

NUMBER OF SURVIVORS IN THE PRESENCE OF A DEMON

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Abstract

This paper complements the analysis of Louchard and Prodinger [LP08] on the number of *rounds* in a coin-flipping selection algorithm that occurs in the presence of a demon. We precisely analyze a very different aspect of the selection algorithm, using different methods of analysis. Specifically, we obtain precise descriptions of the distribution and all moments of the number of *participants* ultimately selected during the algorithm. The selection algorithm is robust in at least two significant ways. The presence of a demon allows for the precise analysis even when errors may occur between the rounds of the selection process. (The analysis also handles the more traditional case, in which no demon is involved.) The selection algorithm can also use either biased or unbiased coins.

1. INTRODUCTION

We precisely analyze the number of *survivors* in a selection process that occurs in the presence of a demon. Louchard and Prodinger [LP08] recently utilized a different methodology (for extreme value distributions, a.k.a. “Gumbel distributions”) to analyze the number of *rounds* required to perform the selection algorithm.

The inclusion of a “demon” can be viewed as a generalization of traditional selection algorithms. The demon represents errors which might occur between rounds of the process. Another interpretation is that participants might be likely to drop out of the selection process for reasons unrelated to the coin flips in the selection process itself. In any case, a traditional selection algorithm (with no demon involved) is just a special case (using $\nu = 0$) of our very general analysis. The special case $\nu = 0$ (i.e., with no demon involved) is a selection process using a traversal of binary retrieval trees (tries), where a coin flip of “heads” is analogous to descending one direction in the trie, and a flip of “tails” corresponds to descending in the other direction. The involvement of a demon makes the present algorithm more complicated and realistic than the traditional trie algorithm. As another feature of the generality of our analysis, the coins used in the selection can be either unbiased (i.e., $p = q = 1/2$) or biased ($p, q \neq 1/2$).

We are able to give the complete distribution and all moments of the number of survivors in a selection algorithm that occurs in the presence of a demon.

2. SELECTION ALGORITHM

At the start of the selection algorithm, n people are present. Each person flips a coin with probability q of heads and p of tails. If all n people flip tails, then they are all selected by the algorithm and the selection process is finished. If $j > 0$ people flip heads, then these j people remain in play, and the other $n - j$ (who flipped tails) are eliminated from further play.

Then a demon arrives and, with probability ν , removes exactly one of the survivors, so $j - 1$ remain; he leaves the j survivors alone with probability $\mu = 1 - \nu$. If he leaves the j survivors alone, then these j survivors begin another round of coin flipping. If he removes a survivor and $j - 1 = 0$ (i.e., no survivors remain) then the selection process is finished and nobody is selected. If he removes a survivor and $j - 1 > 0$ (i.e., some survivors remain) then these $j - 1$ survivors begin another round of coin flipping.

The end of the algorithm can occur in two possible ways:

- (1.) During a round of coin flipping, one or more people remain. All of the remaining people simultaneously flip tails at this stage, and the algorithm ends. All of the people at this last stage are selected by the algorithm.
- (2.) During a visit by the demon, only one person is present, and this person is removed by the demon. In this case, zero people are selected by the algorithm.

Here is a pseudocode interpretation of the selection process:

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begin round with  $n$  people
  all  $n$  people flip coins
  if all  $n$  get tails, then
    all  $n$  are selected, and the algorithms ends;
  else if  $j > 0$  get heads and  $n - j$  get tails, then
    with probability  $\mu = 1 - \nu$ :
      the demon leaves the  $j$  survivors alone;
      begin a new round with  $j$  people
    otherwise (i.e., with probability  $\nu$ ):
      the demon removes one of the  $j$  survivors;
      if  $j - 1 = 0$  then 0 people are selected, and the algorithm ends;
      else (i.e., if  $j - 1 > 0$ ) then begin a new round with  $j - 1$  people
  end if
end round

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All of the coin flips are conducted independently. Each person's flips are independent of every other person's flips, and subsequent flips by each person are independent too.

Example 2.1. Suppose that the probability of heads is $q = 1/3$ and the probability of tails is $p = 2/3$. Suppose that the demon appears with probability $\nu = 1/5$. Then the selection algorithm might proceed as follows:

Initially 100 people are present. Exactly 31 of them flip heads (happens with probability $\binom{100}{31}q^{31}p^{69}$), and then the demon arrives (happens with probability $1/5$) and removes one of the 31 survivors. So the next round begins with 30 people.

Exactly 12 of the remaining 30 people flip heads (happens with probability $\binom{30}{12}q^{12}p^{18}$), and then the demon leaves the survivors alone (happens with probability $4/5$). So the next round begins with 12 people.

Exactly 3 of the remaining 12 people flip heads (happens with probability $\binom{12}{3}q^3p^9$), and then the demon leaves the survivors alone (happens with probability $4/5$). So the next round begins with 3 people.

All 3 remaining people flip tails (happens with probability $\binom{3}{0}q^0p^3$), and all three are selected by the algorithm.

