The Dice Race : results and open problems

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1 The problem

The following dice game has been partially analyzed in the litterature. A player rolls a fair die successively (this is a round), accumulating the scores so long as the outcome 1 does not occur. But should a 1 arise, the accumulated score is wiped out, and the turn ends. At any stage after a roll, she can choose to end her turn and bank her accumulated score. Several gamblers in turn perform series of rolls. The winner is the first gambler to reach some fixed target n. We present some results and open problems on optimal strategies and winning probability in a two players game.

2 One round

We consider a simple problem: the single round optimization. A round is a series of dice rolling by one player. Two strategies (or more?) are possible in order to get a maximum mean sum (recall that 1 leads to 0 sum).

2.1 Strategy 1 (Threshold Strategy)

The player waits until getting at least the sum k (if possible). Let $P(k,i)$ be the probability that the player obtains the sum $i \in [0, k, k+1, \dots, k+5]$ with the plan to reach at least k. Then

$$P(k,0) = \frac{1}{6} + \frac{1}{6}\sum_{i=2}^{6} P(k-i,0),$$

$$P(k,k+\delta) = \sum_{i=2}^{6} \frac{1}{6} P(k-i,k-i+\delta), \quad \delta = 0,\dots,5,$$

and, by convention,

$$P(-k,.) = 0, \quad k \geq 0,$$
$$P(.,1) = 0.$$

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More precisely,

$$P(k,j) = \frac{1}{6} + \frac{1}{6}\sum_{i=2}^{k-1} P(k-i, j-i), \quad k=1, \quad j=2,\dots,6 \quad \text{or} \quad k=2,\dots,6, \quad j=k,\dots,6,$$

$$P(k,k+\delta) = \sum_{i=2}^{6} \frac{1}{6} P(k-i, k-i+\delta), \quad k=1,\dots,6, \quad 7-k \le \delta \le 5,$$

$$P(k,k+\delta) = \sum_{i=2}^{6} \frac{1}{6} P(k-i, k-i+\delta), \quad k>6, 0 \le \delta \le 5.$$

The probability of getting k (or more) is given by $1 - P(k,0)$. The mean gain is given by

$$G(k) = P(k,0).0 + \sum_{i=k,\dots,k+5} P(k,i).i,$$

maximum for $k = 20$,$k = 21$, and

$$m := G(20) = \frac{492303203}{60466176} = 8.141794894\dots$$

Note that the two probability distributions $P(20,i), P(21,i)$ are different.

2.2 Strategy 1: mean number of rolls during a round

It is interesting to compute the mean number of rolls, $\bar{T}(k)$, during a round for Strategy 1 with threshold k. We have

$$\bar{T}(1) = 1,$$

$$\bar{T}(k) = \frac{1}{6} + \sum_{i=2}^{6} \frac{1}{6}(1 + \bar{T}(k-i)),$$

$$\bar{T}(k) = 0, k \le 0, \text{ by convention.}$$

$\bar{T}(k)$ increases from 1 and converges to $\bar{T} = 6$, fixed part solution of

$$\bar{T} = \frac{1}{6} + \frac{5}{6}(1 + \bar{T}).$$

Of course, $\bar{T}$ corresponds to the time until a 1 appears. Also

$$\bar{T}(20) = 3.747245007\dots,$$

$$\bar{T}(21) = 3.846957993\dots.$$

2.3 Strategy 2

The player plays ℓ steps (if possible). We use now the notation $\tilde{P}(\ell, u)$. Then

$$\tilde{P}(\ell,0) = \frac{1}{6} + \frac{1}{6}\sum_{i=2}^{6} \tilde{P}(\ell-1,0),$$

$$\tilde{P}(\ell,u) = \frac{1}{6}\sum_{i=2}^{6} \tilde{P}(\ell-1,u-i), \quad u = 2\ell,\dots,6\ell,$$

with suitable initial conditions. The mean gain is given by

$$\tilde{G}(l) = \tilde{P}(\ell,0).0 + \sum_{u} \tilde{P}(\ell,u).u,$$

maximum for $\ell = 5, 6$, and $\tilde{G}(5) = \frac{15625}{1944} = 8.037551440\dots$ So Strategy 1 is better. We expected this since Strategy 2 is not history-dependent and can therefore not be optimal, given the forced-stop loss is history-dependent. However it is interesting to see that the difference is not large.

3 Target n, the mean

We now have a target n and our objective is to reach it with a minimum number of rounds. For large n, it seems intuitive that we should use Strategy 1 with $k = 20$ or $k = 21$. But, as we shall see, when we approach the target, the strategy must be adapted. Let $C(n, k)$ be the number of rounds necessary to reach at least n, starting the first round with Strategy 1 and k. Let

$$E(n, k) := \mathbb{E}[C(n, k)].$$

Then

$$E(n, k) = 1 + \sum_{i \in [0,k,\ldots,k+5]} P(k, i)\bar{E}(n - i), \tag{1}$$

where $\bar{E}(n)$ is the *optimal* expected number of rounds necessary to reach at least n. This implies

$$\bar{E}(n) = \min_k E(n, k),$$

as well as the existence of an optimal value for k, $\bar{k}(n)$ say. We must solve the recurrence (1) for each n in some range. To be more precise, we must solve

$$E(n, k) = \sum_{j \geq n} P(k, j).1 + P(k, 0)(1 + E(n, k)) + \sum_{2 \leq j < n} P(k, j)(1 + \bar{E}(n - j))$$

for $E(n, k)$ and take the minimum on k. The first values of $\bar{k}(n)$ for $n = 1, \ldots, 35$ obtained by this are

$$[1, 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 14, 15, 15, 16, 16, 17]$$

Let us mention that, apart from $n = 1, 2$, the values of $\bar{k}(n)$ are unique. Note that, for $k \leq 29$, the player tries one round, for $k > 29$, he tries more then one round. Note also that this argument is valid since there is only one player with target n. In a competitive environment the situation is, as we shall see, more complicated.

We see in Figure 1 (with $n = 1, \ldots, 1000$, as in all our next figures) that, after some time, there is periodic stabilization of $\bar{k}$ between 20 and 21, with period $\mathcal{P}(20) = 12$ for 20 and $\mathcal{P}(21) = 10$ for 21.

PROBLEM 1 : PROVE THIS

In figure 2, we compare $\bar{E}(n)$ with n/m (recall that m is the mean gain with $k = 20$ or 21).

The difference $\bar{E}(n) - n/m$ is given in Figure 3 and Figure 4: there appears some convergence (with damping oscillations) to some constant K_1 with $0.22275 < K_1 < 0.22295$. The asymptotic period is 22 as expected.

PROBLEM 2: COMPUTE THIS CONSTANT K_1

This constant depends on initial values. To check this, we have used the following Strategy 3: for $n = 1, \ldots, 54$, we use $\bar{k}(n)$ and $\bar{E}(n)$ as computed previously. For $n = 55..1000$, we use $\bar{k}(n) = 20, \bar{k}(n) = 21$, with periods $12, 10$. For $\bar{E}(n)$, the behaviour is similar to Figures 2 and 3. Figure 5 shows $\bar{E}(n) - n/m$ in the neighbourhood of $n = 1000$. This shows again an asymptotic period of 22 with a constant K_2, such that $0.22645 < K_2 < 0.22680$ and $K_2 > K_1$.

