Recent studies on the Dice Race Problem and its connections

Guy Louchard *

March 13, 2016

Dedicated to F. Thomas Bruss on the occasion of his retirement from the Université Libre de Bruxelles as full professor and chair of Service Mathématiques Générales.

Abstract

The following type of dice games has been mentioned and/or studied in the literature. Players take turns in rolling a fair die successively, each player accumulating his or her scores as long as the outcome 1 does not occur. If the result 1 turns up, the accumulated score is wiped out, and the turn ends, that is the player gives the die to the next player. At any stage after a roll, the player (she, say) can choose to end her turn and bank her accumulated score. The winner is the first player to reach some fixed target $n \in \mathbb{N}$. We present some new results on optimal strategies and winning probability in a one or two players game. For just one player there is no competition of course, and in this case we suppose that the player simply wants to minimize her total expected number of tosses over all possible banking strategies.

Keywords: Dice game, Optimal strategies, One or two players game, Number of rounds, Winning probability.

2010 Mathematics Subject Classification: 60G40, 62L15, 91A05, 91A60.

1 Introduction

Players take turns in rolling a fair die successively, each player accumulating his or her scores as long as the outcome 1 does not occur. If the result 1 turns up, the accumulated score is wiped out, and the turn ends, that is the player gives the die to the next player. At any stage after a roll, the player (she, say) can choose to end her turn and bank her accumulated score. The winner is the first player to reach some fixed target $n \in \mathbb{N}$. We present some new results on optimal strategies and winning probability in a one or two players game. For just one player without competition we suppose that she wants to minimise her total expected number of tosses to reach n.

Related papers in the literature are: Roters, [15], Haigh and Roters, [9], Neller and Presser [11], Neller and Presser [12], Neller et al. [13], Crocce and Mordecki [5], where this list may not be complete. In this paper, we present some new results and open problems on optimal strategies and winning probability in a one or two players game.

The paper is organised as follows: in Sec.2, we consider the one round problem, where a round is a sequence of tosses which is ended either if the player decides to bank (and declares the end of the current round) or else if the result 1 occurs.

Sec.3 is devoted to the mean number of rounds with target n according to the strategy she applies. In Sec.4 we analyse the variance of the corresponding number of rounds with optimal strategy and target n. Then we consider the total number of runs and total number of rolls to reach the target n

*Université Libre de Bruxelles, Département d'Informatique, CP 212, Boulevard du Triomphe, B-1050 Bruxelles, Belgium, `louchard@ulb.ac.be`

in Sec.5. I the next section, Sec.6, the asymptotic winning probability for the first player is studied, still without taking the adversary's situation into account.

Sec.7 then looks at how the adversary's situation must be taken into account.

Finally, a certain strategy due to Haigh and Roters is considered in Sec.8 and in Sec.9 we present Ferguson's look (private communication) at the problem as game with some payoff.

2 One round

We consider here first the single round optimisation. A round is a sequence of dice rolling by one player ended by a decision to bank or by a forced stop. What strategy maximizes the expected payoff of a round? Two strategies (or more?) are possible in order to get a maximum mean sum. Recall that the occurrence of a 1 leads to 0 gain.

2.1 Strategy 1 :Threshold Strategy

The player rolls the die until getting at least the sum k, say, if possible. Let J_k denote the corresponding random payoff and let $P_J(k,j)$ be the probability that the player obtains the gain $j \in [0, k, k+1, \ldots, k+5]$ with the plan to reach at least k. Then

$$P_J(k,0) = \frac{1}{6} + \frac{1}{6}\sum_{i=2}^{6} P_J(k-i,0),$$

$$P_J(k,k+\delta) = \sum_{i=2}^{6} \frac{1}{6} P_J(k-i,k-i+\delta), \quad \delta = 0,\ldots,5,$$

with the obvious initial conditions

$$P_J(-k,.) = 0, \quad k \geq 0,$$
$$P_J(.,1) = 0.$$

More precisely,

$$P_J(k,j) = \frac{1}{6} + \frac{1}{6}\sum_{i=2}^{k-1} P_J(k-i,j-i), \quad k=1, \quad j=2,\ldots,6 \quad \text{or} \quad k=2,\ldots,6, \quad j=k,\ldots,6,$$

$$P_J(k,k+\delta) = \sum_{i=2}^{6} \frac{1}{6} P_J(k-i,k-i+\delta), \quad k=1,\ldots,6, \quad 7-k \leq \delta \leq 5,$$

$$P_J(k,k+\delta) = \sum_{i=2}^{6} \frac{1}{6} P_J(k-i,k-i+\delta), \quad k > 6, 0 \leq \delta \leq 5.$$