3. NOTATION TABLE

Most of the following definitions are already embedded at the appropriate places in the analysis. For the reader's convenience, we also summarize many of the terminologies used, in one succinct location. For the convenience of someone who already read [LP08], we preserve some of Louchard and Prodingers' earlier notation.

n := number of people present at the start of the selection algorithm
 q := probability that a coin flip shows heads,
 so the corresponding person advances
 p := $1 - q$, probability that a coin flip shows tails,
 so the corresponding person is eliminated
 ν := probability that, during a visit by the demon, one survivor is removed

$\mu := 1 - \nu$, probability that the demon does not remove a survivor during a visit
 $X_n :=$ number of people selected by the algorithm, with n initial participants
 $\pi(n, m, j) :=$ probability the algorithm selects m of the initial n people and requires j rounds
 by convention, $\pi(0, 0, 1) = 1$, and otherwise $\pi(0, m, j) = 0$; thus $F_0(u, v) = v$
 $\pi(n, m) := \mathbf{P}(X_n = m)$
 = probability that the algorithm selects m of the initial n people;
 by convention, $\pi(0, 0) = 1$, and $\pi(0, m) = 0$ for $m \neq 0$; thus $F_0(u) = 1$
 $F_n(u, v) := \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \pi(n, m, j) u^m v^j$
 $F_n(u) := F_n(u, 1) = \sum_{m=0}^{\infty} \pi(n, m) u^m = \sum_{m=0}^{\infty} \mathbf{P}(X_n = m) u^m$
 $Q := 1/q$
 $L := \ln Q$
 $\chi_l := 2l\pi i/L$
 $H_j := \sum_{k=1}^j \frac{1}{k}$ is the j th harmonic number
 $x^{\underline{j}} := \prod_{l=0}^{j-1} (x - l) = (x)(x-1)(x-2)\cdots(x-j+1)$ is the j th falling power of x
 $\mathbb{E}[X_n^{\underline{j}}] := \mathbb{E}\left[\prod_{l=0}^{j-1} (X_n - l)\right] = E[(X_n)(X_n-1)(X_n-2)\cdots(X_n-j+1)]$
 is the j th factorial moment of the random variable X_n
 i.e., the expected value of the j th falling power of X_n

We utilize some concepts from the theory of q -analysis. Since the value of q is fixed, we suppress the dependence on q . For positive integers n , we use the q -Pochhammer symbol

$$(x)_n := \prod_{j=0}^{n-1} (1 - xq^j) = (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}).$$

We also define $(x)_{\infty} := \prod_{j=0}^{\infty} (1 - xq^j) = \lim_{n \rightarrow \infty} (x)_n$. For complex-valued z , we define

$$(x)_z := \frac{(x)_{\infty}}{(xq^z)_{\infty}}.$$

4. RESULTS

The s th factorial moment $\mathbb{E}[X_n^{\underline{s}}]$ of the number of people selected at the end of the selection algorithm has the form $\sim \text{const} + \delta(\log_Q n) + o(1)$, where δ is a fluctuating function.

Theorem 4.1. *The s th factorial moment of the number of people X_n selected by the algorithm, when beginning with n participants, is*

$$E[X_n^s] = \frac{(Qp)^s}{L} \left[\frac{(\mu q)_\infty}{(q)_\infty} (s-1)! + s(-1)^{s-1} \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{l \geq 1} \frac{\mu q^{j+l}}{1 - \mu q^{j+l}} - \sum_{l \geq 1} \frac{q^{j+l}}{1 - q^{j+l}} \right) + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right] + \sum_{l \neq 0} \phi_{s,l}(n) + O(n^{-1}),$$

where

$$\phi_{s,l}(n) = \frac{(Qp)^s (-1)^s}{L} \left(\frac{(\mu q)_\infty}{(q)_\infty} \chi_l^s + s \sum_{j \geq 0} \left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) \left(j^{s-1} - (j + \chi_l)^{s-1} \right) \right) \times \Gamma(-\chi_l) e^{2l\pi i \log_Q n}.$$

Note that $e^{2l\pi i \log_Q n}$ is fluctuating, with $|e^{2l\pi i \log_Q n}| = 1$.

The distribution $P(X_n > r)$ of the number of people selected at the end of the selection algorithm also has the form $\sim \text{const} + \delta(\log_Q n) + o(1)$, where δ is a fluctuating function.

Theorem 4.2. *The distribution of the number of people X_n selected by the algorithm, when beginning with n participants, is*

$$P(X_n > r) = \frac{(\mu q)_\infty}{L(q)_\infty} \left[L - \sum_{s=1}^r \frac{p^s}{s} - \frac{\nu p^r}{r+1} - Q(-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r(m-1)L}}{(1-q^{m-1})^{r+1}} \right) \right] + \sum_{l \neq 0} \Phi_{r,l}(n) + O(n^{-1}),$$

where

$$\Phi_{r,l}(n) = \frac{Q}{L} \frac{(\mu q)_\infty}{(q)_\infty} \left(-q \sum_{s=1}^r \frac{\chi_l^s}{s!} (-p)^s + (-p)^r \nu q \frac{\chi_l^{r+1}}{(r+1)!} + (-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{\chi_l^s}{s!} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} \right) \right) \times \Gamma(-\chi_l) e^{2l\pi i \log_Q n}.$$

Note that $e^{2l\pi i \log_Q n}$ is fluctuating, with $|e^{2l\pi i \log_Q n}| = 1$.

