A tight analysis of the maximal matching heuristic

Jean Cardinal[1], Martine Labbé[1], Stefan Langerman[1] ⋆, Eythan Levy[1] ⋆⋆, and Hadrien Mélot[2]

[1] Université Libre de Bruxelles, Brussels – Belgium,
[2] Université de Mons-Hainaut – Belgium.
{jcardin,mlabbe,slanger,elevy}@ulb.ac.be, hadrien.melot@umh.ac.be

Abstract. We study the algorithm that iteratively removes adjacent vertices from a simple, undirected graph until no edge remains. This algorithm is a well-known 2-approximation to three classical NP-hard optimization problems: MINIMUM VERTEX COVER, MINIMUM MAXIMAL MATCHING and MINIMUM EDGE DOMINATING SET. We show that the worst-case approximation factor of this simple method can be expressed in a finer way when assumptions on the density of the graph is made. For graphs with an average degree at least ϵn, called *weakly ϵ-dense graphs*, we show that the asymptotic approximation factor is $\min\{2, 1/(1-\sqrt{1-\epsilon})\}$. For graphs with a minimum degree at least ϵn – *strongly ϵ-dense graphs* – we show that the asymptotic approximation factor is $\min\{2, 1/\epsilon\}$. These bounds are obtained through a careful analysis of the tight examples.

1 Introduction

The maximal matching heuristic is a textbook algorithm that provides a 2-approximation for the MINIMUM VERTEX COVER, MINIMUM MAXIMAL MATCHING and MINIMUM EDGE DOMINATING SET problems, three classical NP-hard problems [7]. It is perhaps one of the simplest and best-known approximation algorithms. It consists in finding a maximal collection of disjoint edges (a maximal matching) by iteratively removing adjacent vertices until no more edges are left in the graph. Tightness of the 2-approximation is witnessed by a number of examples, for instance by the family of complete bipartite graphs in the case of MINIMUM VERTEX COVER. This paper addresses the question of expressing the approximation ratio in a finer way, as a function of well-chosen graph parameters. We show that density parameters are good candidates for this purpose. Actually, the approximation ratio of the maximal matching heuristic is strictly less than 2 for graphs with a sufficiently high number of edges or sufficiently high minimum degree. We characterize precisely the asymptotic approximation ratio as a function of these parameters, together with tight examples. This is, to our knowledge, the tightest analysis ever done of this algorithm. This study shows that even simple heuristics might deserve nontrivial analyses. It was initiated using GraPHedron, a newly developed software for the investigation of relations between graph invariants (see [3] and [15]).

In the MINIMUM VERTEX COVER problem, one is asked to find a set of vertices of minimum cardinality such that each edge has at least one of its endpoints in the chosen set. A maximal matching is a collection of disjoint edges that cannot be augmented by the adjunction of any new edge. It is easy to see that the vertices of any maximal matching form a vertex cover the size of which is at most twice the size of the minimum cardinality vertex cover, since each one of the disjoint edges of the matching needs to be covered.

⋆ Chercheur qualifié du F.N.R.S.
⋆⋆ Corresponding author.

From an approximation point of view, the minimum vertex cover problem is 2-approximable using the maximal matching heuristic, but no polynomial time algorithm with constant approximation ratio better than 2 is known. The problem is further known to be APX-complete [17] and not approximable within a factor of $7/6$ [10]. Monien and Speckenmeyer [16] and Bar-Yehuda and Even [1] provide algorithms that achieves a ratio of $(2-(\ln\ln n)/\ln n)$, where n is the number of vertices in the graph. Karakostas [13] later reduced the approximation ratio to $2-\Theta(1/\sqrt{\log n})$. For graphs with maximum degree Δ, Halperin [8] provides an aproximation algorithm with a ratio of $2-(1-o(1))2\ln\ln\Delta/\ln\Delta$. The problem has further been studied under the hypothesis that the input graph is dense. We say that a graph G is *weakly ϵ-dense* if its *average* degree is at least ϵn, i.e. if $m \geq \epsilon n^2/2$, with m being the number of edges in the graph, and *strongly ϵ-dense* if its *minimum* degre is at least ϵn. It has been shown [4] that the MINIMUM VERTEX COVER problem restricted to strongly ϵ-dense graphs is APX-complete. Eremeev [6] shows that it is NP-hard to approximate the minimum vertex cover within a ratio less than $(7+\epsilon)/(6+2\epsilon)$ in strongly ϵ-dense graphs. Nagamochi and Ibaraki ([11]) provide an approximation algorithm with a ratio of $2-8m/(13n^2+8m)$, where m is the number of edges in the graph. This algorithm can also be seen as an approximation algorithm that achieves an approximation ratio that is asymptotic to $2-4\epsilon/(13+4\epsilon)$ for weakly ϵ-dense graphs. Karpinski and Zelikovsky [14] propose an algorithm that achieves a better ratio of $2/(2-\sqrt{1-\epsilon})$ for weakly ϵ-dense graphs, and a ratio of $2/(1+\epsilon)$ for strongly ϵ-dense graphs. Finally, Imamura and Iwama [12] recently proposed a randomized approximation algorithm which, with high probability, yields an approximation factor of $2/(1+\gamma(G))$, where $\gamma(G)$ is a function of the maximum and the average degree, and runs in polynomial time if Δ, the maximum degree of the graph, is $\Omega(n\log\log n/\log n)$.

