p} = \langle p_1, \ldots, p_k\rangle$):

- $succ^T(root^T)$ containts all nodes $\langle s, v\rangle$ such that $s \in succ^S(\top^S)$ and $v = p_1 \cdot val(s)$;
- for all nodes $\langle s, v\rangle$ of the ℓ-th layer ($\ell < k$) $succ^T(\langle s, v\rangle) = \{ \langle s', v'\rangle$ such that $s' \in succ^S(s)$ and $v' = v + p_{\ell+1} \cdot val(s') \}$; note that if $p_i = 0$ then $\langle s', 0\rangle \in succ^T(\langle s, v\rangle)$ if and only if $s' \in succ^S(s)$.
- for all nodes $\langle s, v\rangle$ of the k-th layer $succ^T(\langle s, v\rangle) = \{ end^T \}$.

The resulting pre-sharing tree has the following property: If $s' \in succ^S(s)$ then for each value $v \in \{0, \ldots, M\}$, there is *at most one* v' such that $\langle s, v'\rangle \in succ^T(\langle s, v\rangle)$. Furthermore, $\mathbf{t} \in elem(T)$ if and only if $\mathbf{t} \in elem(S)$ and $\mathbf{p}^T \cdot \mathbf{t} = \mathbf{p}^T \cdot \mathbf{m_0}$. The construction of T is done in time *polynomial* in the size of S and $\mathbf{p}$. Finally, to turn T into a CST we have to enforce the maximal sharing of suffixes in T. This can also be done in *polynomial* time using the algorithm proposed in Fig. 26. □

10.2 Approximation of the Structural Heuristic

In the previous section, we presented a polynomial algorithm to remove all the *useless* paths of a CST according to a structural invariant. We present now an approximated algorithm that can be used as an heuristics to remove *useless* paths. The new algorithm is more efficient in practical cases. To describe the heuristics, we need the following definitions. Let $\mathbf{S}$ be a CST , and let $e = (\mathbf{v}, \mathbf{w})$ be an edge of $\mathbf{S}$ connecting nodes of two adjacent layers. We define $elem_e(\mathbf{S})$ as the set of tuples from $elem(\mathbf{S})$ denoted by paths of $\mathbf{S}$ passing through e. Formally, $\mathbf{m} = \langle val(v_1), \ldots, val(v_n)\rangle \in elem_e(\mathbf{S})$ iff there exists a path $\langle \top, v_1, \ldots, \mathbf{v}, \mathbf{w}, \ldots, v_n, \bot\rangle$ in $\mathbf{S}$ such that $e = (\mathbf{v}, \mathbf{w})$. Consider now a *structural invariant*, say $\mathcal{I}$, having the form $\mathbf{p}^T \cdot \mathbf{x} = \mathbf{p}^T \cdot \mathbf{m_0}$.

Our heuristics works by removing an edge e of $\mathbf{S}$, whenever we can prove that the markings in $elem_e(\mathbf{S})$ denote cones that do not intersect with the structural invariant $\mathcal{I}$. To check this condition on the edge e connecting a node of layer $\mathbf{i}$ and a node of layer $\mathbf{i+1}$, we first compute the two values $min_{prec}(e)$ and $min_{\succ}(e)$ defined as follows: $min_{\prec}(e)$ is the minimal value of *prefixes* $\langle m_1, \ldots, m_{\mathbf{i}}\rangle$ of tuples in $elem_e(\mathbf{S})$ evaluated on the function $\mathbf{x} \rightsquigarrow \mathbf{p}^T \cdot \mathbf{x}$; symmetrically, $min_{\succ}(e)$ is computed for *suffixes* $\langle m_{\mathbf{i+1}}, \ldots, m_n\rangle$. Specifically, we define

$$min_{\prec}(e) = min \{ \mathbf{p}^T \cdot \langle m_1, \ldots, m_{\mathbf{i}}, 0, \ldots, 0\rangle \mid \langle m_1, \ldots, m_n\rangle \in elem_e(\mathbf{S})\}$$

$$min_{\succ}(e) = min \{ \mathbf{p}^T \cdot \langle 0, \ldots, 0, m_{\mathbf{i+1}}, \ldots, m_n\rangle \mid \langle m_1, \ldots, m_n\rangle \in elem_e(\mathbf{S})\}.$$

The following two properties characterize our heuristics.