The probability of getting k (or more) is given by $1 - P_J(k,0)$. The mean gain becomes

$$\bar{J}(k) = P_J(k,0).0 + \sum_{j=k,\ldots,k+5} P_J(k,j).j,$$

which is maximum for $k = 20$ or $k = 21$, and

$$m_J := \bar{J}(20) = \mathbb{E}(J_{20}) = \frac{492303203}{60466176} = 8.141794894\ldots$$

This value has also been obtained by Roters in [15]. Note that the two probability distributions $P_J(20,j), P_J(21,j)$ are different. This leads to the two variances $V_J(20) = 111.0712987\ldots$ and $V_J(21) = 119.2145260\ldots$.

2.2 Strategy 1: Total number of rolls during a round

Let R be the total number of rolls until k is reached or exceeded. We now compute the mean of R , $\bar{R}(k)$, during a round for Strategy 1 with threshold k. We have

$$\begin{aligned}
\bar{R}(1) &= 1, \\
\bar{R}(k) &= \frac{1}{6} + \sum_{i=2}^{6} \frac{1}{6}(1 + \bar{R}(k-i)), \\
\bar{R}(k) &= 0, k \leq 0, \text{ by convention.}
\end{aligned}$$

$\bar{R}(k)$ increases from 1 and converges as $k \to \infty$ to $\bar{R} = 6$, which is the fixed part solution of

$$\bar{R} = \frac{1}{6} + \frac{5}{6}(1 + \bar{R}).$$

Of course, $\bar{R}$ corresponds to the time until a 1 appears. Also

$$\begin{aligned}
\bar{R}(20) &= 3.747245007\ldots, \\
\bar{R}(21) &= 3.846957993\ldots.
\end{aligned}$$

Let $P_R(k, \ell)$ be the probability that, for given target number k, we have $R = \ell$. This yields the recurrence equations

$$\begin{aligned}
P_R(-k, .) &= 0, \quad k \geq 0, \\
P_R(1, 1) &= 1, \\
P_R(k, 1) &= \frac{1}{6}(8 - k), 2 \leq k \leq 6, \\
P_R(k, 1) &= \frac{1}{6}, k > 6, \\
P_R(k, \ell) &= \sum_{i=2}^{6} \frac{1}{6} P_R(k - i, \ell - 1), k > 6, \ell > 1.
\end{aligned}$$

from which we obtain for $k = 20$ the variance $V_R(20) = 3.25139253$.

2.3 Strategy 1: gain and number of tosses during a round

It will be useful to obtain the joint distribution $P_{JR}(k, j, \ell)$ of J and R. This is given by the following recurrences

$$
\begin{aligned}
P_{JR}(-k, ., .) &= 0, \quad k \geq 0, \\
P_{JR}(k, 1, \ell) &= 0, \\
P_{JR}(k, 0, 1) &= \frac{1}{6}, \\
P_{JR}(k, 0, \ell) &= \sum_{i=2}^{6} \frac{1}{6} P_{JR}(k-i, 0, \ell-1), \ell > 1, \\
P_{JR}(1, j, 1) &= \frac{1}{6}, j = 2..6 \\
P_{JR}(k, j, 1) &= \frac{1}{6}, k = 2..6, j = k..6 \\
P_{JR}(k, j, \ell) &= \sum_{i=2}^{k-1} \frac{1}{6} P_{JR}(k-i, j-i, \ell-1), k = 2..6, j = k..6, \ell > 1, \\
P_{JR}(k, k+\delta, \ell) &= \sum_{i=2}^{k-1} \frac{1}{6} P_{JR}(k-i, k-i+\delta, \ell-1), k = 1..6, 7-k \leq \delta \leq 5, \ell > 1, \\
P_{JR}(k, k+\delta, \ell) &= \sum_{i=2}^{6} \frac{1}{6} P_{JR}(k-i, k-i+\delta, \ell-1), k > 6, 0 \leq \delta \leq 5, \ell > 1,
\end{aligned}
$$

which allows us now to compute the correlation between J and R, namely $C(20) = .6764271127$.