In the MINIMUM MAXIMAL MATCHING problem, one is asked to find a maximal matching of minimum cardinality, i.e. a minimum-cardinality set of disjoint edges that cannot be augmented. It is fairly easy to see that any maximal matching has a size that is at most twice the size of the minimum maximal matching. Much less is known about the minimum maximal matching problem than about the minimum vertex cover problem. Chlebik and Chlebikova [2] do nevertheless show that it is NP-hard to approximate the problem within a constant factor better than $7/6$.

In the MINIMUM EDGE DOMINATING SET problem (also known as the Minimum Edge Cover problem), one is asked to find a minimum-cardinality set of edges such that each edge of the graph either is in the set or is adjacent to some edge of the set. As noted by Yannakakis and Gavril [18], the sizes of the minimum maximal matching and the minimum edge dominating set are equal for each graph, and any minimum edge dominating set can be transformed into a minimum maximal matching in polynomial time. Note that the converse transformation is trivial since a maximal matching is by definition an edge dominating set. They further show [18] that the minimum edge dominating set problem remains NP-hard even when restricted to planar or bipartite graphs of maximum degree 3.

Our results. We study the worst-case approximation ratio of the maximal matching heuristic for the MINIMUM VERTEX COVER and MINIMUM MAXIMAL MATCHING problems in weakly and strongly ϵ-dense graphs. For all three MINIMUM VERTEX COVER, MINIMUM MAXIMAL MATCHING and MINIMUM EDGE DOMINATING SET problems in weakly ϵ-dense graphs, we characterize the exact worst-case approximation ratio as a function of ϵ and obtain a function that is asymptotic to 2 when $\epsilon \leq 3/4$ and to $1/(1-\sqrt{1-\epsilon})$ otherwise. In the case of strongly ϵ-dense graphs, again we characterize the exact worst-case approximation ratio as a function of ϵ and obtain a function that is asymptotic to 2 when $\epsilon \leq 1/2$ and to $1/\epsilon$ otherwise. It is interesting to compare the approximation ratios we obtain for MINIMUM VERTEX COVER with the ones obtained by Zelikovsky and Karpinski: we note that in both the weakly and the strongly ϵ-dense cases the ratios differ only by one unit in the numerator and one in the denominator. Furthermore, the approximation ratios obtained for MINIMUM MAXIMAL MATCHING are, to the best of our knowledge, the best ones known for this problem since it does not seem to have been yet studied under density constraints.

Section 2 is devoted to graph-theoretic preliminaries. In section 3 we study the worst-case aproximation ratio for the MINIMUM VERTEX COVER problem in weakly and strongly ϵ-dense graphs. The same kind of analysis is performed for MINIMUM MAXIMAL MATCHING in section 4.

2 Preliminaries

In the sequel we shall use the classical definition of a simple, loopless, undirected graph $G = (V, E)$, with vertex set V and edge set E. We denote by $\mathcal{G}_{n,m}$ the set of all non isomorphic graphs having n vertices and m edges. We use $n(G)$ to denote $|V|$, $m(G)$ to denote $|E|$ and $\delta(G)$ for the minimum degree of G. We also use the classical notions of *complete graph*, *empty graph*, *independent set*, *clique*, *complete bipartite graph* $K_{a,b}$, *matching*, *perfect matching* and *augmenting path*. Readers that are not familiar with these are referred to standard graph theory texts such as Diestel [5]. The *join* of two graphs G_1 and G_2 with vertex sets respectively V_1 and V_2 is the graph having $V_1 \cup V_2$ as vertex set and containing all edges of G_1, G_2, and all edges between vertices in V_1 and V_2. We denote by $\tau(G)$ the size of a minimum cardinality vertex cover of G, by $\nu(G)$ the size of a maximum cardinality matching of G, and by $\mu(G)$ the size of a minimum maximal matching of G.

We shall make extensive use of the following family of graphs, that arise as extremal graphs for several graph invariants (see [9] and [3]). A *complete split graph* $\Psi_{n,\alpha}$ with $1 \leq \alpha \leq n-1$, is a graph that can be decomposed in an independent set of size α and a clique of size $n-\alpha$, with each vertex of the independent set being adjacent to each vertex in the clique.

We shall also use the following basic results on the value of invariants of complete split graphs. Proofs are simple and omitted.