Theorem 17. *Given the initial marking* $\mathbf{m_0}$*, the CST* $\mathbf{S}$*, the structural property* $\mathbf{p}^T \cdot \mathbf{x} = \mathbf{p}^T \cdot \mathbf{m_0}$*, and the edge* e *of* $\mathbf{S}$*, if* $min_{\prec}(e) + min_{\succ}(e) > \mathbf{p}^T \cdot \mathbf{m_0}$*, then* $cone(\mathbf{m}) \cap \mathcal{O}(\mathbf{m_0}, \mathbf{p}) = \emptyset$ *for any* $\mathbf{m} \in elem_e(\mathbf{S})$

Theorem 18. *Given a CST* S*, and the invariant* $\mathbf{p}^T.\mathbf{x} = \mathbf{p}^T.\mathbf{m_0}$ *such that* $\mathbf{p}^T.\mathbf{m_0} \in \mathbb{Z}$*, there exists a linear algorithm that computes* $min_{\prec}$ *and* $min_{\succ}$ *for each* $e = \langle n, n'\rangle$ *(*$n, n \in N^S$*) such that* $n' \in succ(n)$*.*

Proof. The following algorithm computes $min_{\succ}$ on a CST S for all $e = \langle n, n'\rangle$ such that $n' \in succ^S(n)$.

Function $min_{\succ}(S, e)$;

For each $e = \langle n, n'\rangle$ **do** $min_{\succ}(e) = 0$ **endf**
For each layer **from** the last **to** the first **do**
  For each node n of the $l - th$ layer **do**
    $minprec(n) = \infty$
    For each $e = \langle n, n'\rangle$ such that $n' \in succ^S(n)$ **do**
      $min_{\succ}(e) = minprec(n')$
      $minprec(n) = min(minprec(n), minprec(n'))$
    $minprec(n) = minprec(n) + \mathbf{p_l}.val(n)$
For each $e = \langle \top, n\rangle$ such that $n \in succ^S(\top)$ **do**
  $min_{\succ}(e) = minprec(n)$
end.

The algorithm is linear in the size of the input CST. In fact, it amounts to a breadth-first search with constant time operations at each node and edge. The algorithm that computes $min_{\prec}$ can be defined starting from the algorithm computing $min_{\succ}$ by replacing the relation $succ^S$ by the relation $father^S$, $min_{\succ}$ by $min_{\prec}$ and traversing the CST from the first layer to the last one.

Based on this results, the algorithm that approximates the removing of the *useless* paths is defined as follow.

procedure ApproxSH(S : CST, $\mathbf{p}^T.\mathbf{m_0}$: integer)
  Compute $min_{\succ}$ on S
  Compute $min_{\prec}$ on S
  Remove $e = \langle n, n'\rangle | min_{\succ}(e) + min_{\prec}(e) > \mathbf{p}^T.\mathbf{m_0}$

From Theorem 18, we obtain the following proposition.

Proposition 9. *The algorithm* **ApproxSH** *is computed in linear time in the size of the input CST.*

As an example of application of the structural heuristics, consider the CST of fig. 17. The CST $\mathbf{S}$ contains the elements obtained at iteration 1, the pairs of value on the arcs are the values $min_{\prec}(e)$ and $min_{\succ}(e)$ for the place invariant $x_2 + x_5 = 1$, the dashed edges can be removed and thus the useless element $\langle 1, 1, 1, 0, 1\rangle$ is removed from the CST. Note that if we use the invariant $x_3 + x_4 = 1$ then the last element can also be eliminated. The heuristics simply traverses a CST layer by layer, removing all edges that satisfy the hypothesis of Theorem 17.

Based on the previous ideas, the CST-based symbolic reachability algorithm of Fig. 13 can be enriched as shown in Fig. 16.

Given a set of place invariants computed a priori via a static analisys of a Petri Net, the procedure

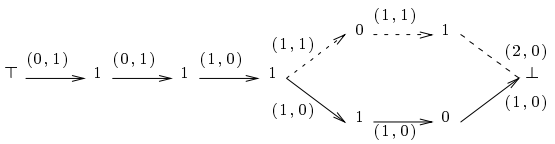

Fig. 17. Application of Heuristics for Structural Invariants to a CST.