5. ASYMPTOTIC MOMENTS OF THE NUMBER OF SURVIVORS

5.1. Derivation of Generating Functions. We next establish an exact formula for the bivariate generating function $F_n(u, v)$ that describes the probabilities associated with the number of survivors and the number of rounds in the entire algorithm.

Lemma 5.1. *Let $F_n(u, v)$ be a bivariate generating function such that the coefficient of $u^m v^j$ is the probability that, in the algorithm, exactly m people are ultimately selected and exactly j rounds are used to complete the election. Then*

$$(1) \quad F_n(u, v) = \sum_{k=0}^n \binom{n}{k} \left(v(-q)^k \frac{(v)_k}{(\mu v q)_k} + (-q)^{k-1} \frac{(vq)_{k-1}}{(\mu q v)_k} \sum_{j=0}^{k-1} \frac{(1-pu)^j p v (u-1) (\mu q v)_j}{q^j (vq)_j} \right).$$

The proof of Lemma 5.1 utilizes some recurrences associated with $F_n(u, v)$. The proof is given in Section 7.

Corollary 5.2. *Setting $u = 1$ in Lemma 5.1, we obtain*

$$(2) \quad F_n(1, v) = \sum_{k=0}^n \binom{n}{k} v(-q)^k \frac{(v)_k}{(\mu v q)_k}.$$

This verifies that our results about the number of rounds agrees with the results from our previous paper.

During the remainder of the paper, we no longer pay attention to the number of rounds. We focus exclusively on the number of survivors.

Lemma 5.3. *The s th factorial moment of X_n is*

$$(3) \quad E[X_n^{\underline{s}}] = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} \varphi_s(k),$$

with

$$(4) \quad \varphi_s(z) = q^z \frac{(q)_{z-1}}{(\mu q)_z} s(Qp)^s (-1)^{s-1} \psi_s(z),$$

and

$$\psi_s(z) = \frac{(\mu q)_\infty}{(q)_\infty} \frac{z^{\underline{s}}}{s} + \sum_{j \geq 0} \left[\left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) j^{s-1} - \left(\frac{(\mu q)_{j+z}}{(q)_{j+z}} - \frac{(\mu q)_\infty}{(q)_\infty} \right) (j+z)^{s-1} \right].$$

5.2. Asymptotics. Now we turn our attention to the asymptotic moments of the number of survivors in the algorithm as the number n of initial participants grows large. We note that $(q)_{z-1} = \frac{(q)_\infty}{(1-q^z)(q^{z+1})_\infty}$, so $\varphi_s(z)$ has a simple pole at each of the locations of the form $z = m + \frac{2l\pi i}{L}$ for $l, m \in \mathbb{Z}$ with $m \leq 0$. By Theorem 2 of [FS95], we can restrict attention to the poles where $m = 0$, i.e., where $z = \chi_l$ for $l \in \mathbb{Z}$. Thus

$$E[X_n^{\underline{s}}] = \sum_{l \in \mathbb{Z}} \operatorname{Res}_{z=\chi_l} \left[\varphi_s(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] + O(n^{-1}).$$

We need the local expansion of $\varphi_s(z)$ and thus $\psi_s(z)$ around $z = 0$ to two terms, since $\varphi_s(z) \frac{n!}{(z)(z-1)\cdots(z-n)}$ has a double pole at $z = 0$, but only a simple pole at $z = \chi_l$ for $l \neq 0$. As $z \rightarrow 0$,

$$\frac{(\mu q)_{j+z}}{(q)_{j+z}} \sim \frac{(\mu q)_j}{(q)_j} \left[1 - zL \sum_{l \geq 1} \frac{\mu q^{j+l}}{1 - \mu q^{j+l}} + zL \sum_{l \geq 1} \frac{q^{j+l}}{1 - q^{j+l}} \right],$$

and

$$(j+z)^{\underline{s-1}} \sim j^{\underline{s-1}} [1 + z(H_j - H_{j-s+1})].$$

Thus

$$\begin{aligned} \psi_s(z) &\sim \frac{(\mu q)_\infty}{(q)_\infty} \frac{(-1)^{s-1}(s-1)!}{s} z + \sum_{j \geq s-1} \left[\left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) j^{s-1} \right. \\ &\quad \left. - \left(\frac{(\mu q)_j}{(q)_j} \left[1 - zL \sum_{l \geq 1} \frac{\mu q^{j+l}}{1 - \mu q^{j+l}} + zL \sum_{l \geq 1} \frac{q^{j+l}}{1 - q^{j+l}} \right] - \frac{(\mu q)_\infty}{(q)_\infty} \right) \right. \\ &\quad \left. \times j^{s-1} [1 + z(H_j - H_{j-s+1})] \right]. \end{aligned}$$

More simply, as $z \rightarrow 0$,

$$\begin{aligned} \psi_s(z) &\sim z \left[\frac{(\mu q)_\infty}{(q)_\infty} \frac{(-1)^{s-1}(s-1)!}{s} + \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{l \geq 1} \frac{\mu q^{j+l}}{1 - \mu q^{j+l}} - \sum_{l \geq 1} \frac{q^{j+l}}{1 - q^{j+l}} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right]. \end{aligned}$$