Lemma 2.1. $m(\Psi_{n,\alpha}) = \binom{n}{2} - \binom{\alpha}{2} = (n-\alpha)(n+\alpha-1)/2$.

Lemma 2.2. $\nu(\Psi_{n,\alpha}) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \textit{if } \alpha \leq n/2, \\ n-\alpha & \textit{otherwise}. \end{cases}$

Lemma 2.3. $\tau(\Psi_{n,\alpha}) = n - \alpha$.

Note that $\tau(\Psi_{n,\alpha}) = n - \alpha$ is obtained only by choosing all vertices in the clique as a vertex cover.

Lemma 2.4. $\mu(\Psi_{n,\alpha}) = \lceil \frac{n-\alpha}{2} \rceil$.

The following two simple properties shall also be useful.

Lemma 2.5. *For any graph G, we have $\tau(G) \geq \delta(G)$.*

Lemma 2.6. *For any graph G, we have $\mu(G) \geq \left\lceil \frac{\delta(G)}{2} \right\rceil$.*

3 Minimum Vertex Cover

We analyze the worst-case behavior of the maximal matching heuristic when applied to the MINIMUM VERTEX COVER problem. We first consider weakly ϵ-dense graphs, which amounts to express the approximation ratio as a function of the number of edges.

3.1 Approximation Ratio vs Number of Edges

Lemma 3.1. *Let n and m be positive integers such that $m(\Psi_{n,\alpha+1}) < m \leq m(\Psi_{n,\alpha})$ for some α. The minimum value of $\tau(G)$ attained by a graph G in $\mathcal{G}_{n,m}$ is $\tau(\Psi_{n,\alpha}) = n - \alpha$.*

Proof. The proof is in two steps: we first show that a graph G in $\mathcal{G}_{n,m}$ cannot have $\tau(G) < \tau(\Psi_{n,\alpha})$, and second that there is a graph G in $\Psi_{n,\alpha}$ having $\tau(G) < \tau(\Psi_{n,\alpha})$. The first step is by contradiction. We show that if $\tau(G) < \tau(\Psi_{n,\alpha})$, then $m \leq m(\Psi_{n,\alpha+1})$, contradicting our hypothesis on m:

$$m(G) \leq \sum_{i=1}^{\tau(G)}(n-i) \leq \sum_{i=1}^{n-\alpha-1}(n-i) = \binom{n}{2} - \binom{\alpha+1}{2} = m(\Psi_{n,\alpha+1}).$$

The first inequality above is obtained by maximizing the number of edges covered by each vertex of the vertex cover: the first vertex in the cover can cover at most $n-1$ edges, the second at most $n-2$ edges, and so on. The second step is by construction. Let G be a graph formed by removing $m(\Psi_{n,\alpha}) - m$ edges from the edges joining the clique and the independent set of a $\Psi_{n,\alpha}$. We show that $\tau(G) = \tau(\Psi_{n,\alpha})$. The number of edges removed is between 0 and $\alpha - 1$ since $m(\Psi_{n,\alpha}) - m(\Psi_{n,\alpha+1}) = \alpha$. Thus each vertex in the clique has at least one neighbor in the independent set. We now show $\tau(G) = \tau(\Psi_{n,\alpha}) = n - \alpha$. The vertices of the clique clearly form a vertex cover of size $n - \alpha$ for G. Any smaller set of vertices would contain x vertices in the independent set and y vertices in the clique, with $x + y < n - \alpha$. If $x = 0$ then at least one edge joining a vertex from the clique to the vertices of the independent set would not be covered. If $x > 0$, then at least some edge in the clique will not be covered since we would have $y < n - \alpha - 1$. We thus have $\tau(G) = n - \alpha$. □

Lemma 3.2. *Let n and m be positive integers such that $m(\Psi_{n,\alpha+1}) < m \leq m(\Psi_{n,\alpha})$ for some α. There exists a graph G in $\mathcal{G}_{n,m}$ such that $\tau(G) = \tau(\Psi_{n,\alpha})$ and $\nu(G) = \nu(\Psi_{n,\alpha})$.*

Proof. Let G be a graph formed by removing $m(\Psi_{n,\alpha}) - m$ edges from the edges joining the clique and the independent set of a $\Psi_{n,\alpha}$. The number of edges removed is between 0 and $\alpha - 1$ since $m(\Psi_{n,\alpha}) - m(\Psi_{n,\alpha+1}) = \alpha$. We show that $\nu(G) = \nu(\Psi_{n,\alpha})$. When $\alpha \leq n/2$, a perfect matching for G can be obtained in the following way: From Hall's condition (see [5]) any set of α vertices of the clique can be perfectly matched with the vertices of the independent set, and the remaining vertices of the clique can be matched inside the clique. When $\alpha > n/2$, from Hall's condition again, each vertex of the clique can be matched with a vertex of the independent set, yielding a matching of size $n - \alpha = \nu(\Psi_{n,\alpha})$. Furthermore, we have $\tau(G) = \tau(\Psi_{n,\alpha})$ by the proof of lemma 3.1. □