Size of the Petri Net
P=Number of places;
T = Number of transitions.
Verification problem (VP) (only the type of property)
ME=Mutual exclusion property;
C=Covering for a random marking;
SL=Semi-liveness.
Use of heuristics
I=Invariant-based reductions (structural heuristic rule);
S=Simulation-based reductions.
Statistics: execution
EX=Execution time (in seconds) on an AMD Athlon 900 Mhz;
NI=N. of iterations before termination (with *=before stopping the execution).
Quality of analysis
R=An initial state has been reached.
Statistics: use of memory
MaxE(N)=N. of *elements* (*nodes*) of the biggest CST associated to **S** of Fig. 13;
NE(N)=Number of *elements* (*nodes*) of the CST for the fixpoint;
Ratio of memory saving (using CSTs)
RM=MAX-N/(MAX-E × P) in pct.;
RN=NN/(NE × P) in pct.

Fig. 18. Parameters of the experimental evaluation.

CST_structural_reduction implements the *reduction algorithm* based on structural heuristics of Theorem 16 to removes all *useless* generators from the input CST. To complete the scenario, we need to compute *automatically* the structural invariants. This can be done using specialized libraries to compute systems of generators for place invariants like the one available with GreatSPN [CFGR95], or PEP [Gra97]. .

11 Experimental Evaluation

In the previous sections, we have presented several algorithms for the backward analysis of Petri nets. We have also proposed a heuristic based on structural invariants to prune the backward search space. In this section, we evaluate the practical behavior of the different algorithms and the structural heuristic on several examples of infinite state Petri nets. More precisely, we have considered *mutual exclusion* properties for the parameterized, concurrent and production systems like the Mesh 2x2 of [ABC+95] (Fig. 130, p. 256), its extension to the 3x2 case, and for an extension of the Readers-Writers example given in [Rei86] in which we use several large buffers (e.g. with 45 slots). Furthermore, we have considered *semi-liveness* and *coverability* problems for the PNCSA communication protocol analyzed in [BF99, Fin93]. We also compare our work with alternative approaches.

In Section 11.1, we report on results obtained with the exact symbolic algorithms that we propose and an intelligent implementation of the standard enumerative algorithm. In Section 11.2, we compare the exact symbolic algorithm with a version where the simulation based heuristics are used. In Section 11.3, we show experimental evidences of the benefits of using structural invariants to guide the backward search. Finally, in Section 11.4, we compare our prototype with other tools for the analysis of infinite state and finite state Petri nets. We present some links that exists with other symbolic structures that can be used to represent infinite set of states and in particular upward-closed sets.

11.1 Enumerative versus Symbolic

In the previous sections we have shown that we are able to explore the backward reachable states of an infinite state Petri net starting from an upward-closed set of final markings without enumerating the minimal points of the successive upward-closed sets computed in the fixpoint computation. We have given an algorithm that manipulates directly the graph structure of the CSTs representing compactly those minimal points.

```
Proc Enumerative_B S(S : CST, P : CST)
 F := S;   Frontier
 T := ∅;   Alreadyvisited
 While F ≠ ∅ do
   T := T ∪ F; R := ∅;
   foreach s ∈ F do
     foreach transition t do
       s' := Pre_min(t, s);
       if ¬∃s'' ∈ T ∪ S ∪ R : s'' ≤ s'
         then R := R ∪ {s'};
     EndDo;
   EndDo;
   T := T ∪ F;
   F := R;
 EndDo
```

Fig. 19. The enumerative algorithm for the backward search.

We have also seen that some problems on CSTs, related to the symbolic manipulation of minimal elements of an upward-closed set, have a relatively high theoretical complexity (coNP-completness for example). We show in this subsection that despite the theoretical complexities of the symbolic operations required for the backward search, the CSTs based algorithm outperforms a good implementation of the enumerative algorithm. Note that the term "enumerative" can be misleading: only the finite number of minimal elements of the infinite upward-closed sets are enumerated, not all the elements of those sets.

Before presenting our results, we need to be more specific about the enumerative version of the algorithm that we have used. A high level version of the algorithm that we have used is presented in figure 11.1.

The sets R, F, and T are maintained as CSTs. We do that in order to isolate as much as possible the effect of enumerating the elements of the frontier instead of manipulating these sets of minimal elements symbolically. Note that this way, the test $\neg\exists \mathbf{s}'' \in T \cup S \cup R$ s.t. $\mathbf{s}'' \leq \mathbf{s}'$ can be done efficiently using the algorithm given in the proof of theorem 2. So, the enumerative search is not naive and the comparison with the symbolic one is fair.