Notice the absence of the constant term! Substituting into the definition of $\varphi_s(z)$ in (4), it follows that

$$\begin{aligned} \varphi_s(z) &\sim q^z \frac{(q)_{z-1}}{(\mu q)_z} s(Qp)^s (-1)^{s-1} \times z \left[\frac{(\mu q)_\infty}{(q)_\infty} \frac{(-1)^{s-1}(s-1)!}{s} \right. \\ &\quad \left. + \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{l \geq 1} \frac{\mu q^{j+l}}{1 - \mu q^{j+l}} - \sum_{l \geq 1} \frac{q^{j+l}}{1 - q^{j+l}} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right] \end{aligned}$$

as $z \rightarrow 0$. Also $z(q)_{z-1} \sim 1/L$ and $(\mu q)_z \sim 1$, so

$$\begin{aligned} \varphi_s(z) &\sim \frac{(Qp)^s}{L} \left[\frac{(\mu q)_\infty}{(q)_\infty} (s-1)! + s(-1)^{s-1} \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{l \geq 1} \frac{\mu q^{j+l}}{1 - \mu q^{j+l}} - \sum_{l \geq 1} \frac{q^{j+l}}{1 - q^{j+l}} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right] \end{aligned}$$

as $z \rightarrow 0$. Also $\frac{n!(-1)^n}{(z-1)\cdots(z-n)} \sim 1$ as $z \rightarrow 0$. Therefore

$$\begin{aligned} \operatorname{Res}_{z=0} \left[\varphi_s(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] &= \lim_{z \rightarrow 0} \varphi_s(z) \\ &= \frac{(Qp)^s}{L} \left[\frac{(\mu q)_\infty}{(q)_\infty} (s-1)! + s(-1)^{s-1} \sum_{j \geq s-1} j^{s-1} \right. \\ &\quad \times \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{l \geq 1} \frac{\mu q^{j+l}}{1 - \mu q^{j+l}} - \sum_{l \geq 1} \frac{q^{j+l}}{1 - q^{j+l}} \right) \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right]. \end{aligned}$$

So the s th factorial moment $\mathbb{E}[X_n^s]$ of the number of people selected by the algorithm, when the algorithm begins with n participants, is

$$E[X_n^s] = \frac{(Qp)^s}{L} \left[\frac{(\mu q)_\infty}{(q)_\infty} (s-1)! + s(-1)^{s-1} \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{l \geq 1} \frac{\mu q^{j+l}}{1 - \mu q^{j+l}} - \sum_{l \geq 1} \frac{q^{j+l}}{1 - q^{j+l}} \right) + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right] + \sum_{l \neq 0} \tilde{\phi}_{s,l}(n) + O(n^{-1}),$$

where

$$\begin{aligned} \tilde{\phi}_{s,l}(n) &= \operatorname{Res}_{z=\chi_l} \left[\varphi_s(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] \\ &= \operatorname{Res}_{z=\chi_l} \left[(q)_{z-1} \frac{q^{\chi_l} s p^s}{(\mu q)_{\chi_l} q^s} (-1)^{s-1} \psi_s(\chi_l) \frac{n!(-1)^n}{(\chi_l)(\chi_l-1)\cdots(\chi_l-n)} \right] \\ &= \frac{(Qp)^s (-1)^s}{L} \left(\frac{(\mu q)_\infty}{(q)_\infty} \chi_l^s + s \sum_{j \geq 0} \left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) \left(j^{s-1} - (j + \chi_l)^{s-1} \right) \right) \\ &\quad \times \Gamma(-\chi_l) e^{2l\pi i \log_Q n} (1 + O(n^{-1})). \end{aligned}$$

Note that $e^{2l\pi i \log_Q n}$ is fluctuating, with $|e^{2l\pi i \log_Q n}| = 1$. This completes the proof of Theorem 4.1.

6. ASYMPTOTIC DISTRIBUTION OF THE NUMBER OF SURVIVORS

6.1. Derivation of the Distribution of the Number Of Survivors. Now we derive an exact formula for the distribution of the number of survivors selected at the end of the algorithm.

Lemma 6.1. *Let $r \geq 0$. The probability that strictly more than r out of n initial participants are selected at the end of the algorithm is*

$$(5) \quad P(X_n > r) = \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}.$$

The proof of Lemma 6.1 utilizes the q -binomial theorem; see Section 7.

Lemma 6.2. *The distribution of X_n has the form*

$$(6) \quad P(X_n > r) = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} \varrho_r(k),$$

with

$$(7) \quad \varrho_r(z) = q^{z-1} \frac{(q)_{z-1}}{(\mu q)_z} p^{r+1} (-1)^r \Psi_r(z),$$

and

$$\begin{aligned}\Psi_r(z) &= \frac{(\mu q)_\infty}{(q)_\infty} \left(\frac{q}{(-p)^{r+1}} \left(1 - \sum_{s=0}^r \frac{z^s}{s!} q^{-z} (-p)^s \right) + \frac{(\mu-1)q}{p} \frac{z^{r+1}}{(r+1)!} \right. \\ &\quad \left. + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{z^s}{s!} \frac{q^{(m-1)(z+r-s)}}{(1-q^{m-1})^{r-s+1}} \right) \right).\end{aligned}$$