The above results enable us to state Theorem 3.1:

Theorem 3.1. *Let $\beta(G)$ be the worst-case approximation ratio for graph G. Let $\beta(m,n)$ be the worst approximation ratio attained by a graph in $\mathcal{G}_{n,m}$. We have:*

$$\beta(m,n) = \beta(\Psi_{n,\alpha^*(m,n)}) = \begin{cases} 2 & \textit{if } \alpha^*(m,n) > n/2, \\ \frac{2\lfloor \frac{n}{2} \rfloor}{n-\alpha^*(m,n)} & \textit{otherwise}, \end{cases}$$

where $\alpha^*(m,n) = \left\lfloor 1/2 + \sqrt{n(n-1)+1/4-2m} \right\rfloor$ *is the integer value α such that* $m(\Psi_{n,\alpha+1}) < m \leq m(\Psi_{n,\alpha})$.

Proof. Lemma 3.2 implies the existence of a graph in $\mathcal{G}_{n,m}$ on which the ratio is given by the expression of the theorem. When $\alpha^*(m,n) > n/2$, a value of 2 trivially maximizes the ratio of our 2-approximation algorithm. When $\alpha^*(m,n) \leq n/2$, the ratio is maximized since the numerator is maximized (the matching is perfect) and the denominator is minimized (by Lemma 3.1). □

The above theorem gives a tight upper bound on the approximation ratio of the maximal matching heuristic to the minimum vertex cover problem in graphs of n vertices and m edges, in the form of a discrete step function of m. The function equals 2 when $\alpha > n/2$ and begins to decrease afterwards.

Corollary 3.1. *Let $\tilde{\beta}(\epsilon,n)$ be the worst approximation ratio attained by a graph with n vertices and an average degree at least ϵn. We have:*

$$\lim_{n\to\infty} \tilde{\beta}(\epsilon,n) = \begin{cases} 2 & \textit{if } \epsilon \leq 3/4, \\ \frac{1}{1-\sqrt{1-\epsilon}} & \textit{otherwise}. \end{cases}$$

Proof. Since $\beta(m,n)$ is a monotonically decreasing function of m for each fixed n, $\tilde{\beta}(\epsilon,n)$ can be defined as $\beta(\lceil \epsilon n^2/2 \rceil, n)$. This yields:

$$\tilde{\beta}(\epsilon,n) = \begin{cases} 2 & \text{if } \alpha^*(\lceil \epsilon n^2/2 \rceil, n) > n/2, \\ \frac{2\lfloor \frac{n}{2} \rfloor}{n-\alpha^*(\lceil \epsilon n^2/2 \rceil, n)} & \text{otherwise}. \end{cases} \tag{1}$$

We now compute the asymptotics of the second above expression:

$$\begin{aligned} \lim_{n\to\infty} \frac{2\lfloor n/2 \rfloor}{n-\alpha^*(\lceil \epsilon n^2/2 \rceil)} &= \lim_{n\to\infty} \frac{2\lfloor n/2 \rfloor}{n - \left\lfloor 1/2 + \sqrt{n(n-1)+1/4-2\lceil \epsilon n^2 \rceil} \right\rfloor} \\ &= \frac{1}{1-\sqrt{1-\epsilon}} \end{aligned} \tag{2}$$

Let us now take a closer look at the condition $\alpha^*(\lceil \epsilon n^2/2 \rceil, n) \leq n/2$:

$$\begin{aligned} &\alpha^*(\lceil \epsilon n^2/2 \rceil, n) \leq n/2 \\ \Leftrightarrow\ &\lfloor 1/2 + \sqrt{n(n-1)+1/4-2\lceil \epsilon n^2/2 \rceil} \rfloor \leq n/2 \\ &\Leftrightarrow \epsilon \geq 3/4 + O(1/n). \end{aligned} \tag{3}$$

Results (1), (2), and (3) immediately imply our corollary. □

This asymptotic result is to be compared with the results of [11] and [14] quoted in the introduction (see figure 2).

3.2 Approximation Ratio vs Minimum Degree

Let $A_{n,\alpha}$ be the set of all graphs of minimum degree $n-\alpha$ that can be expressed as the join of an independent set of order α and a graph of order $n-\alpha$. Note that $A_{n,\alpha}$ contains $\Psi_{n,\alpha}$.

Lemma 3.3. *For all $G \in A_{n,\alpha}$ we have* $\nu(G) = \begin{cases} n-\alpha & \text{if } \alpha \geq n/2, \\ \lfloor \frac{n}{2} \rfloor & \text{otherwise.} \end{cases}$

Proof. If $\alpha \geq n/2$, then G has an independent set of order α joined to a graph of lower or equal order $n-\alpha$. At most $n-\alpha$ vertices of the independent set can be matched, using all $n-\alpha$ vertices of the other graph.