4 Target n, the variance with optimal strategy

4.1 Computation of the variance of $C(n, \bar{k}(n))$

Recall that $C(n, \bar{k}(n))$ is the number of rounds necessary to reach at least n, with the optimal starting value $\bar{k}(n)$. To compute its variance $\mathbb{V}(n)$, set $D(n) := \mathbb{E}[C(n, \bar{k}(n))^2]$. We have

$$D(n) := \sum_{j \geq n} P(\bar{k}(n), j).1 + P(\bar{k}(n), 0)[1 + 2\bar{E}(n) + D(n)] + \sum_{2 \leq j < n} P(\bar{k}(n), j)[1 + 2\bar{E}(n-j) + D(n-j)], \tag{2}$$

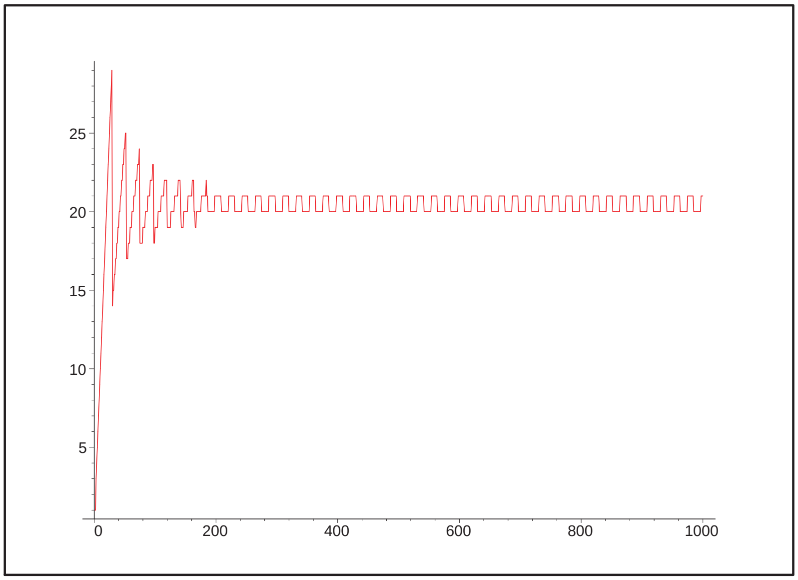

Figure 1: $\bar{k}(n)$

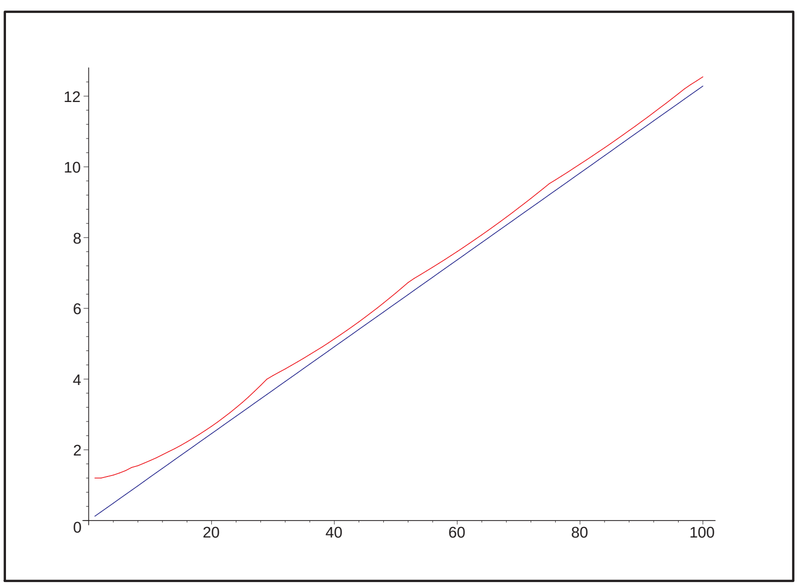

Figure 2: $\bar{E}(n)$ (red), n/m (blue)