2.4 Strategy 2

We now look at an alternative strategy, Strategy 2, which lets the player play exactly ℓ steps in a round (if possible). Using the notation $\tilde{P}(\ell, u)$ we get

$$
\begin{aligned}
\tilde{P}(\ell, 0) &= \frac{1}{6} + \frac{1}{6} \sum_{i=2}^{6} \tilde{P}(\ell-1, 0), \\
\tilde{P}(\ell, u) &= \frac{1}{6} \sum_{i=2}^{6} \tilde{P}(\ell-1, u-i), \quad u = 2\ell, \ldots, 6\ell,
\end{aligned}
$$

with suitable initial conditions. The mean gain is now given by

$$
\tilde{J}(l) = \tilde{P}(\ell, 0).0 + \sum_{u} \tilde{P}(\ell, u).u,
$$

which is maximum for $\ell = 5$ or 6, and $\tilde{J}(5) = \frac{15625}{1944} = 8.037551440\ldots$ So Strategy 1 is better. We had expected that this would be the case because Strategy 2 is not history-dependent whereas the forced-stop loss imposes history-dependence by definition. However it is interesting to see that the difference is not large.

3 Target n, the mean number of rounds

We now have a target n and our objective is to reach it with a minimum number of rounds. For large n, it seems intuitive that we should use Strategy 1 with $k = 20$ or $k = 21$. But, as we shall see, when we approach the target, the strategy must be adapted. Let $C(n, k)$ be the number of rounds necessary to reach at least n, starting the first round with Strategy 1 and round-target k. With $\mathbb{E}(n,k) := \mathbb{E}[C(n,k)]$ we obtain

$$E(n,k) = 1 + \sum_{i \in [0,k,\ldots,k+5]} P(k,i)\bar{E}(n-i), \tag{1}$$

where $\bar{E}(n)$ is the *optimal* expected number of rounds necessary to reach at least n. This implies

$$\bar{E}(n) = \min_k E(n,k),$$

as well as the existence, for each n, of an optimal value for k, $\bar{k}(n)$ say. We must solve the recurrence (1) for each n in some range. To be more precise, we must solve

$$E(n,k) = \sum_{j \geq n} P(k,j).1 + P(k,0)(1+E(n,k)) + \sum_{2 \leq j < n} P(k,j)(1+\bar{E}(n-j))$$

for $E(n,k)$ and take the minimum on k. The first values of $\bar{k}(n)$ for $n = 1, \ldots, 35$ obtained by this are

$$[1,1,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,$$
$$22,23,24,25,26,27,28,29,14,15,15,16,16,17]$$

Let us mention that, apart from $n = 1, 2$, the values of $\bar{k}(n)$ are unique. Note that, for $k \leq 29$, the player tries one round, for $k > 29$, he tries more then one round. Note also that this argument is valid since there is only one player with target n. In a competitive environment the situation is, as we shall see, more complicated.

We see from Figure 1 (with $n = 1, \ldots, 1000$, as in all our next figures) that, after some time, there is a periodic stabilisation of $\bar{k}$ between 20 and 21, with period $\mathcal{P}(20) = 12$ for 20 and $\mathcal{P}(21) = 10$ for 21. However, the proof of this is not evident. The phenomenon is related with a feature of "non-convergence" encountered in a problem solved by Bruss and O'Cinneide [4] and this is worth as being stated as the first

OPEN PROBLEM 1 : Prove that $\bar{k}$ *stabilises periodically* between 20 and 21, with period $\mathcal{P}(20) = 12$ for 20 and $\mathcal{P}(21) = 10$ for 21.