There remains to show that if $\alpha < n/2$, then G has a perfect matching. We show that any non-perfect matching M for G can be augmented.

Let us look at a matching and fix the following notations (see figure 1): let I be the independent set of size α in G, let U be the set of unmatched vertices in I, let W be the set of matched vertices in I, let X be the set of unmatched vertices in $V \backslash I$, let Y be the set of vertices in $V \backslash I$ that are matched inside $V \backslash I$, and finally let Z be the set of vertices in $V \backslash I$ that are matched with vertices of I. Our matching not being perfect, there are at least two unmatched vertices. Three cases must be distinguished.

Case 1. Both U and X are non-empty. In that case the matching is augmented by adding to it an edge joining a vertex from U to a vertex from X.

Case 2. U is empty and X is non-empty. In that case we first prove that any vertex $z \in Z$ always has a neighbor in Z or in X (note that Z is non-empty in this second case). If z has no neighbors in Z nor in X, it can have at most $(n-\alpha-1)-(\alpha-1)-|X|$ neighbors in $V \backslash I$, where $n-\alpha-1$ is the number of other vertices in $V \backslash I$ and $\alpha - 1$ is the number of other vertices in Z. This quantity is upper bounded by $n-2\alpha-2$ since $|X| \geq 2$. But our minimum degree hypothesis implies that z needs to have at least $n-2\alpha$ neighbors in $V \backslash I$, a contradiction. We now consider the two possible cases. If z has a neighbor z' in Z, let wz and $w'z'$ be the edges in M having z and z' as endpoints, and let $\{x, x'\} \subseteq X$. We can augment the matching by replacing edges wz and $w'z'$ in M by edges zz', xw, and $x'w'$. Otherwise, z has a neighbor x in X. Let now x' be another node in X, and wz be the edge in M having z as an endpoint. We augment the matching by replacing edge wz with edges xz and $x'w$.

Case 3. X is empty and U is non-empty. Let T be a set of $\lfloor \frac{|U|}{2} \rfloor$ edges taken in Y. Augment the matching by removing these edges and joining each one of their endpoints with a vertex in U (leaving one unmatched vertex in U if $|U|$ is odd). Note that the matching thus obtained is perfect. $\square$

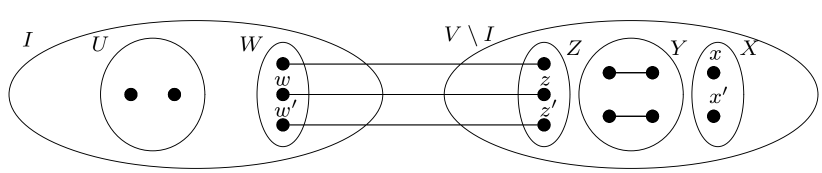


Fig. 1. Notations for the perfect matching construction

Lemma 3.4. *For all $G \in A_{n,\alpha}$ we have $\tau(G) = n - \alpha$. Furthermore, among all graphs with n vertices and minimum degree $n - \alpha$, this value of τ is minimal, and is attained only by graphs in $A_{n,\alpha}$.*

Proof. Let I be an independent set of size α in G. We know from Lemma 2.5 that a graph cannot have a vertex cover of size lower than its minimum degree. Since $V(G)\backslash I$ is a vertex cover of size $\delta(G)$, we have $\tau(G) = n - \alpha$. There remains to show that this value is attained by graphs in $A_{n,\alpha}$ only. This is done by construction. Let G' be a graph having $\tau(G') = \delta(G')$. We show that such a graph must be in $A_{n,\alpha}$ with $\alpha = n - \delta(G')$. Let X be the vertices of a minimum vertex cover of G'. Since X is a vertex, $V \backslash X$ is an independent set. But since $|X| = \delta(G')$, each vertex of the independent set must be adjacent to each vertex in X. This is the definition of a graph in $A_{n,\alpha}$ with $\alpha = n - \delta(G')$. □

Theorem 3.2. *Let $\gamma(\delta, n)$ be the worst approximation ratio attained by a graph with n vertices and minimum degree δ. We have:*

$$\gamma(\delta, n) = \begin{cases} 2 & \text{if } \delta \leq \frac{n}{2}, \\ \frac{2}{\delta}\lfloor \frac{n}{2} \rfloor & \text{otherwise.} \end{cases}$$