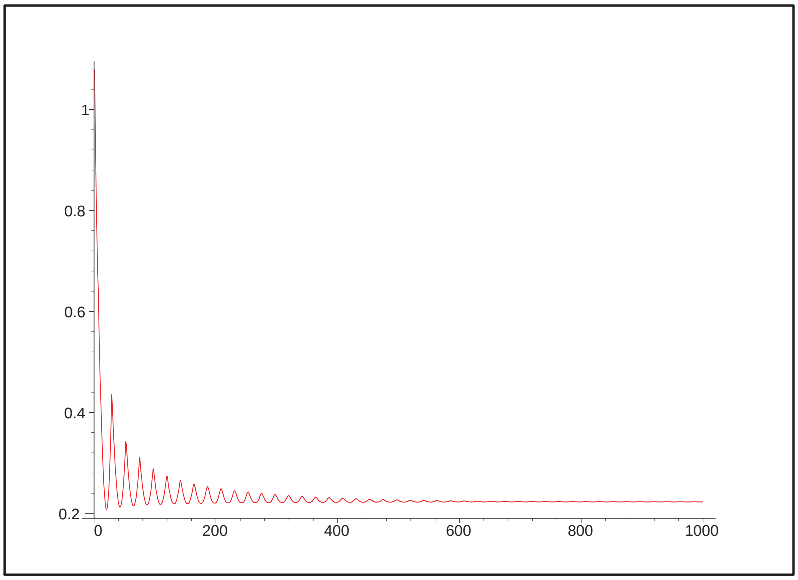

Figure 3: $\bar{E}(n) - n/m$

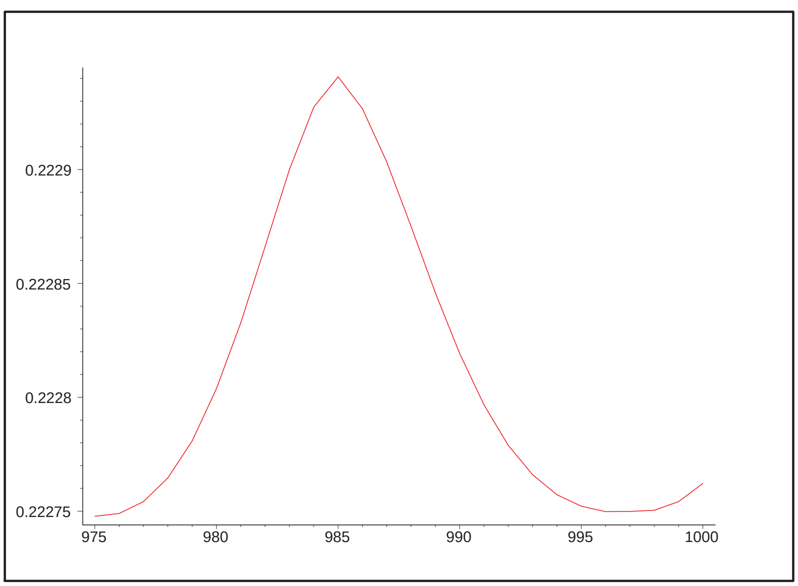

Figure 4: $\bar{E}(n) - n/m$

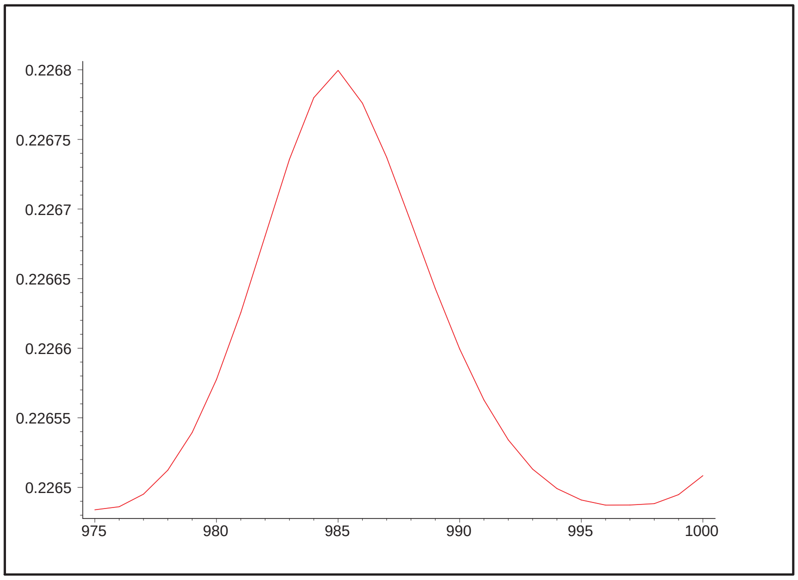

Figure 5: $\bar{E}(n) - n/m$, Strategy 3

and

$$\mathbb{V}(n) := D(n) - \bar{E}(n)^2.$$

To compute the asymptotic variance, we face a problem: if we always use $\bar{k} = 20$, we compute $\tilde{\mathbb{V}}(20) = 111.0712987\ldots$ from $P(20, i)$ and, by the renewal theorem, $C(n) := C(n, \bar{k}(n))$ is asymptotically Gaussian, with mean n/m and variance $n\tilde{\mathbb{V}}(20)/m^3$. But we have asymptotically, $\bar{k} = 20, \bar{k} = 21$, with, alternatively, asymptotic period $12, 10$. So we should be tempted to use

$$s_1^2 := \frac{12\tilde{\mathbb{V}}(20) + 10\tilde{\mathbb{V}}(21)}{22m^3} = 0.2126563592\ldots,$$

with $\tilde{\mathbb{V}}(21) = 119.2145260\ldots$ computed from $P(21, i)$. In Figure 6, we compare $\mathbb{V}(n)$ with ns_1^2.

The difference $\mathbb{V}(n) - ns_1^2$ is given in Figure 7: there appears some permanent oscillations with amplitude 2 and again a period 22. But the mean is far from 0.

Curiously enough, if we use the simple mean

$$s_2^2 := \frac{\tilde{\mathbb{V}}(20) + \tilde{\mathbb{V}}(21)}{2m^3} = 0.2133421842\ldots,$$

the fit seems better, as shown in Figures 8 and 9.

4.2 A Markov chain approach

Actually, we can compute the asymptotic variance by analyzing the stationary distribution of the following random walk. We start at position 0, at each round we make a jump of i, $i \in [0, k(\ell), k(\ell) + 1, \ldots, k(\ell) + 5]$, where ℓ is the distance to the target. We know that k is either 21 or 20. We stop as soon as we reach (or go beyond) the barrier n.

If we assume that, for n large, we are alternatively in a range $k = 21$ (of length 10), denoted by I, or $k = 20$ (of length 12), denoted by II, we can compute the stationary distribution $\pi_I(j)$ of being in position j of range I and similarly for range II. This leads to the following stationarity equations for

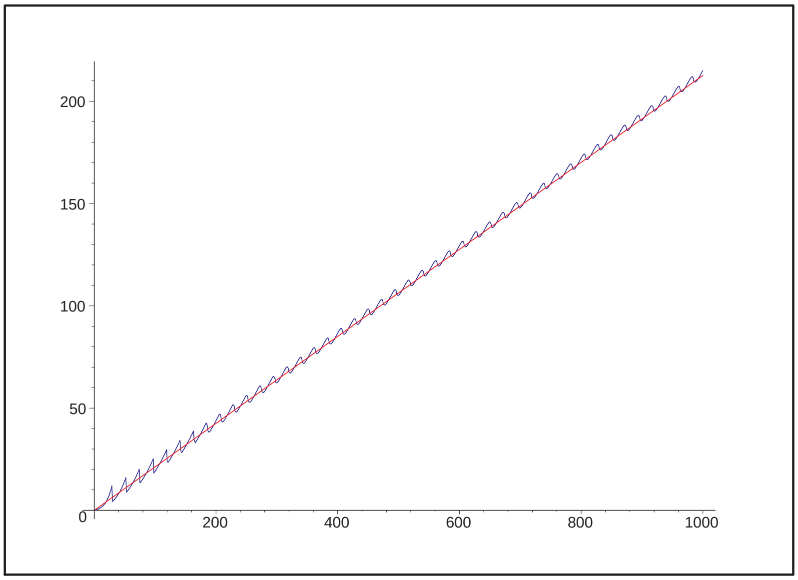

Figure 6: $\mathbb{V}(n)$ (blue), ns_1^2 (red)

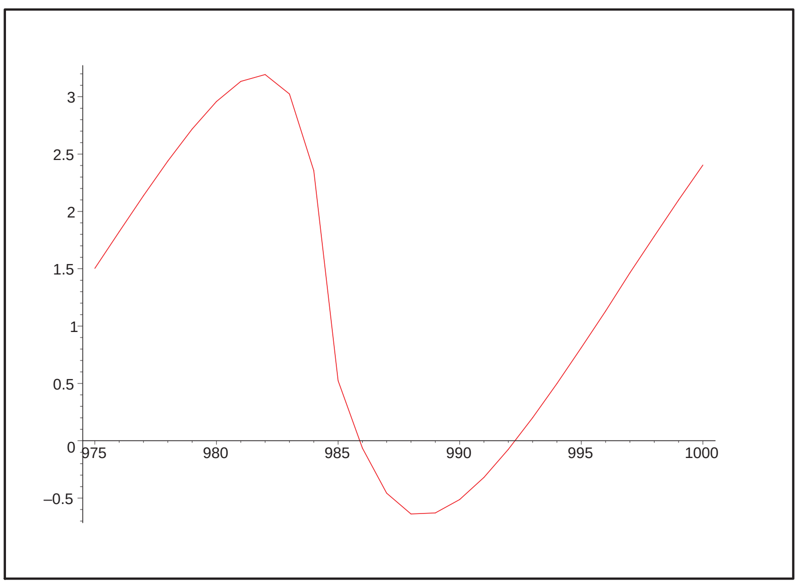

Figure 7: $\mathbb{V}(n) - ns_1^2$

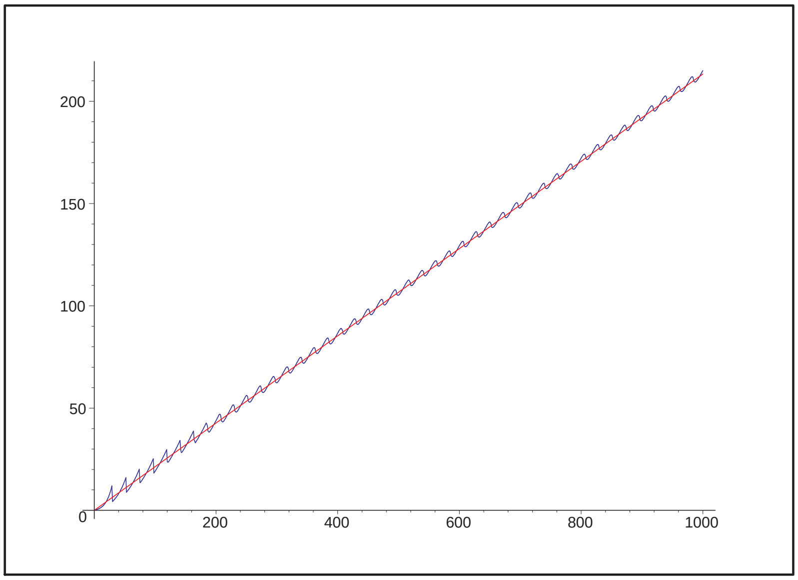

Figure 8: $\mathbb{V}(n)$ (blue), ns_2^2 (red)