For the smaller examples the enumerative search has execution times that are from two to three times larger than the symbolic search. Note that the larger frontier as well as the fixpoint has the same size in the two methods. This is not surprising as the two methods maintain the intermediate results in form of CSTs. For the larger examples, the symbolic method works far better. In fact, for the second property of the PNCSA example, the symbolic method is able to compute the 38 iterations needed to reach the fixpoint, whereas the enumerative methods is only able to compute 11 iterations. The advantage of working symbolically is also apparent in the R/W example where the symbolic method is able to compute in 4 hours 19 iterations of the fixpoint computation and treat a frontier of up to 200 thousands elements when the enumerative algorithm is only able to compute 11 iterations in the same time and seems to be limited to treat frontiers of less than 10 thousands minimal elements in reasonnable time. So working symbolically seems to be the right way to go. In the next two subsections, we will examine the practical behavior of two heuristics that can potentially further improve the efficiency of the symbolic algorithm.

11.2 Simulation-based vs Exact Minimization

As discussed in the previous sections, several basic operations of the *exact* backward symbolic search can not be done in polynomial time unless $P = NP$. However we have proposed exact algorithms that are symbolic and only exponential in the worst-case. The exponential behavior can only occur when the underlying automaton structure of the CST is made deterministic. Experiments should determine if this pathological behavior often occurs in practice. Because of this hardness result, we have proposed polynomial approximation methods based on the notion of simulation. When using the simulation heuristics, the structure remains deterministic and no determinization is necessary. But the simulation based algorithms are not exact and even if in the fixpoint the sets computed at each iterations are the same as in the exact method, the CSTs that represent those sets may contain some redundancy, i.e. the CSTs may contain elements that are not minimal ones. We compare here the practical behavior of the two approaches: the exact method never generate redundant element but use algorithms with exponential worst-case behavior while the simulation based heuristic use polynomial time algorithms (with the exception of the fixpoint detection algorithm) but may generate some redundancy in the CSTs.

In the examples that we have tested so far, the expential worst-case behavior of the exact algorithm does not occur and the performances of the exact algorithm are better than the performances of the simulation based algorithm. In the small examples, we can see that the heuristic does not generate too much redundancy, in fact, CSTs of the maximal frontiers as well as the ones of the fixpoints only contain a small number of redundant elements (difference between the number of elements in the simulation based search and the exact search in the tables of fig. 21 and fig. 22). The simulation based algorithm should not be judged useless, however. In fact, the exact algorithm may have a patological behavior on some problems and it could be worth to test the simulation based algorithm in those cases.

11.3 Practical Evaluation of the Structural Heuristics.

In section 10, we have proposed to use structural invariants to prune the backward search space. We show here

Case-study	P	T	VP	EX(sec)	NI	MAX-E	MAX-N	RM	NE	NN	RN	Mem
Mesh2x2$_{k\geq 1}$	32	32	ME	1s	15	53	323	19.2	427	344	2.5	1676 kb
Mesh3x2$_{k\geq 1}$	52	54	ME	18.84s	21	332	1253	7.6	2224	987	0.8	1984 kb
Finkel	31	36	SL	35.28	17	540	1967	11.7	89	176	6.7	1960 kb
Finkel	31	36	C	>4h	11*	5042	11323	7.2	–	–	–	–
R/W$_{k\geq 1}$	24	22	ME	>4h	12*	15449	5996	1.6	–	–	–	–

Fig. 20. Enumerative Algorithm.

Case-study	P	T	VP	EX(sec)	NI	MAX-E	MAX-N	RM	NE	NN	RN	Mem
Mesh2x2$_{K\geq 1}$	32	32	ME	0.8	15	53	323	19	427	344	2.5	1932 kb
Mesh3x2$_{K\geq 1}$	52	54	ME	6.5	21	332	1253	7.3	2224	987	0.9	3492 kb
PNCSA	31	36	SL	17.65	17	540	2100	12.5	89	176	6.7	3748kb
PNCSA	31	36	C	3h50min	38	8545	17468	6.6	132	368	9	42772kb
R/W$_{K\geq 1}$	24	22	ME	>4h	19*	205102	27112	0.5	–	–	–	–

Fig. 21. Exact Symbolic Algorithm.