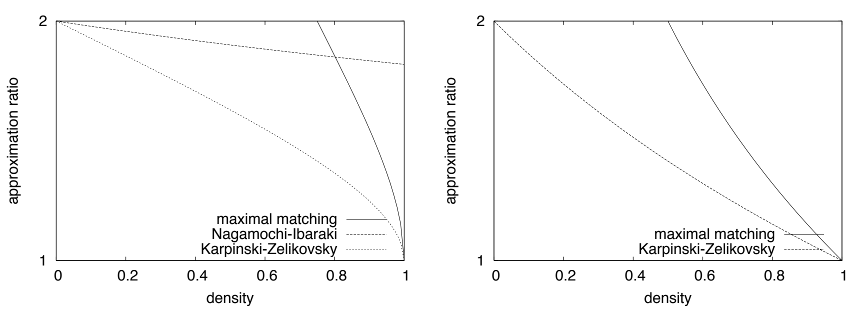
Furthermore, the only graphs that maximize the approximation ratio among all graphs with n vertices and a minimum degree of $n - \alpha$ when $\delta > n/2$ are in $A_{n,\alpha}$.

Proof. The theorem follows directly from Lemmata 3.3 and 3.4. □

Corollary 3.2. *Let $\tilde{\gamma}(\epsilon, n)$ be the worst approximation ratio attained by a graph with n vertices and minimum degree at least ϵn. We have:*

$$\lim_{n \to \infty} \tilde{\gamma}(\epsilon, n) = \begin{cases} 2 & \text{if } \epsilon \leq 1/2, \\ \frac{1}{\epsilon} & \text{otherwise .} \end{cases}$$

This asymptotic result is again to be compared with the result of [14] quoted in the introduction (see figure 2).



(a) weakly ϵ-dense (b) strongly ϵ-dense

Fig. 2. A comparison of the approximation ratios for MINIMUM VERTEX COVER.

4 Minimum Maximal Matching

In this section we show that the analysis proposed for MINIMUM VERTEX COVER can be performed for MINIMUM MAXIMAL MATCHING as well. As this problem is polynomially equivalent to MINIMUM EDGE DOMINATING SET, the results apply to the latter too.

4.1 Approximation Ratio vs Number of Edges

The proof of the following lemma is straightforward and omitted.

Lemma 4.1. *For any fixed k, the only graph G with n vertices that maximizes the number of edges among all graphs having $\mu(G) = k$ is $\Psi_{n,n-2k}$.*

Lemma 4.2. *Let n and m be positive integers such that $m(\Psi_{n,\alpha+1}) < m \leq m(\Psi_{n,\alpha})$ for some α. The minimum value for $\mu(G)$ attained by a graph G in $\mathcal{G}_{n,m}$ is $\mu(\Psi_{n,\alpha}) = \lceil \frac{n-\alpha}{2} \rceil$.*

Proof. We first show that a graph in $\mathcal{G}_{n,m}$ cannot have a maximal matching of size less than $\mu(\Psi_{n,\alpha})$. We then show that the value $\mu(\Psi_{n,\alpha})$ is indeed attained by some graph in $\mathcal{G}_{n,m}$. The first step is by contradiction: we prove that if a graph $G \in \mathcal{G}_{n,m}$ has $\mu(G) < \mu(\Psi_{n,\alpha})$ then $m(G) \leq m(\Psi_{n,\alpha+1})$. We know by Lemma 4.1 that $m(G) \leq m(\Psi_{n,n-2(\mu(\Psi_{n,\alpha})-1)})$. Some basic algebra now gives us the desired result:

$$\begin{aligned} m(G) \leq m(\Psi_{n,n-2(\mu(\Psi_{n,\alpha})-1)}) &= m(\Psi_{n,n+2-2\lceil \frac{n-\alpha}{2} \rceil}) \\ &\leq m(\Psi_{n,n+2-2(\frac{n-\alpha+1}{2})}) = m(\Psi_{n,\alpha+1}) \ . \end{aligned}$$

There remains to present a graph in $\mathcal{G}_{n,m}$ that has a maximal matching of size $\mu(\Psi_{n,\alpha})$. As in Lemma 3.1, let G be a graph formed by removing $m(\Psi_{n,\alpha}) - m$ edges from the edges joining the clique and the independent set of a $\Psi_{n,\alpha}$. The number of edges removed is between 0 and $\alpha - 1$, since $m(\Psi_{n,\alpha}) - m(\Psi_{n,\alpha+1}) = \alpha$. Thus each vertex in the clique has at least one neighbor in the independent set. In case $n - \alpha$ is even, taking any $\frac{n-\alpha}{2}$ disjoint edges in the clique will trivially yield a maximal matching of the desired size. In case $n - \alpha$ is odd, taking $\lfloor \frac{n-\alpha}{2} \rfloor$ disjoint edges in the clique and an additional edge joining the unmatched vertex of the clique and an arbitrary vertex of the independent set clearly yields a maximal matching of the desired size. □

Lemma 4.3. *Let n and m be positive integers such that $m(\Psi_{n,\alpha+1}) < m \leq m(\Psi_{n,\alpha})$ for some α. There exists a graph G in $\mathcal{G}_{n,m}$ such that $\mu(G) = \mu(\Psi_{n,\alpha})$ and $\nu(G) = \nu(\Psi_{n,\alpha})$.*