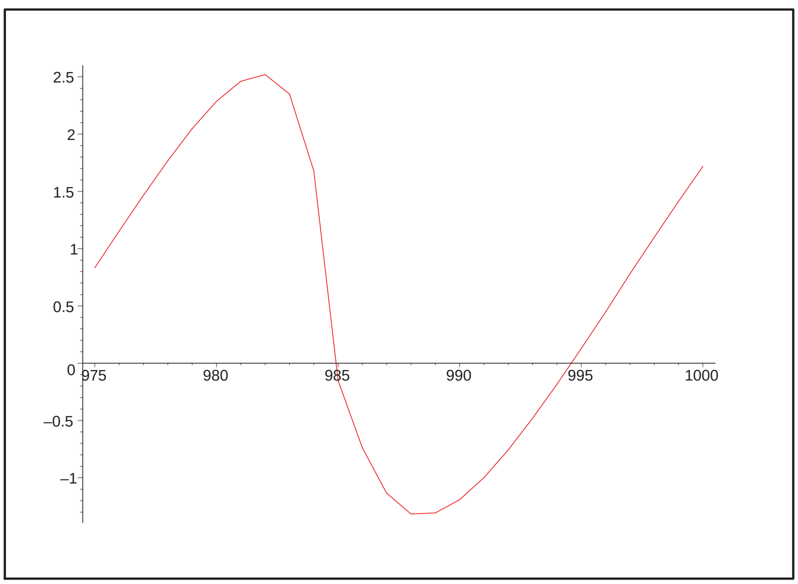

Figure 9: $\mathbb{V}(n) - ns_2^2$

the distributions $\pi_I(j)$ and $\pi_{II}(j)$

$$\pi_I(j) = \sum_{i=1}^{j} \pi_I(i)P(21, j-i) + \sum_{i=1}^{12} \pi_{II}(i)P(20, j+12-i)$$
$$+\sum_{i=1}^{10} \pi_I(i)P(21, j+12+10-i) + \sum_{i=1}^{12} \pi_{II}(i)P(20, j+12+10+12-i), \quad j = 1..10,$$
$$\pi_{II}(j) = \sum_{i=1}^{j} \pi_{II}(i)P(20, j-i) + \sum_{i=1}^{10} \pi_I(i)P(21, j+10-i)$$
$$+\sum_{i=1}^{12} \pi_{II}(i)P(20, j+10+12-i) + \sum_{i=1}^{10} \pi_I(i)P(21, j+10+12+10-i), \quad j = 1..12$$
$$\sum_{i=1}^{10} \pi_I(i) + \sum_{i=1}^{12} \pi_{II} = 1.$$

Note that no more equations are needed.

Solving this system leads to

$$p_I = \sum_{i=1}^{10} \pi_I(i) = 0.3633246365,$$
$$p_{II} = \sum_{i=1}^{12} \pi_{II}(i) = 0.6366753635.$$

Now the stationay variance is given by

$$p_I\tilde{\mathbb{V}}(21) + p_{II}\tilde{\mathbb{V}}(20),$$

and this gives

$$s^2 = \frac{p_I\tilde{\mathbb{V}}(21) + p_{II}\tilde{\mathbb{V}}(20)}{m^3} = 0.2112800054.$$

By the renewal theorem for Markov chain, the asymptotic number of rounds to reach n is Gaussian, with mean n/m and variance ns^2. This will be analyzed in Section 5.

4.3 Effect of the alternating sequence

Let us check the effect of the alternating $\bar{k} = 20, \bar{k} = 21$ sequence: we have tried the following Strategy 4: for $n = 1, \ldots, 250$, we use $\bar{k}(n)$ and $\bar{E}(n)$ as computed previously. At the end of this range, the alternating $\bar{k} = 20, \bar{k} = 21$ sequence appears already. For $n = 251, \ldots, 1000$, we always use $\bar{k}(n) = 20$. For $\bar{E}(n)$, the behaviour is similar to Figures 2 and 3. Figure 10 shows $\bar{E}(n) - n/m$ in the neighbourhood of $n = 1000$. An asymptotic period of 22 is still present, due to the alternating behaviour of $\bar{k}(n)$ at the end of the range 1..250. There appears a constant K_3, such that $0.2291 < K_3 < 0.2300$ and $K_1 < K_2 < K_3$.

Concerning the variance, the behaviour is different: Figures 11 and 12 give $\mathbb{V}(n) - n\tilde{\mathbb{V}}(20)/m^3$. Now some damping oscillations are apparent: the effect of the alternating behaviour of $\bar{k}(n)$ at the end of the range 1..250 is present later on, but gradually amortized.

4.4 Computation of the periodic part

Let us return to the general case with optimal $\bar{k}(n)$. Actually, we can compute the asymptotic periodic contribution in equilibrium as follows. Assume that, asymptotically, we have

$$\bar{E}(n) = \frac{n}{m} + K, \qquad \text{no oscillations},$$
$$\mathbb{V}(n) = ns^2 + U(n) = D(n) - \bar{E}(n)^2,$$