6.2. Asymptotics. Now we turn our attention to the asymptotic distribution of the number of survivors in the algorithm as the number n of initial participants grows large. We follow the derivation for the Rice Method discussed in Section 5. As before, $(q)_{z-1} = \frac{(q)_\infty}{(1-q^z)(q^{z+1})_\infty}$, so $\varrho_r(z)$ has a simple pole at each of the locations of the form $z = m + \frac{2l\pi i}{L}$ for $l, m \in \mathbb{Z}$ with $m \leq 0$. Again, by [FS95], we focus on the poles $z = \chi_l$ for $l \in \mathbb{Z}$. Thus

$$P(X_n > r) = \sum_{l \in \mathbb{Z}} \operatorname{Res}_{z=\chi_l} \left[\varrho_r(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] + O(n^{-1}).$$

Similar to the derivation in Section 5, we need the local expansion of $\varrho_r(z)$ and thus $\Psi_r(z)$ around $z = 0$ to two terms, since $\varrho_r(z) \frac{n!}{(z)(z-1)\cdots(z-n)}$ has a double pole at $z = 0$, but only a simple pole at $z = \chi_l$ for $l \neq 0$. As $z \rightarrow 0$,

$$\begin{aligned}\sum_{s=0}^r \frac{z^s}{s!} q^{-z} (-p)^s &\sim 1 - z \sum_{s=1}^r \frac{p^s}{s} + zL, \\ \frac{z^{r+1}}{(r+1)!} &\sim z \frac{(-1)^r}{(r+1)},\end{aligned}$$

and

$$\begin{aligned}\sum_{s=0}^r \frac{z^s}{s!} \frac{q^{(m-1)(z+r-s)}}{(1-q^{m-1})^{r-s+1}} &\sim \frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} \\ &\quad + z \sum_{s=1}^r \frac{(-1)^{s-1}}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} - z \frac{q^{(m-1)r} L(m-1)}{(1-q^{m-1})^{r+1}}.\end{aligned}$$

Thus

$$\begin{aligned}\Psi_r(z) &\sim \frac{(\mu q)_\infty}{(q)_\infty} \left(\frac{q}{(-p)^{r+1}} \left(z \sum_{s=1}^r \frac{p^s}{s} - zL \right) + \frac{(\mu-1)q}{p} z \frac{(-1)^r}{(r+1)} \right. \\ &\quad \left. + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(-z \sum_{s=1}^r \frac{(-1)^{s-1}}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + z \frac{q^{(m-1)r} L(m-1)}{(1-q^{m-1})^{r+1}} \right) \right).\end{aligned}$$

More simply, as $z \rightarrow 0$,

$$\begin{aligned}\Psi_r(z) &\sim z \frac{(\mu q)_\infty}{(q)_\infty} \left[\frac{q(-1)^r}{p^{r+1}} \left(L - \sum_{s=1}^r \frac{p^s}{s} \right) - \frac{(1-\mu)q}{p} \frac{(-1)^r}{(r+1)} \right. \\ &\quad \left. + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r} (m-1)L}{(1-q^{m-1})^{r+1}} \right) \right].\end{aligned}$$

As before, notice the absence of the constant term. Substitution into (7) yields

$$\begin{aligned} \varrho_r(z) &\sim q^{z-1} \frac{(q)_{z-1}}{(\mu q)_z} p^{r+1} (-1)^r \\ &\times z \frac{(\mu q)_\infty}{(q)_\infty} \left[\frac{q(-1)^r}{p^{r+1}} \left(L - \sum_{s=1}^r \frac{p^s}{s} \right) - \frac{(1-\mu)q}{p} \frac{(-1)^r}{(r+1)} \right. \\ &\quad \left. + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}} \right) \right] \end{aligned}$$

as $z \rightarrow 0$. Also $z(q)_{z-1} \sim 1/L$ and $(\mu q)_z \sim 1$, so

$$\begin{aligned} \varrho_r(z) &\sim \frac{(\mu q)_\infty}{L(q)_\infty} \left[L - \sum_{s=1}^r \frac{p^s}{s} - \frac{\nu p^r}{r+1} \right. \\ &\quad \left. - Q(-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}} \right) \right] \end{aligned}$$

as $z \rightarrow 0$. Also $\frac{n!(-1)^n}{(z-1)\cdots(z-n)} \sim 1$ as $z \rightarrow 0$. Therefore

$$\begin{aligned} \operatorname{Res}_{z=0} \left[\varrho_r(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] &= \lim_{z \rightarrow 0} \varrho_r(z) \\ &= \frac{(\mu q)_\infty}{L(q)_\infty} \left[L - \sum_{s=1}^r \frac{p^s}{s} - \frac{\nu p^r}{r+1} - Q(-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \right. \\ &\quad \left. \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}} \right) \right]. \end{aligned}$$