Proof. The graphs described in the proofs of Lemmata 3.2 and 4.2 satisfy $\mu(G) = \mu(\Psi_{n,\alpha})$ and $\nu(G) = \nu(\Psi_{n,\alpha})$. □

The above results enable us to state the theorem below, for which we shall need the following notations:

Theorem 4.1. *Let $\rho(G)$ be the worst approximation ratio for graph G. Let $\rho(m, n)$ be the worst approximation ratio attained by a graph in $\mathcal{G}_{n,m}$. For each n, m we have:*

$$\rho(m,n) = \rho(\Psi_{n,\alpha^*(m,n)}) = \begin{cases} 2 & \text{if } \alpha^*(m,n) > n/2 + 1, \\ \frac{\lfloor \frac{n}{2} \rfloor}{\lceil \frac{n-\alpha^*(m,n)}{2} \rceil} & \text{otherwise}. \end{cases}$$

Proof. When $\alpha^*(m,n) \leq n/2+1$ or when $\alpha^*(m,n) > n/2+1$ and $n-\alpha$ is even, the proof follows the same scheme as the proof of Theorem 3.1, by using Lemmata 4.2 and 4.3. Otherwise, a value of 2 can easily be shown to be reached by a variant of the class of graphs used in the proof of Lemma 4.2. When $n \geq 4$, this variant consists in removing one edge from the clique and adding one edge st to the independent set such that $s-t-v-u$ is a path for some vertices v in the clique and u in the (formerly) independent set. □

Corollary 4.1. *Let $\tilde{\rho}(\epsilon, n)$ be the worst approximation ratio attained by a graph with n vertices and an average degree at least ϵn. We have:*

$$\lim_{n\to\infty} \tilde{\rho}(\epsilon, n) = \begin{cases} 2 & \text{if } \epsilon \leq 3/4, \\ \frac{1}{1-\sqrt{1-\epsilon}} & \text{otherwise.} \end{cases}$$

4.2 Approximation Ratio vs Minimum Degree

Let $B_{n,\delta}$ be the set of graphs of order n having minimum degree δ and a maximal matching of size $\lceil \delta/2 \rceil$. Note that $B_{n,\delta} = A_{n,n-\delta}$ when δ is even.

Lemma 4.4. *Each graph in $B_{n,\delta}$ with $\lceil \delta/2 \rceil > n/4$ has a perfect matching.*

Proof. Let us partition the vertices of G in two sets A and B such that B contains the endpoints of our maximal matching of size $\lceil \delta/2 \rceil$ (A is thus the complementary independent set, of size $n-2\lceil \delta/2 \rceil$). The formula $\lceil \delta/2 \rceil > n/4$ just expresses the condition $|A| < |B|$. In the case where δ is even, the proof is that of Lemma 3.3. If δ is odd, then $|B| = \delta+1$. We start from M, a perfect matching for B. We show that M can be augmented into a perfect matching. We show below that for any vertices a_i and a_j in A, there exists at most one $m \in M$ that does not provide an augmenting path for a_i and a_j. This implies that M can be augmented into a perfect matching for G by augmenting $\lfloor |A|/2 \rfloor$ disjoint pairs of vertices of A with different edges of M, since, having each element of a set X choose a different element of a set Y, with only one forbidden element in Y for each element of X is always possible when $|Y| > |X|$ (note here that $|M| > \lfloor |A|/2 \rfloor$).

The proof of our first assertion is by contradiction: suppose there are 2 edges $m = b_i b_j$ and $m' = b_k b_l$ that do not provide augmenting paths for a_i and a_j. This implies that in each one of the 4 following pairs of edges, at least one edge of the pair is absent from the graph: $\{(a_i b_i, a_j b_j), (a_i b_j, a_j b_i), (a_i b_k, a_j b_l), (a_i b_l, a_j b_k)\}$ and therefore that: $deg(a_i) + deg(a_j) \leq 2|B| - 4 = 2\delta - 2 < 2\delta$, which contradicts the degree condition. □

Theorem 4.2. *Let $\sigma(\delta, n)$ be the worst approximation ratio attained by a graph with n vertices and minimum degree δ. We have:*

$$\sigma(\delta, n) = \begin{cases} 2 & \text{if } \lceil \delta/2 \rceil \leq n/4 \\ \frac{\lfloor n/2 \rfloor}{\lceil \delta/2 \rceil} & \text{otherwise.} \end{cases}$$

Furthermore, when $\lceil \delta/2 \rceil > n/4$, $B_{n,\delta}$ is the exact set of graphs that maximize the ratio among all graphs with n vertices and minimum degree δ.