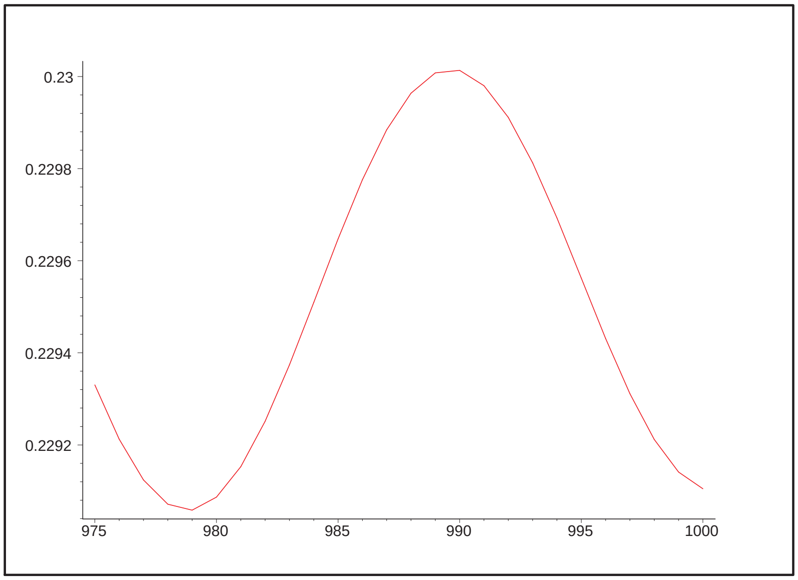

Figure 10: $\bar{E}(n) - n/m$, for Strategy 4

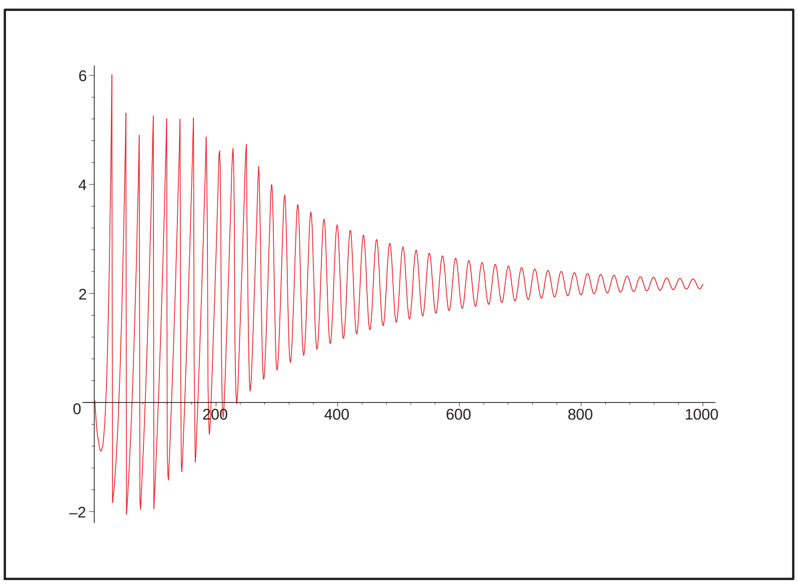

Figure 11: $\mathbb{V}(n) - n\tilde{\mathbb{V}}(20)/m^3$, for Strategy 4

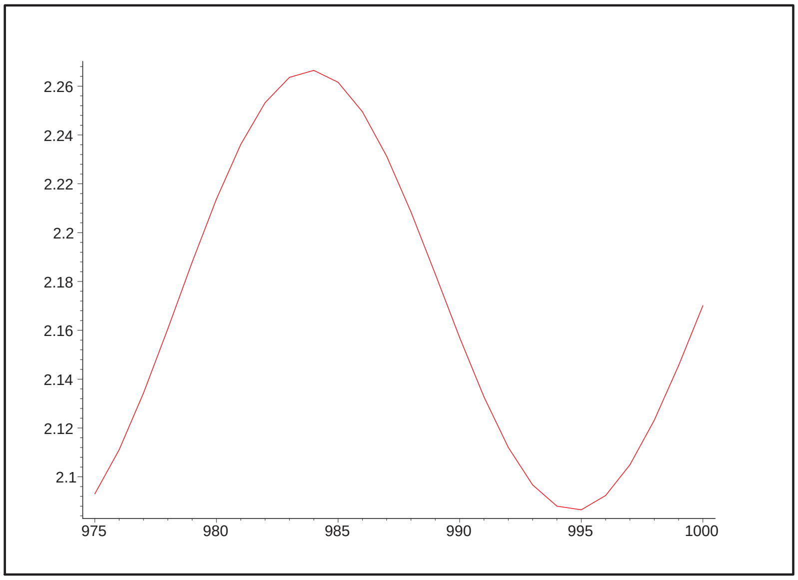

Figure 12: $\mathbb{V}(n) - n\tilde{\mathbb{V}}(20)/m^3$, for Strategy 4

for some constant s^2 and $U(n)$ is periodic with period 22. This gives, from (2),

$$\left(\frac{n}{m}+K\right)^2 + ns^2 + U(n) = P(\bar{k}(n),0)\left[1+2\left(\frac{n}{m}+K\right)+\left(\frac{n}{m}+K\right)^2+ns^2+U(n)\right]$$
$$+\sum_{j\in[\bar{k}(n),\ldots,\bar{k}(n)+5]} P(\bar{k}(n),j)\left[1+2\left(\frac{n-j}{m}+K\right)+(n-j)s^2+U(n-j)+\left(\frac{n-j}{m}+K\right)^2\right],$$

or

$$U(n) = P(\bar{k}(n),0)U(n) + \sum_{j\in[\bar{k}(n),\ldots,\bar{k}(n)+5]} P(\bar{k}(n),j)\left[U(n-j)+\frac{j^2}{m^2}\right] - 1 - ms^2$$
$$= P(\bar{k}(n),0)U(n) + \frac{\tilde{\mathbb{V}}(\bar{k}(n))}{m^2} + \sum_{j\in[\bar{k}(n),\ldots,\bar{k}(n)+5]} P(\bar{k}(n),j)U(n-j) - ms^2.$$

For $n = 1,\ldots,22$, $\bar{k}(n)$ alternatively $20, 21$ (with periods $12, 10$) and $(n-j)$ computed as $(n-j-1) \pmod{22} + 1$, this gives a set of 22 linear equations, which, together with

$$\sum_1^{22} U(n) = 0,$$

give $U(1),\ldots,U(22)$ and $s^2 = 0.2112800055\ldots$. This fits well with s^2 as computed in Section 4.2. Actually, the best numerical fit between $\mathbb{V}(n)$ and $ns^2 + U(n)$ (with $U(n)$ just computed) appears with $s_3^2 = 0.2137$ and is given in Figure 13. Given that we only used $n = 1000$, this is quite good.

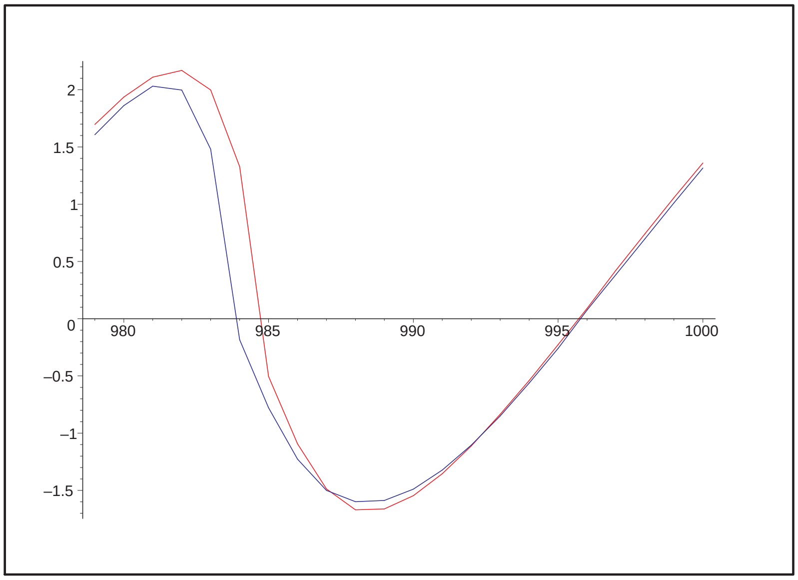

Figure 13: $\mathbb{V}(n) - ns_3^2$ (red), $U(n)$ (blue)

5 The asymptotic winning probability for the first player.

The distribution of the optimal cost $C(n)$ is given by

$$\Pi(n,j) = \mathbb{P}(C(n)=j) = \sum_{i\in[0,\bar{k}(n),\ldots,\bar{k}(n)+5]} P(\bar{k}(n),i)\Pi(n-i,j-1), \quad j\geq 1,$$
$$\Pi(k,0)=1, k\leq 0, \quad \Pi(k,0)=0, k\geq 1.$$

We have computed this probability for n up to 1000. The numerical difference between the computed mean

$$\mathbb{E}[C(n)] := \sum_1^\infty \Pi(n,j)j$$

and $\bar{E}(n)$ is given in Figure 14. This is of order 10^{-7}.

The numerical difference between the computed variance

$$\mathbb{V}_c[C(n)] := \sum_1^\infty \Pi(n,j)j^2 - [\mathbb{E}[C(n)]]^2$$

and $\mathbb{V}(n)$ is given in Figure 15. This is of order 10^{-4}.

The comparison between $\Pi(1000,j)$ and $e^{-\left(\frac{j-1000/m}{\sqrt{1000s}}\right)^2/2}/\sqrt{2\pi 1000 s^2}$ (s^2 as computed in Section 4.2) is given in Figure 16.