We should add saying that the *general* problem of a unique maximum for i.i.d. rv's has been elegantly solved by a different approach in Bruss and Grübel [3]. See also Louchard and Prodinger [10], Brands et al. [1].

It is worth to draw here attention to the interesting fact that stopping games involving competition have a tendancy to favor difficulties stemming from problems of convergence or non-convergence (with narrow fluctuations), as examplified in particular in Bruss, Drmota and Louchard [2] (see in particular pages 419-437 of this paper) which completed open questions in an interesting game studied in a series of papers by Enns and Ferenstein [6], [7],[8].

A related interesting non-zero-sum game version of the secretary problem was studied by Szajowski [17] and another related competitive stopping problem for zero-sum games in Szajowski [16]. Concerning the dice race problem it would be interesting to know whether the optimal strategy for two players approaching both the target ends up to solve the problem of selection of a correlated equilibrium in Markov stopping games as studied by Ramsey and Szajowski [14]. Lacking experience in this field, we have not pursued this question in more depth.

In figure 2, we compare $\bar{E}(n)$ with n/m (recall that m is the mean gain with $k = 20$ or 21).

The difference $\bar{E}(n) - n/m$ is given in Figure 3 and Figure 4: there appears some convergence (with fading but persisting oscillations) to some constant K_1 with $0.22275 < K_1 < 0.22295$. The asymptotic period is 22 as expected.