Proof. **Case 1:** $\lceil \delta/2 \rceil \leq n/4$. $\Psi_{n,n-2\lceil \delta/2 \rceil}$ minimizes $\mu(G)$ among all graphs G with n vertices and m edges (by noticing that $\Psi_{n,n-2\lceil \delta/2 \rceil} \in B_{n,\delta}$ and by applying Lemma

2.6). Further, $\nu(\Psi_{n,n-2\lceil\delta/2\rceil}) = 2\lceil\delta/2\rceil$ (by Lemma 2.2), and thus $\beta(\Psi_{n,n-2\lceil\delta/2\rceil}) = 2\lceil\delta/2\rceil/\lceil\delta/2\rceil = 2$. This is the maximum value for the approximation ratio since our heuristic is a 2-approximation algorithm.

Case 2: $\lceil\delta/2\rceil > n/4$**.** We know by Lemma 2.6 that the graphs in $B_{n,\delta}$ are the only graphs that minimize $\mu(G)$ among all graphs G with n vertices and m edges. We further know, by Lemma 4.4, that these graphs have a perfect matching. The extremal ratio is therefore given by $\lfloor n/2\rfloor/\lceil\delta/2\rceil$ and is attained by the graphs of $B_{n,\delta}$ only. □

Corollary 4.2. *Let $\tilde{\sigma}(\epsilon, n)$ be the worst approximation ratio attained by a graph with n vertices and minimum degree at least ϵn. We have:*

$$\lim_{n\to\infty} \tilde{\sigma}(\epsilon, n) = \begin{cases} 2 & \text{if } \epsilon \leq 1/2, \\ \frac{1}{\epsilon} & \text{otherwise}. \end{cases}$$

References

1. R. Bar-Yehuda and S.Even. A local-ratio theorem for approximating the weighted vertex cover problem. *Ann. Discrete Math.*, 25:27–45, 1985.
2. M. Chlebk and J. Chlebkov. Approximation hardness of minimum edge dominating set and minimum maximal matching. In *Proceedings of 14th International Symposium on Algorithms and Computation (ISAAC)*, Lecture Notes in Computer Science. Springer–Verlag, 2003.
3. J. Christophe, S. Dewez, J. Doignon, S. Elloumi, G. Fasbender, P. Grégoire, D. Huygens, M. Labbé, H. Mélot, and H. Yaman. Linear inequalities among graph invariants: using GraPHedron to uncover optimal relationships. Submitted. Available at http://www.optimization-online.org/DB_HTML/2004/09/964.html, 2004.
4. A. Clementi and L. Trevisan. Improved non-approximability results for vertex cover problems with density constraints. *Theoretical Computer Science*, 225(1–2):113–128, 1999.
5. R. Diestel. *Graph Theory, Second Edition*. Springer-Verlag, 2000.
6. A.V. Eremeev. On some approximation algorithms for dense vertex cover problem. In *Proceedings of SOR*. Springer–Verlag, 1999.
7. M. R. Garey and D. S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness*. Freeman and Company, 1979.
8. E. Halperin. Improved approximation algorithms for the vertex cover problem in graphs and hypergraphs. *Siam Journal on Computing*, 31:1608–1623, 2002.
9. P. Hansen and H. Mélot. Variable neighborhood search for extremal graphs 9. Bounding the irregularity of a graph. To appear in S. Fajtlowicz *et al*. (Eds.), Graphs and Discovery, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, (forthcoming), Providence, American Mathematical Society.
10. J. Hastad. Some optimal inapproximability results. In *Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing*, pages 1–10, 1997.
11. T. Ibaraki and H. Nagamochi. An approximation of the minimum vertex cover in a graph. *Japan J. Indust. Appl. Math.*, 16:369–375, 1999.
12. T. Imamura and K. Iwama. Approximating vertex cover on dense graphs. In *Proceedings of the 16th ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2005.
13. G. Karakostas. A better approximation ratio for the vertex cover problem. *ECCC Report TR04-084*, 2004.
14. M. Karpinski and A. Zelikovsky. Approximating dense cases of covering problems. In P. Pardalos and D. Du, editors, *Proceedings of the DIMACS Workshop on Network Design: Connectivity and Facilites Location*, volume 40 of *DIMACS series in Disc. Math. and Theor. Comp. Sci.*, pages 169–178, 1997.
15. H. Mélot. Facets defining inequalities among graph invariants: the system GraPHedron. In preparation.
16. B. Monien and E. Speckenmeyer. Ramsey numbers and an approximation algorithm for the vertex cover problem. *Acta Inf.*, 22:115–123, 1985.
17. C. H. Papadimitriou and M. Yannakakis. Optimization, approximation, and complexity classes. *J. Comput. System. Sci.*, 43(3):425–440, 1991.
18. M. Yannakakis and F. Gavril. Edge dominating sets in graphs. *SIAM J. Appl. Math.*, 38(3):364–372, 1980.