So

$$\begin{aligned} P(X_n > r) &= \frac{(\mu q)_\infty}{L(q)_\infty} \left[L - \sum_{s=1}^r \frac{p^s}{s} - \frac{\nu p^r}{r+1} - Q(-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \right. \\ &\quad \left. \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r}(m-1)L}{(1-q^{m-1})^{r+1}} \right) \right] + \sum_{l \neq 0} \tilde{\Phi}_{r,l}(n) + O(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} \tilde{\Phi}_{r,l}(n) &= \operatorname{Res}_{z=\chi_l} \left[\varrho_r(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] \\ &= \operatorname{Res}_{z=\chi_l} \left[(q)_{z-1} \frac{q^{\chi_l-1} p^{r+1}}{(\mu q)_{\chi_l}} (-1)^r \Psi_r(\chi_l) \frac{n!(-1)^n}{(\chi_l)(\chi_l-1)\cdots(\chi_l-n)} \right] \\ &= \frac{Q}{L} \frac{(\mu q)_\infty}{(q)_\infty} \left(-q \sum_{s=1}^r \frac{\chi_l^s}{s!} (-p)^s + (-p)^r \nu q \frac{\chi_l^{r+1}}{(r+1)!} \right. \\ &\quad \left. + (-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{\chi_l^s}{s!} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} \right) \right) \\ &\quad \times \Gamma(-\chi_l) e^{2l\pi i \log_Q n} (1 + O(n^{-1})). \end{aligned}$$

Note that $e^{2l\pi i \log_Q n}$ is fluctuating, with $|e^{2l\pi i \log_Q n}| = 1$. This completes the proof of Theorem 4.2

7. PROOFS

Proof. (Proof of Lemma 5.1.)

When starting with n participants, if all n participants are simultaneously eliminated by coin flipping, then these n participants are selected by the algorithm; this corresponds to the term $\binom{n}{0}q^0p^n u^n v$ in recurrence (8) below. If exactly j participants obtain heads, with $1 \leq j \leq n$, then the demon arrives and removes one additional participant with probability ν , or leaves the j remaining participants alone with probability μ . This phenomenon corresponds to $v \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} (\nu F_{j-1}(u, v) + \mu F_j(u, v))$ in formula (8). (Note that, since $F_0(u, v) = v$, the recurrence below also holds when $n = 0$.) So the recurrence

$$(8) \quad F_n(u, v) = \binom{n}{0} q^0 p^n u^n v + v \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} (\nu F_{j-1}(u, v) + \mu F_j(u, v))$$

holds for all integers $n \geq 0$. More simply,

$$(9) \quad F_n(u, v) = v \left(p^n u^n + \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \right).$$

Next we define the exponential generating function

$$G(z, u, v) := \sum_{n=0}^{\infty} F_n(u, v) \frac{z^n}{n!}.$$

From the recurrence in (9), it follows that

$$\begin{aligned} G(z, u, v) &= v \sum_{n=0}^{\infty} \left(p^n u^n + \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \right) \frac{z^n}{n!} \\ &= v \left(e^{puz} + \sum_{j=1}^{\infty} q^j z^j (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \sum_{n=j}^{\infty} \binom{n}{j} p^{n-j} \frac{z^{n-j}}{n!} \right) \\ &= v \left(e^{puz} + e^{pz} \sum_{j=1}^{\infty} \frac{(qz)^j}{j!} (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \right) \\ (10) \quad &= v \left(e^{puz} + e^{pz} \left(\nu q \int G(qz, u, v) dz + \mu G(qz, u, v) - \mu v \right) \right). \end{aligned}$$

The generating function $G(z, u, v)$ becomes simpler if we replace the fixed number n of people present at the start of the algorithm by a Poisson number of participants with mean z . For this reason, we replace $G(z, u, v)$ by the Poissonized exponential generating function,

$$D(z, u, v) := G(z, u, v) e^{-z} = \sum_{n=0}^{\infty} D_n(u, v) \frac{z^n}{n!}.$$

From (10), it follows that

$$(11) \quad D(z, u, v) = v e^{(pu-1)z} + v e^{-qz} \left(\nu q \int G(qz, u, v) dz + \mu G(qz, u, v) - \mu v \right).$$

We use a succinct notation for differentiation with respect to the first of three variables:

$$D'(z, u, v) := \frac{d}{dz} D(z, u, v)$$

and

$$G'(z, u, v) := \frac{d}{dz} G(z, u, v).$$

Differentiating both sides of (11) with respect to z yields

$$\begin{aligned} D'(z, u, v) &= (pu - 1)ve^{(pu-1)z} - vqe^{-qz} \left(\nu q \int G(qz, u, v) dz + \mu G(qz, u, v) - \mu v \right) \\ &\quad + ve^{-qz} (\nu q G(qz, u, v) + \mu q G'(qz, u, v)). \end{aligned}$$

It follows that

$$D'(z, u, v) + qD(z, u, v) = \mu qv D'(qz, u, v) + qv D(qz, u, v) + e^{(pu-1)z} pv(u-1).$$

For $n \geq 1$, extracting the coefficient of $\frac{z^{n-1}}{(n-1)!}$ from $D(z, u, v) = \sum_{n=0}^{\infty} D_n(u, v) \frac{z^n}{n!}$ yields

$$D_n(u, v) + qD_{n-1}(u, v) = \mu qv D_n(u, v) q^{n-1} + qv D_{n-1}(u, v) q^{n-1} + (pu-1)^{n-1} pv(u-1),$$

or equivalently,

$$D_n(u, v) = D_{n-1}(u, v) \frac{vq^n - q}{1 - \mu vq^n} + \frac{(pu-1)^{n-1} pv(u-1)}{1 - \mu vq^n}.$$