PROBLEM 3: EXPLAIN THE SLIGHT DISCREPANCY WITH THE GAUSSIAN

The shape is identical when we use $\bar{E}(1000)$ and $\mathbb{V}(1000)$ instead of n/m and $1000s^2$.

The winning probability Pw for the first player, if both players use the same threshold stategy, without looking at each other's position, is given by

$$Pw := \sum_k \Pi(n,k) \sum_{i\geq k} \Pi(n,i) = \frac{1}{2}\left[1 + \sum_k \Pi(n,k)^2\right], \tag{3}$$

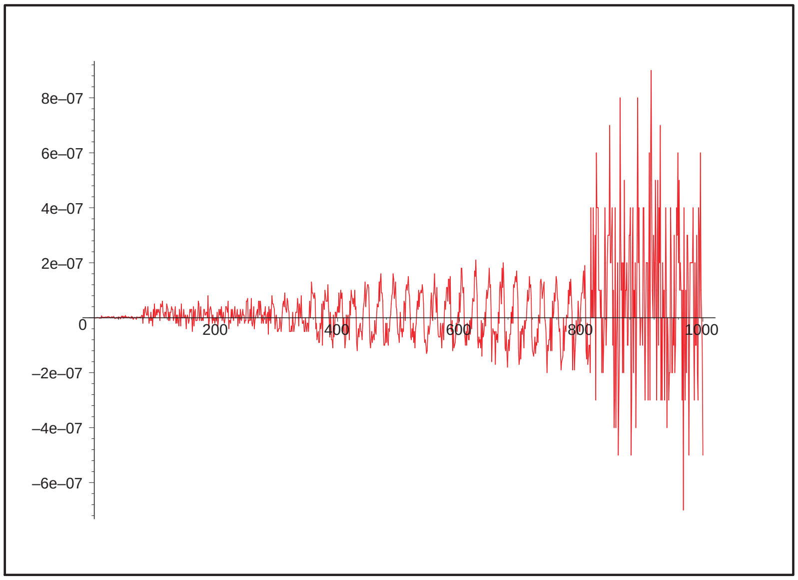

Figure 14: $\mathbb{E}[C(n)] - \bar{E}(n)$

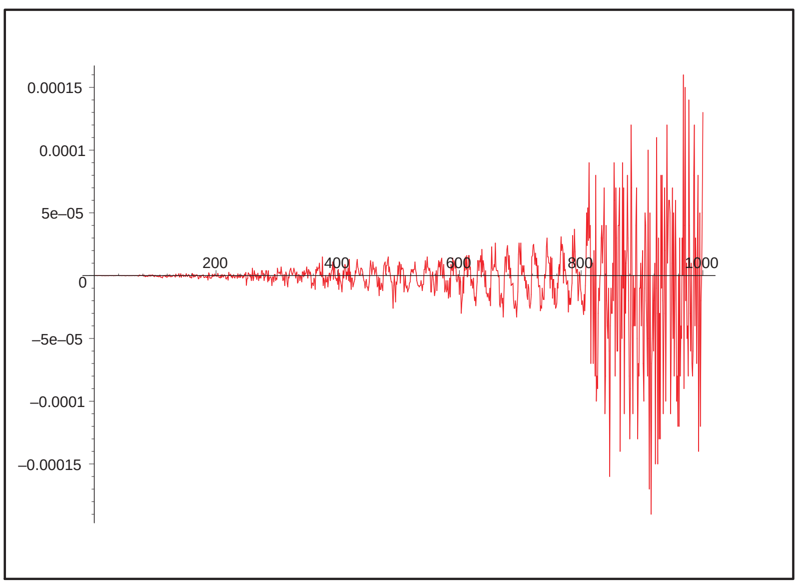

Figure 15: $\mathbb{V}_c[C(n)] - \mathbb{V}(n)$

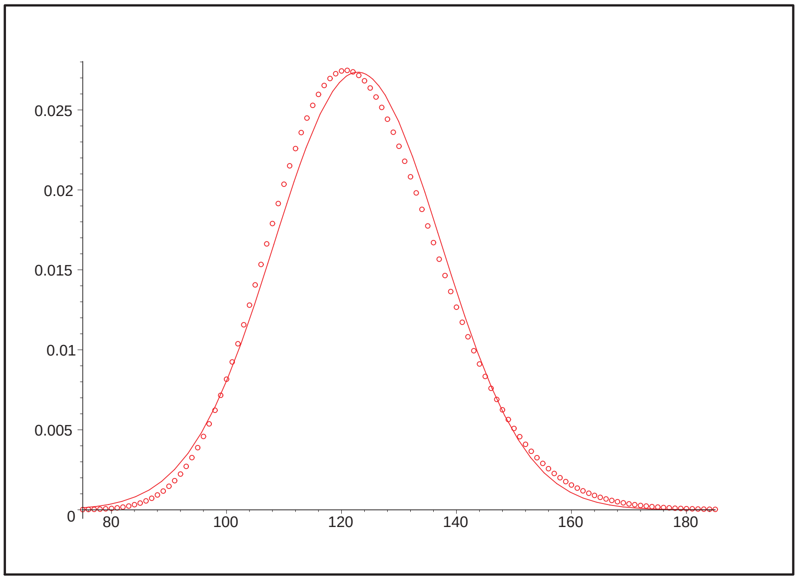

Figure 16: $\Pi(1000,j)$ (circle), $e^{-\left(\frac{j-1000/m}{\sqrt{1000s}}\right)^2/2}/\sqrt{2\pi 1000 s^2}$ (line)

and, by Euler-Maclaurin

$$Pw \sim \frac{1}{2}\left[1+\int_{-\infty}^{\infty} e^{-2\left(\frac{k-n/m}{\sqrt{ns}}\right)^2/2}/(2\pi n s^2)dk\right] = \frac{1}{2}\left[1+\frac{1}{2\sqrt{n\pi s}}\right] = \frac{1}{2}\left[1+\frac{0.613713\ldots}{\sqrt{n}}\right].$$

For $n = 1000$, $Pw = 0.5097043\ldots$ as given by (3), and our asymptotics gives $0.5097037\ldots$

6 Taking the adversary's situation into account

Let $P(i,j,k)$ be the player's probability of winning if the player's score is i, the opponent's score is j, and the player's turn total is k. In the case where $i+k \geq n, P(i,j,k) = 1$ because the player can simply hold and win. In the general case where $0 \leq i,j < n$ and $k < n-i$, the probability of an optimal player winning is

$$\begin{aligned} P(i,j,k) &= \max(P(i,j,k,r), P(i,j,k,h)),\\ P(i,n,0) &= 0,\\ P(i,j,k) &= 1, i+k \geq n, \forall j,\\ P(n,j,0) &= 1. \end{aligned}$$

where $P(i,j,k,r)$ and $P(i,j,k,h)$ are the probabilities of winning if one rolls and holds, respectively. These probabilities are given by:

$$\begin{aligned} P(i,j,k,r) &= \frac{1}{6}\left((1-P(j,i,0)) + \sum_{u=2}^{6} P(i,j,k+u)\right),\\ P(i,j,k,h) &= 1-P(j,i+k,0). \end{aligned}$$

The probability of winning after rolling a 1 or holding is the probability that the other player will not win beginning with the next turn. Using iteration techniques, Nellner and Presser, [2] and [3], computed numerically, for $n = 100$, the solution as a surface which is the boundary between states where player 1 should roll (below the surface) and states where player 1 should hold (above the surface). This is given in Figure 17.