Case-study	P	T	VP	EX(sec)	NI	MAX-E	MAX-N	RM	NE	NN	RN	Mem
Mesh2x2$_{k\geq 1}$	32	32	ME	1s	15	54	319	18.4	432	338	2.4	2148 kb
Mesh3x2$_{k\geq 1}$	52	54	ME	12.94s	21	334	1198	6.9	2444	1088	0.9	4112 kb
Finkel	31	36	SL	30.05s	17	548	1972	11.6	89	176	6.4	5092 kb
Finkel	31	36	C	> 14400	15*	23700	29120	4	–	–	–	–
R/W$_{k\geq 1}$	24	22	ME	> 14400	16*	129086	24037	0.8	–	–	–	–

Fig. 22. Simulation Heuristic

that this heuristic is very effective in practice and allow us to verify systems much more efficiently and more importantly systems that are out of reach of the method without this heuristic.

In all the examples, the use of the structural heuristic improves dramatically the performance of the exact symbolic backward search. The covering property for the PNCSA is solved using the heuristic within $43.09s$ instead of $3h50m$ (that is around 320 time faster). The gain when verifying the R/W example is even clearer. The exact symbolic algorithm without the invariant heuristic is only able to compute the first 19 iterations of the fixpoint computation in 4 hours when the 118 iterations necessary to reach the fixpoint are computed in less than $10m$ when the structural heuristic is used.

11.4 Comparison with HyTech

HyTech has been developed for the forward and backward analysis of linear hybrid automata. HyTech uses polyhedra in the reals to represent possible valuations for the continuous variables of the hybrid automata. The linear operations involved in the backward analysis of Petri nets can equivalently be evaluated in the positive reals or the natural numbers. Futhermore, cones are special case of (unbounded) polyhedra. As a consequence, we can use HyTech to verify infinite state Petri nets. We have run HyTech on the examples and include invariants to cut the search space. The results are given in the table of Fig. 24.

The performances of HyTech on this class of infinite state systems are far worst than the performance of our prototype. The comparison allow us to conclude that a general data structure like polyhedra used in HyTech can not compete with our data structure specialized for manipulating complex upward-closed sets. The main problem with the polyhedra library seems the memory consumption : every example but the simpler leads to a memory overflow. More than $500Mb$ are used by the algorithm based on polyhedra after a few iterations while our algorithm use less than $35Mb$ to solve completely the largest example.

11.5 Comparaison with finite state verification.

Several of the parameterized systems that we have verified are finite state systems when the initial markings is fixed. Those instantiated systems can them be verified using the forward reachability algorithm of finite state model-checkers. We have run several example with GreatSpin [CFGR95] (a state of the art model-checker for finite state Petri nets). In all the cases, even for small values of the parameters (small numbers of processes), the CST verification of the parameterized problem is much more efficient. For example, when run on the Mesh example with $K = 2$ (only two processes), GreatSpin takes $46.15s$ to compute the $200,544$ reachable states

Case-study	P	T	VP	EX(sec)	NI	MAX-E	MAX-N	RM	NE	NN	RN	Mem
Mesh2x2$_{K\geq 1}$	32	32	ME	0.8	15	53	323	19	424	340	2.5	2580Kb
Mesh3x2$_{K\geq 1}$	52	54	ME	6.5	21	326	1231	7.3	2209	985	1	4996Kb
PNCSA	31	36	SL	2.86	17	85	510	19.3	89	176	6.4	2596kb
PNCSA	31	36	C	43.09	38	690	1875	8.8	132	368	9	7720kb
R/W$_{K\geq 1}$	24	22	ME	9m31s	118	16402	3842	1	29119	1936	0.3	34008Kb

Fig. 23. With invariants.

Case-study	P	T	VP	EX(sec)	NI	Mem
Mesh2x2$_{K\geq 1}$	32	32	ME	144.65	15	948kb
Mesh3x2$_{K\geq 1}$	52	54	ME	–	–	–
PNCSA	31	36	SL	>319	11*	>500MB
PNCSA	31	36	C	>325	8*	>500MB
R/W$_{K\geq 1}$	24	22	ME	> 1058.04	15*	>500MB

Fig. 24. Hytech.

and is not able to compute the reachability set for $K = 3$ (three processes). In comparison, we were able to verify the Mesh for any number of processes (K left as a parameter) in $0.8s$. As foreseen by Bultan in [Bul00], in most of the cases proving a parameterized property turns out to be more efficient than proving its finite-state instantes.