OPEN PROBLEM 2: compute this constant K_1

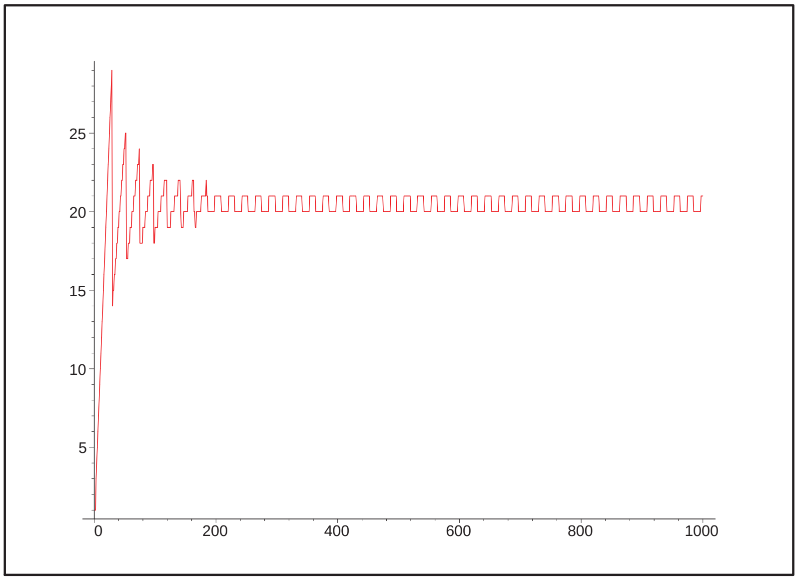

Figure 1: $\bar{k}(n)$

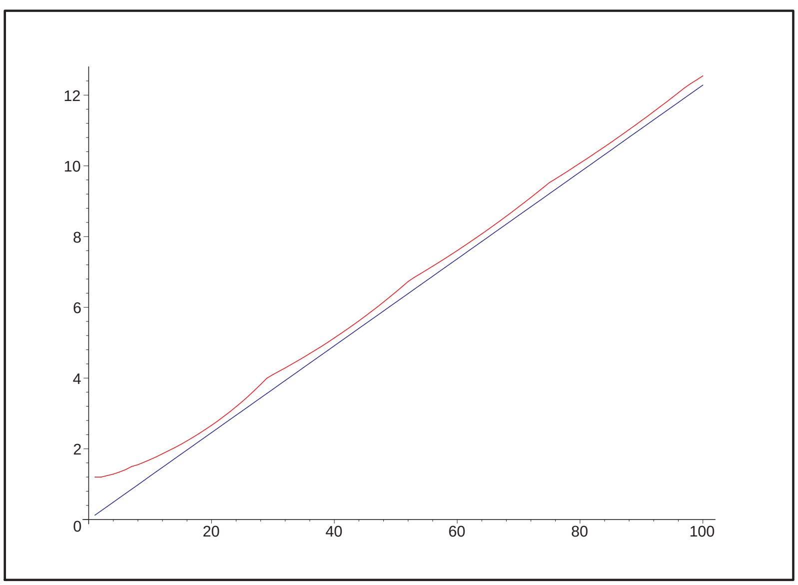

Figure 2: $\bar{E}(n)$ (red), n/m (blue)

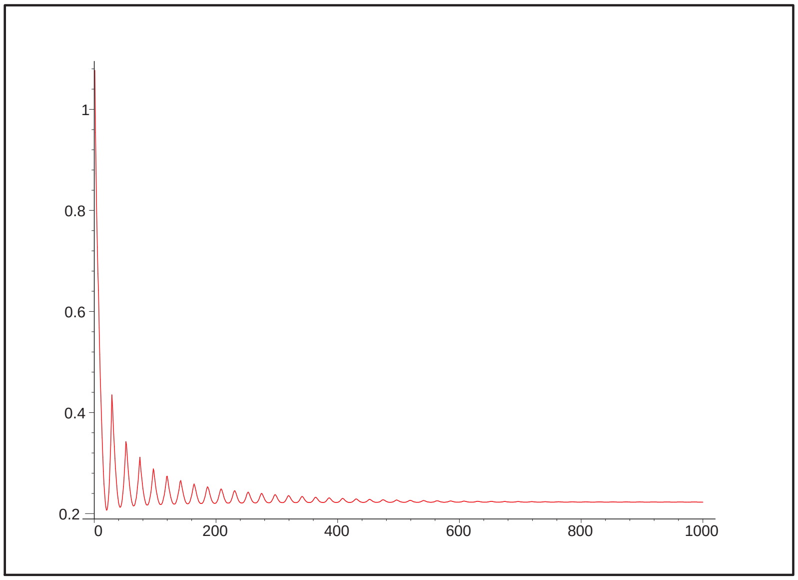

Figure 3: $\bar{E}(n) - n/m$

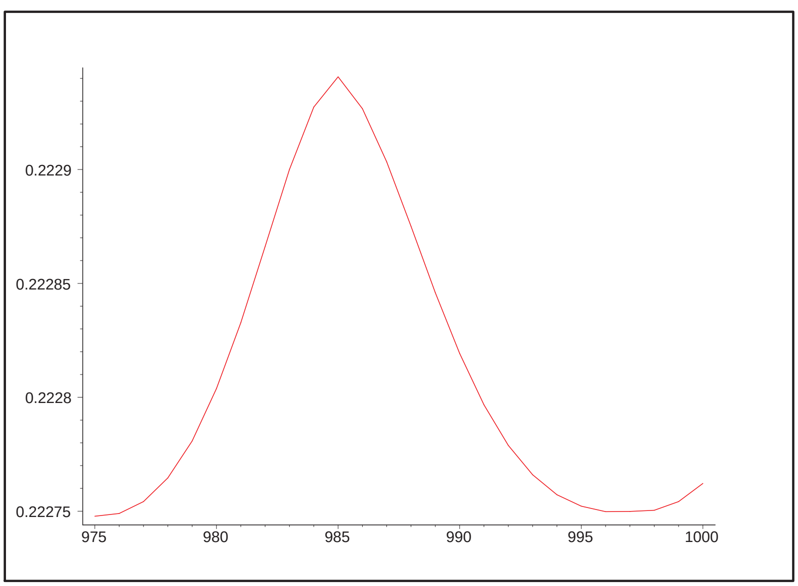

Figure 4: $\bar{E}(n) - n/m$

This constant depends on initial values. To check this, we have used the following Strategy 3: for $n = 1, \ldots, 54$, we use $\bar{k}(n)$ and $\bar{E}(n)$ as computed previously. For $n = 55..1000$, we use $\bar{k}(n) = 20, \bar{k}(n) = 21$, with periods $12, 10$. For $\bar{E}(n)$, the behaviour is similar to Figures 2 and 3. Figure 5 shows $\bar{E}(n) - n/m$ in the neighbourhood of $n = 1000$. This shows again an asymptotic period of 22 with a constant K_2, such that $0.22645 < K_2 < 0.22680$ and $K_2 > K_1$.