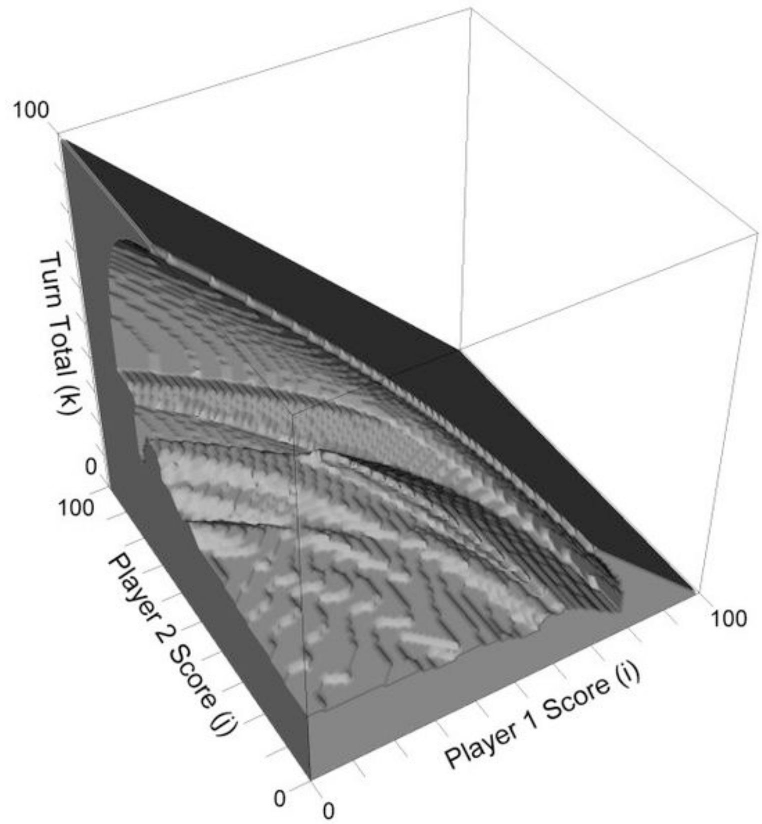

Figure 17: Two players optimal strategy

PROBLEM 4: OBTAIN A DIRECT NUMERICAL SOLUTION FOR $P(i, j, k)$ OR BETTER AN EXPLICIT FORM .

Let us start with

$$P(i, j, 0, r) = \frac{1}{6}(1 - P(j, i, 0)) + \frac{1}{6}\left(\sum_{u=2}^{6} P(i, j, u)\right),$$
$$P(i, j, 0, h) = 1 - P(j, i, 0).$$

For $i \geq n - 2$, we have

$$P(i, j, 0, r) = \frac{1}{6}(1 - P(j, i, 0)) + \frac{5}{6},$$
$$P(i, j, 0, h) = 1 - P(j, i, 0).$$

and, for $i \geq n - 2, \forall j$,

$$P(i, j, 0) = \frac{1}{6}(1 - P(j, i, 0)) + \frac{5}{6},$$

so, for $i, j \geq n - 2$,

$$P(i, j, 0) = \frac{6}{7}.$$

Now, consider the case $k = 1$. We have

$$P(i, j, 1, r) = \frac{1}{6}(1 - P(j, i, 0)) + \frac{1}{6}\left(\sum_{u=2}^{6} P(i, j, 1 + u)\right),$$
$$P(i, j, 1, h) = 1 - P(j, i + 1, 0).$$

For $i \geq n-3$, we have

$$P(i,j,1,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{5}{6},$$
$$P(i,j,1,h) = 1 - P(j,i+1,0).$$

This allows computing $P(i,j,1)$ for $i,j \geq n-2$.

For $i = n-3$, we have

$$P(n-3,j,1,r) = \frac{1}{6}\left(1 - P(j,n-3,0)\right) + \frac{5}{6},$$
$$P(n-3,j,1,h) = 1 - P(j,n-2,0).$$

For $j \geq n-2$, we still need $P(j,n-3,0)$, this is computed below.

Next, consider the case $k=2$. We have

$$P(i,j,2,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{1}{6}\left(\sum_{u=2}^{6} P(i,j,2+u)\right),$$
$$P(i,j,2,h) = 1 - P(j,i+2,0).$$

For $i \geq n-4$, we have

$$P(i,j,2,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{5}{6},$$
$$P(i,j,2,h) = 1 - P(j,i+2,0).$$

This allows computing $P(i,j,2)$ for $i,j \geq n-2$.

For $i = n-3, j \geq n-2$, we have

$$P(n-3,j,2,r) = \frac{1}{6}\left(1 - P(j,n-3,0)\right) + \frac{5}{6},$$
$$P(n-3,j,2,h) = 1 - P(j,n-1,0).$$

So we need $P(j,n-3,0)$. Hence we consider

$$P(n-3,j,0,r) = \frac{1}{6}\left(1 - P(j,n-3,0)\right) + \frac{1}{6}P(n-3,j,2) + \frac{4}{6},$$
$$P(n-3,j,0,h) = 1 - P(j,n-3,0).$$

and, for $j \geq n-2$,

$$P(j,n-3,0,r) = \frac{1}{6}\left(1 - P(n-3,j,0)\right) + \frac{5}{6},$$
$$P(j,n-3,0,h) = 1 - P(n-3,j,0).$$

We have already three pairs of equations and three max-operators!

PROBLEM 5: OBTAIN AN ASYMPTOTIC FORM FOR $P(i,j,k)$.

7 Another Strategy 6

In [1], Haigh and Roters propose a Strategy 6 slighly different from Strategy 1 (see also Roters [4]). If the player uses k at position n and reaches $k+5$, he continues until reaching at least $k+5+C$ (if possible) for some constant $C > 0$. This amounts to use $\tilde{P}(k,i)$ such that

$$\tilde{P}(k,j) = P(k,j), \quad k \leq j \leq k+4,$$
$$\tilde{P}(k,k+5) = 0,$$
$$\tilde{P}(k,0) = P(k,0) + P(k,k+5)P(C,0),$$
$$\tilde{P}(k,k+5+C+\delta) = P(k,k+5)P(C,C+\delta), \quad \delta = 0,\ldots,5.$$

The authors show that this gives a better value then $\bar{E}(n), \bar{E}_6(n)$, say, if we choose Strategy 6, for $n = 53, k(53) = 17$ (as in Strategy 1), and $C = 4$, and for $n = 75, k(75) = 18$ (as in Strategy 1), and $C = 5$.

They conjecture (using only $n = 200$) that these are the only values where it is better to use Strategy 6. Figures 18 and 19 give $\bar{E}(n) - \bar{E}_6(n)$ in the starting range and in the neighbourhood of $n = 1000$. The positive effect is gradually amortized and becomes rapidly negligible. The mean $\tilde{G}(k) = \sum_j \tilde{P}(k,j)j$ is maximum for $k = 20, C = 1$ and gives $\tilde{G}(20) = 8.127484455\ldots$. But $G(20) - \tilde{G}(20) = 0.014310439\ldots$ which is much greater then the maximum variation of K_1 at $n = 1000$ (see Section 3). So Strategy 6 will never beat Strategy 1 for large n.