12 Conclusions and Related Works

Parametrized verification has recently attracted a lot of attention in the computer aided verification community. When a counting abstraction is applicable to a parameterized verification problem, (infinite state) Petri nets are a very useful abstract model. Therefore the need for an efficient analysis of infinite state Petri nets is an important issue for proposing practically applicable verification methods for parameterized systems. In this paper, we have proposed an new efficient data structure called Covering Sharing Trees tailored to the representation of infinite upward-closed set of integer vectors and thus of upward-closed set of Petri net markings.

We have studied in details the theoretical complexities of the problems related to the symbolic manipulation of upward-closed sets of integer vectors represented as CSTs.We have shown that the ability of CSTs to represent compactly complex upward-closed sets of integer vectors, i.e. with a large numbers of minimal elements, has a theoretical cost: several useful operations have no polynomial time algorithms unless $P = NP$.

We have shown that CSTs can be used to efficiently solve coverability problems of practically interesting infinite state Petri nets with a parametric initial marking. The algorithm compute symbolically the backward reachable states of the Petri net from an upward-closed set of markings. We have also established that structural invariants can be very effective for cutting the backward search space and allowed us to verify all the examples that we have considered with dramatically better execution time and more importantly to solve problems that are not in the reach of the procedure without that heuristic.

This paper includes and extends with proofs and new experiments the preliminary work presented in [DR00, DRV01].

12.1 Other data structures for parametrized verification.

Other efficient data structures have been proposed for the symbolic representation of infinite sets of integer values.

Numerical Decision Diagrams (NDD) [BW98] can be used to represent any set of vectors of integers (as well as real) numbers definable in the arithmetic of Presburger. NDDs are thus fairly general and can be used to define any upward-closed set. They are thus a possible candidate as a data structure for the backward analysis of infinite state Petri nets. As the NDD is a very expressive data structure, it remains to be tested if it will be usable to encode very complex upward-closed sets. We have shown in the previous subsection that a tool like HyTech that uses polyhedra, another fairly expressive data-structure, encounter rapidely the symbolic state explosion problem when apply to medium size infinite state Petri nets.

Difference Decision Diagrams (DDDs) [MLAH99] has been introduced as an efficient data structure to represent large disjunction of a special class of polyhedra encountered during the verification of timed automata. Again this structure can be used to define infinite set of vectors of integer (as well as real) numbers. The DDDs are able to represent all sets definable by boolean combination of formula of the form : $x-y \sim c$ and $x \sim c$ where x, y are variable that ranges over the integer numbers, c is an integer constant, and $\sim \in \{<, \leq, =, \geq, >\}$.

Differently from NDDs and DDDs, the structure of CST provide for several optimizations (e.g. working at the level of single layers, or using simulation-based reductions) that seem to perform well on practical examples. In order to have a significative comparison of the three data structures, we are currently working on a common software platform in which to run the same set of examples by taking the same set of optimizations and assumptions. Note, in fact, that in our experiments we exploited several domain specific optimizations (e.g. structural heuristics). We are currently investigating the applicability of similar optimizations to the other data structures. The results of this comparison will be made available in a separate work.

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```
Proc Adjust_Condition5(S : pre−CST)
  for each layers from {root^S} to the last one do
    for each node n of the current layer N_i do
    newsucc = ∅;
      for each value c such that ∃n' ∈ succ^S(n) ∧ val^S(n) = c do
        Construct a new node newson_c with val^S(n'') = c;
        for each n' ∈ succ^S(n') ∧ val^S(n') do
          succ^S(newson_c) = succ^S(newson_c) ∪ succ^S(n');
        if ∃n' ∈ N_{i+1} : val^S(n') = c ∧ SameSons(n', newson_c) then
          newsucc = newsucc ∪ {n'};
        else
          newsucc = newsucc ∪ {mewson_c};
          N_{i+1} = N_{i+1} ∪ {mewson_c};
      succ^S(n) = newsucc;
```

Fig. 25. Algorithm that ensures condition 5

```
Proc Adjust_Condition6(S : pre−CST)
  for each n ∈ N^S do
    remove(n) = false;
  for each layers from the last one to the first one do
    for each node n of the current layer N_i^S with remove(n) = false do
      for each node n' ≠ n of the current layer N_i^S
      such that val^S(n) = val^S(n') ∧ SameSons(n, n') do
        for each node n'' such that n' ∈ succ^S(n'') do
          succ^S(n'') \ {n'} ∪ {n};
          remove(n') = true;
    for each n ∈ N_i^S such that remove(n) = true do
      N_i^S \ {n};
```

Fig. 26. Algorithm that ensures condition 6

A Algorithms for adjusting conditions 5 and 6