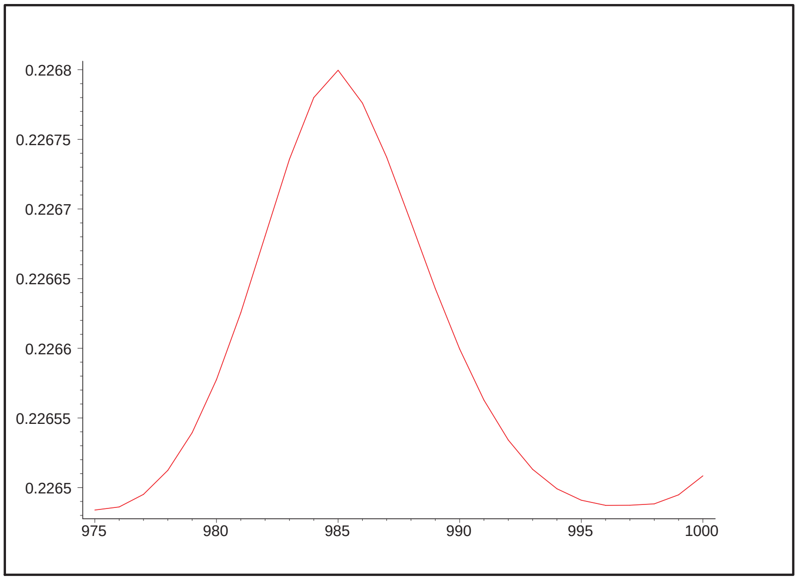

Figure 5: $\bar{E}(n) - n/m$, Strategy 3

4 Target n, the variance of the number of rounds with optimal strategy

4.1 Computation of the variance of $C(n, \bar{k}(n))$

Recall that $C(n, \bar{k}(n))$ is the number of rounds necessary to reach at least n, with the optimal starting value $\bar{k}(n)$. To compute its variance $\mathbb{V}(n)$, set $D(n) := \mathbb{E}[C(n, \bar{k}(n))^2]$. We have

$$D(n) := \sum_{j \geq n} P(\bar{k}(n), j).1 + P(\bar{k}(n), 0)[1 + 2\bar{E}(n) + D(n)] + \sum_{2 \leq j < n} P(\bar{k}(n), j)[1 + 2\bar{E}(n-j) + D(n-j)], \quad (2)$$

and

$$\mathbb{V}(n) := D(n) - \bar{E}(n)^2.$$

To compute the asymptotic variance, we face a problem: if we always use $\bar{k} = 20$, we compute $\tilde{\mathbb{V}}(20) = 111.0712987\ldots$ from $P(20, j)$ and, by the renewal theorem, $C(n) := C(n, \bar{k}(n))$ is asymptotically Gaussian, with mean n/m and variance $n\tilde{\mathbb{V}}(20)/m^3$. But we have asymptotically, $\bar{k} = 20, \bar{k} = 21$, with, alternatively, asymptotic period $12, 10$. So we should be tempted to use

$$s_1^2 := \frac{12\tilde{\mathbb{V}}(20) + 10\tilde{\mathbb{V}}(21)}{22m^3} = 0.2126563592\ldots,$$

with $\tilde{\mathbb{V}}(21) = 119.2145260\ldots$ computed from $P(21, j)$. In Figure 6, we compare $\mathbb{V}(n)$ with ns_1^2.