Iterating this recurrence yields

$$\begin{aligned} D_n(u, v) &= v(-q)^n \frac{(v)_n}{(\mu vq)_n} + \sum_{j=0}^{n-1} \frac{(pu-1)^j pv(u-1) \prod_{k=j+2}^n (vq^k - q)}{\prod_{l=j+1}^n (1 - \mu vq^l)} \\ &= v(-q)^n \frac{(v)_n}{(\mu vq)_n} + (-q)^{n-1} \frac{(vq)_{n-1}}{(\mu qv)_n} \sum_{j=0}^{n-1} \frac{(1-pu)^j pv(u-1)(\mu qv)_j}{q^j (vq)_j}. \end{aligned}$$

Note that $F_n(u, v) = \sum_{k=0}^n \binom{n}{k} D_k(u, v)$, so Lemma 5.1 follows. \square

Proof. (Proof of Lemma 5.3.) Setting $v = 1$ in Lemma 5.1, it follows that

$$\begin{aligned} F_n(u) &= F_n(u, 1) \\ &= \sum_{k=0}^n \binom{n}{k} \left((-q)^k \frac{(1)_k}{(\mu q)_k} + (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} \sum_{j=0}^{k-1} \frac{(1-pu)^j p(u-1)(\mu q)_j}{q^j (q)_j} \right) \\ (12) \quad &= 1 + \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} \sum_{j=0}^{k-1} \frac{(1-pu)^j p(u-1)(\mu q)_j}{q^j (q)_j}. \end{aligned}$$

It follows that, for $s \geq 1$,

$$(13) \quad F_n^{(s)}(1) = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} q^k \frac{(q)_{k-1}}{(\mu q)_k} s(Qp)^s (-1)^{s-1} \sum_{j=0}^{k-1} \frac{(\mu q)_j}{(q)_j} j^{s-1}.$$

Dissecting the summation over j in (13), we obtain

$$\begin{aligned}
\sum_{j=0}^{k-1} \frac{(\mu q)_j}{(q)_j} j^{s-1} &= \frac{(\mu q)_\infty}{(q)_\infty} \sum_{j=0}^{k-1} j^{s-1} + \sum_{j=0}^{k-1} \left[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right] j^{s-1} \\
&= \frac{(\mu q)_\infty}{(q)_\infty} \frac{k^s}{s} + \sum_{j \geq 0} \left[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right] j^{s-1} - \sum_{j \geq k} \left[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right] j^{s-1} \\
&= \frac{(\mu q)_\infty}{(q)_\infty} \frac{k^s}{s} + \sum_{j \geq 0} \left[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right] j^{s-1} - \sum_{j \geq 0} \left[\frac{(\mu q)_{j+k}}{(q)_{j+k}} - \frac{(\mu q)_\infty}{(q)_\infty} \right] (j+k)^{s-1} \\
&= \frac{(\mu q)_\infty}{(q)_\infty} \frac{k^s}{s} + \sum_{j \geq 0} \left[\left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) j^{s-1} - \left(\frac{(\mu q)_{j+k}}{(q)_{j+k}} - \frac{(\mu q)_\infty}{(q)_\infty} \right) (j+k)^{s-1} \right].
\end{aligned}$$

Finally, we observe that, since $F_n(u) = \sum_{m=0}^{\infty} \pi(n, m) u^m$, then $F_n^{(s)}(1) = \sum_{m=0}^{\infty} m^s \pi(n, m) = E[X_n^s]$. Thus $E[X_n^s]$ has the representation given in the statement of Lemma 5.3. \square

Proof. (Proof of Lemma 6.1.) First of all,

$$P(X_n > r) = \sum_{m>r} \pi(n, m) = 1 - \sum_{m=0}^r \pi(n, m).$$

Note that $\sum_{m=0}^r \pi(n, m) = [u^r] \frac{F_n(u)}{1-u}$, and of course $1 = [u^r] \frac{1}{1-u}$, so

$$\begin{aligned}
P(X_n > r) &= [u^r] \frac{1 - F_n(u)}{1 - u} \\
&= [u^r] \frac{F_n(u) - 1}{u - 1} \\
&= [u^r] \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} \sum_{j=0}^{k-1} \frac{(1-pu)^j p (\mu q)_j}{q^j (q)_j} \quad \text{by equation (12)} \\
(14) \quad &= \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \sum_{j=0}^{k-1} \binom{j}{r} \frac{(\mu q)_j}{q^j (q)_j}.
\end{aligned}$$

We focus on the second summation in (14). Recall that $(x)_z := (x)_\infty / (xq^z)_\infty$, so

$$(15) \quad \frac{(\mu q)_j}{(q)_j} = \frac{(\mu q)_\infty}{(q)_\infty} \frac{(q^{j+1})_\infty}{(\mu q^{j+1})_\infty}.$$

Also, the q -binomial theorem states $\frac{(az)_\infty}{(z)_\infty} = \sum_{m \geq 0} \frac{(a)_m}{(q)_m} z^m$. Specifying $z = \mu q^{j+1}$ and $a = 1/\mu$,

$$(16) \quad \frac{(q^{j+1})_\infty}{(\mu q^{j+1})_\infty} = \sum_{m \geq 0} \frac{(1/\mu)_m}{(q)_m} (\mu q^{j+1})^m.$$