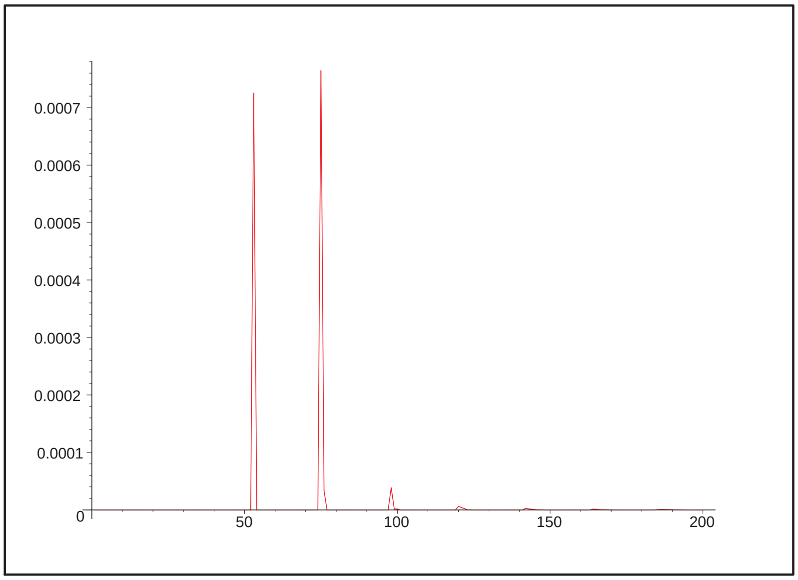

Figure 18: $\bar{E}(n) - \bar{E}_6(n), n = 1..150$

8 Ferguson's solution (private communication)

Ferguson proposes to look at the problem as game with some payoff. This will allow an easy proof of the convergence of the numerical iteration procedure. First of aIl, it is convenient to count down rather than up. More precisely, we assume that both players start with scores of n and are trying to reduce their scores to zero. So let x (resp. y) denote the number of points player A (resp. B) needs to finish. In the notations of Sec. 6, $x = n - i, y = n - j$. Secondly, the formulae look simpler if we use a payoff function of $+1$ for a win and -1 for a loss, and try to maximize the expected payoff. The probability of win is then one-half of one more than the expected payoff, Y say. Indeed,

$$\mathbb{E}(Y) = \mathbb{P}(win) - \mathbb{P}(loss) = 2\mathbb{P}(win) - 1,$$
$$\mathbb{P}(win) = \frac{1 + \mathbb{E}(y)}{2}.$$

Let $v(x, y)$ represent the expected payoff to A under optimal play of both players, if A has x to go, B has y to go and it is A's turn. This corresponds to $v(x, y) = 2P(n - x, n - y, 0) - 1$. Suppose we have a situation with x to go for A and y to go for B, and suppose that A has rolled the dice a few times (without rolling a 1) with a total so far equal to k. If A stops after this, the situation would be changed

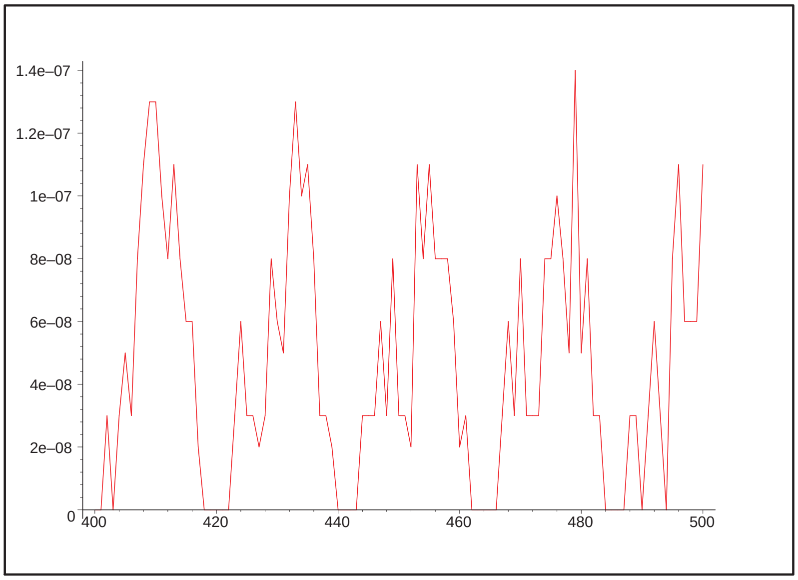

Figure 19: $\bar{E}(n) - \bar{E}_6(n), n = 900..1000$

to one in which A has $x-k$ to go, B has y to go, and it is B's turn. The value to A of this situation is $-v(y, x-k)$. Let $s = x-k$ and let $w(s,x,y)$ denote A's expected payoff in this situation under optimal play of both players. Note that $v(x,y) = w(x,x,y)$ and $w(s,x,y) = 2P(n-x, n-y, x-s) - 1$. Then

$$w(s,x,y) = \max\left(-v(y,s), -\frac{1}{6}v(y,x) + \frac{1}{6}\sum_{j=2}^{6} w(s-j,x,y)\right) \tag{4}$$

for $x \geq 1, y \geq 1$ and $-5 \leq s \leq x$. The boundary conditions on w are

$$w(s,x,y) = +1 \text{ for } x \geq 1, y \geq 1 \text{ and } s \leq 0. \tag{5}$$

We may solve these equations recursively. If we are given $v(x,s)$ for all $s < y$ and $v(s,y)$ for all $s < x$, we can use (4) and the corresponding equations with x and y interchanged to find $v(x,y)$ and $v(y,x)$. One method of accomplishing this is by iteration. Suppose we make a guess at $v(y,x)$. Then we can use (4) to find $w(s,x,y)$ recursively for $s = 1,2,...,x$. Then we put $v(x,y) = w(x,x,y)$ and use (4) with x and y interchanged to compute $w(s,y,x)$ recursively for $s = 1,2,...,y$. This gives us a new value for $v(y,x)$, namely $w(y,y,x)$, which may be used in the next iteration. To show that this method converges, it is sufficient to show that the mapping $v(x,y) \to w(y,y,x)$ given by this method has a fixed point, and that the derivative of the map is in absolute value less than 1.

Let $g(z)$ denote the map that maps $z = v(y,x)$ into $z' = w(x,x,y)$ using (4), and let $f(z')$ denote the corresponding map that maps $z' = v(x,y)$ into $w(y,y,x)$. The desired iteration is then $h(z) = f(g(z))$. There will exist a fixed point if the map is continuous and maps $[-1,1]$ into $[-1,1]$.

Let us prove that the map $h(z)$ js continuous nondecreasing from $[-1,1]$ into $[-1,1]$.

Since inductively, each $w(s,x,y)$ is continuous and nonincreasing in $z = v(y,x)$, the map $g(z)$ is nonincreasing. Similarly the map $f(z')$ is continuous and nonincreasing. Therefore the map $h(z)$ is continuous and nondecreasing. Moreover, each $w(s,x,y)$ is equal either to $-v(y,s)$, or to the average of six numbers inductively between -1 and $+1$, and in either case is in the interval $[-1,1]$.

References

[1] J. Haigh and M. Roters. Optimal stopping on a dice game. *Journal of Applied Probability*, 37:1110–1116, 2000.

[2] T.W. Neller and C.G.M. Presser. Optimal play of the dice game pig. *The UMAP Journal*, 25(1):25–47, 2004.

[3] T.W. Neller and C.G.M. Presser. Pigtail: A pig addendum. *The UMAP Journal*, 26(4):443–458, 2005.

[4] M. Roters. Optimal stopping on a dice game. *Journal of Applied Probability*, 35:229–235, 1998.