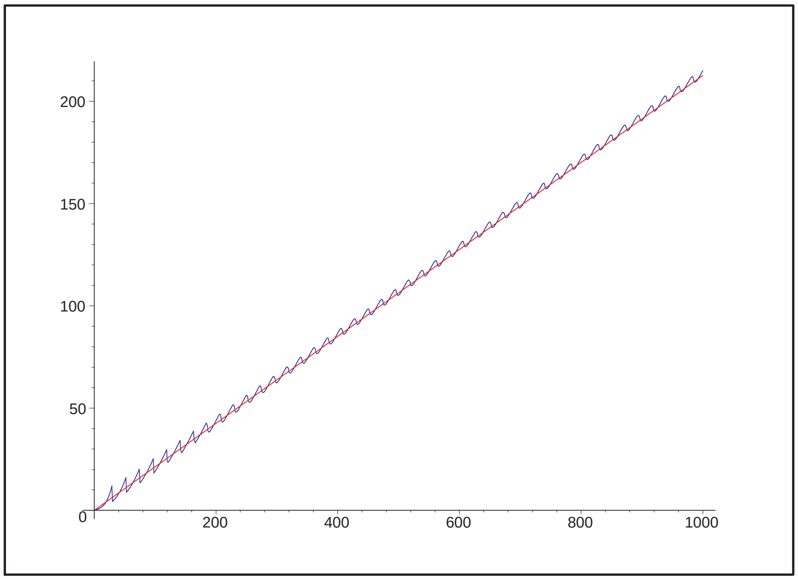

Figure 6: $\mathbb{V}(n)$ (blue), ns_1^2 (red)

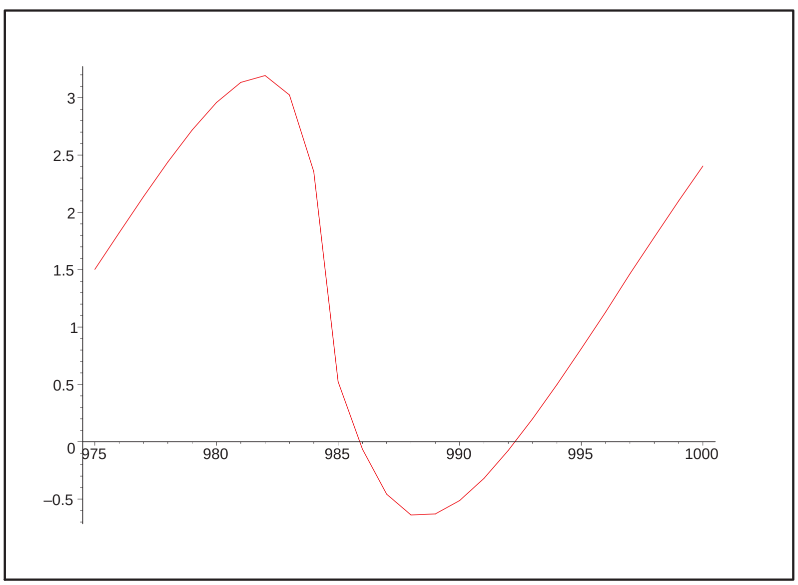

Figure 7: $\mathbb{V}(n) - ns_1^2$

The difference $\mathbb{V}(n) - ns_1^2$ is given in Figure 7: there appears some permanent oscillations with amplitude 2 and again a period 22. But the mean is far from 0.

Curiously enough, if we use the simple mean

$$s_2^2 := \frac{\tilde{\mathbb{V}}(20) + \tilde{\mathbb{V}}(21)}{2m^3} = 0.2133421842\ldots,$$

the fit seems better, as shown in Figures 8 and 9.