Combining (15) and (16) yields

$$\begin{aligned}
\sum_{j=0}^{k-1} \binom{j}{r} \frac{(\mu q)_j}{q^j (q)_j} &= \sum_{j=0}^{k-1} \binom{j}{r} q^{-j} \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m}{(q)_m} (\mu q^{j+1})^m \\
(17) \qquad \qquad \qquad &= \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}.
\end{aligned}$$

Thus

$$P(X_n > r) = \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1},$$

as claimed in the Lemma. \square

Proof. (Proof of Lemma 6.2.) Let $r \geq 0$. The probability that strictly more than r out of n initial participants are selected at the end of the algorithm is

(18)

$$P(X_n > r) = \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}.$$

We handle the sum over m in Lemma 6.1 in three parts, $m = 0$, $m = 1$, or $m \geq 2$, as follows

$$\begin{aligned}
&\sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1} \\
&= \sum_{j=0}^{k-1} \binom{j}{r} q^{-j} + \frac{(1-1/\mu)(\mu q)}{(1-q)} \sum_{j=0}^{k-1} \binom{j}{r} + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1} \\
(19) \qquad \qquad \qquad &= \sum_{j=0}^{k-1} \binom{j}{r} q^{-j} + \frac{(\mu-1)q}{p} \frac{k^{r+1}}{(r+1)!} + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}.
\end{aligned}$$

Now we focus attention on the sums of the form $\sum_{j=0}^{k-1} \binom{j}{r} x^j$. Writing $D = \frac{d}{dx}$, we note

$$(20) \quad \sum_{j=0}^{k-1} \binom{j}{r} x^j = \frac{x^r}{r!} \sum_{j=0}^{k-1} j^r x^{j-r} = \frac{x^r}{r!} \sum_{j=0}^{k-1} D^r x^j = \frac{x^r}{r!} D^r \sum_{j=0}^{k-1} x^j = \frac{x^r}{r!} D^r \frac{1-x^k}{1-x}.$$

The remainder of the analysis does not depend on k being an integer. We have

$$\begin{aligned}
\frac{x^r}{r!} D^r \frac{1-x^k}{1-x} &= \frac{x^r}{r!} \sum_{s=0}^r \binom{r}{s} D^s (1-x^k) \cdot D^{r-s} \left(\frac{1}{1-x} \right) \\
&= \frac{x^r}{r!} (1-x^k) \cdot \frac{r!}{(1-x)^{r+1}} - \frac{x^r}{r!} \sum_{s=1}^r \binom{r}{s} k^s x^{k-s} \cdot \frac{(r-s)!}{(1-x)^{r-s+1}} \\
&= \frac{x^r}{(1-x)^{r+1}} - \frac{x^r x^k}{(1-x)^{r+1}} - \sum_{s=1}^r \frac{k^s}{s!} \frac{x^{k+r-s}}{(1-x)^{r-s+1}} \\
(21) \qquad \qquad \qquad &= \frac{x^r}{(1-x)^{r+1}} - \sum_{s=0}^r \frac{k^s}{s!} \frac{x^{k+r-s}}{(1-x)^{r-s+1}}.
\end{aligned}$$

Thus, combining (20) and (21) with $x = q^{m-1}$, we can simplify the “ $m \geq 2$ ” term of (19), as follows:

$$(22) \quad \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1} = \frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{k^s}{s!} \frac{q^{(m-1)(k+r-s)}}{(1-q^{m-1})^{r-s+1}}.$$

For $m = 0$, the analogous equation is

$$(23) \quad \begin{aligned} \sum_{j=0}^{k-1} \binom{j}{r} q^{-1} &= \frac{q^{-r}}{(1-q^{-1})^{r+1}} - \sum_{s=0}^r \frac{k^s}{s!} \frac{q^{-(k+r-s)}}{(1-q^{-1})^{r-s+1}} \\ &= \frac{q}{(-p)^{r+1}} \left(1 - \sum_{s=0}^r \frac{k^s}{s!} q^{-k} (-p)^s \right). \end{aligned}$$

Plugging the results from (22) and (23) into (19), we get

$$\begin{aligned} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1} &= \frac{q}{(-p)^{r+1}} \left(1 - \sum_{s=0}^r \frac{k^s}{s!} q^{-k} (-p)^s \right) + \frac{(\mu-1)q}{p} \frac{k^{r+1}}{(r+1)!} \\ &\quad + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} \\ &\quad - \sum_{s=0}^r \frac{k^s}{s!} \frac{q^{(m-1)(k+r-s)}}{(1-q^{m-1})^{r-s+1}}. \end{aligned}$$

Finally, a substitution into the form of $P(X_n > r)$ in Lemma 6.1 yields Lemma 6.2. \square

8. FUTURE PROBLEMS

A key problem for future analysis involves a more robust demon, who might be able to remove more than one participant at a time. Another problem to be studied in the future might involve replacing the 2-outcome coins (heads versus tails) with a coin that itself involves some uncertainty. Another interpretation of this extension is that the parameters p and q are unknown before the coin is flipped. Many other possibilities exist for generalizing the present algorithm.

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