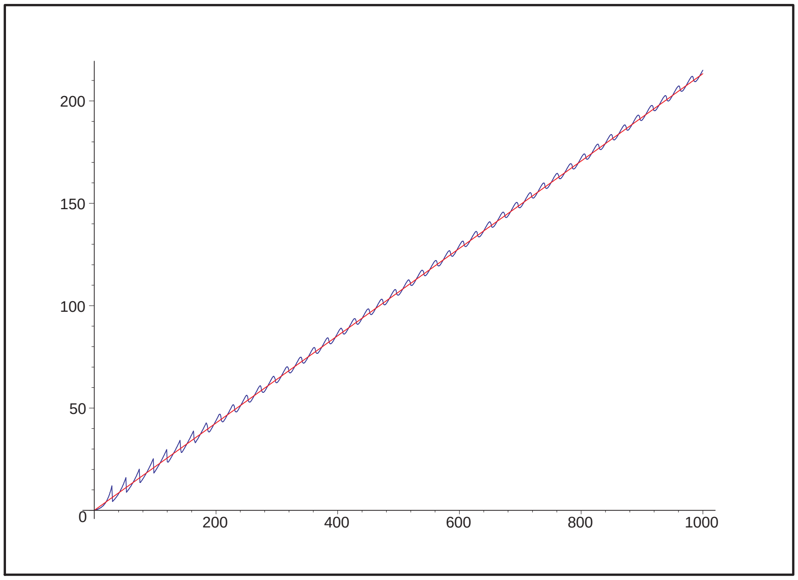

Figure 8: $\mathbb{V}(n)$ (blue), ns_2^2 (red)

4.2 A Markov chain approach

Actually, we can compute the asymptotic variance for n large by analyzing the stationary distribution of the following random walk. We start at position 0, at each round we make a jump of i, $i \in [0, k(\ell), k(\ell)+1, \ldots, k(\ell)+5]$, where ℓ is the distance to the target. We know that k is either 21 or 20. We stop as soon as we reach (or go beyond) the barrier n.

If we assume that, for n large, we are alternatively in a range $k = 21$ (of length 10), denoted by I, or $k = 20$ (of length 12), denoted by II, we can compute the stationary distribution $\pi_I(j)$ of being in position j of range I and similarly for range II. This leads to the following stationary equations for

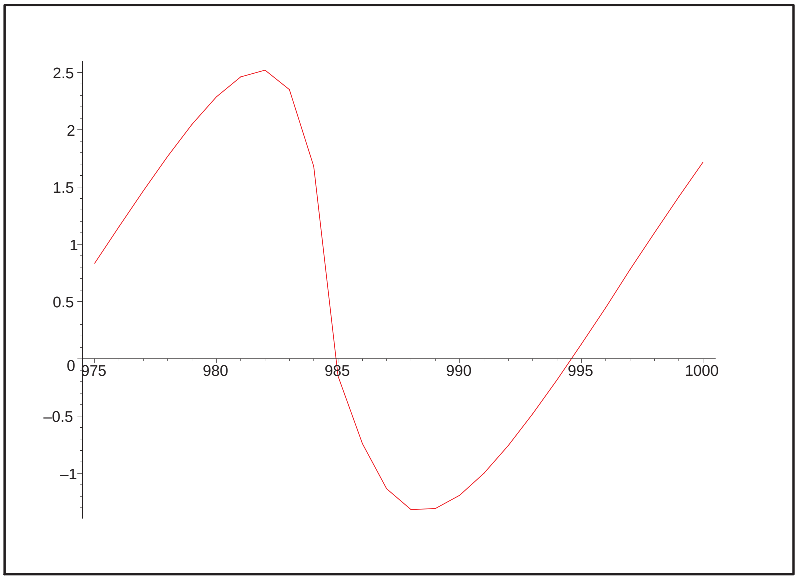

Figure 9: $\mathbb{V}(n) - ns_2^2$

the distributions $\pi_I(j)$ and $\pi_{II}(j)$

$$\begin{aligned}
\pi_I(j) &= \sum_{i=1}^{j} \pi_I(i) P(21, j-i) + \sum_{i=1}^{12} \pi_{II}(i) P(20, j+12-i) \\
&\quad + \sum_{i=1}^{10} \pi_I(i) P(21, j+12+10-i) + \sum_{i=1}^{12} \pi_{II}(i) P(20, j+12+10+12-i), \quad j = 1..10, \\
\pi_{II}(j) &= \sum_{i=1}^{j} \pi_{II}(i) P(20, j-i) + \sum_{i=1}^{10} \pi_I(i) P(21, j+10-i) \\
&\quad + \sum_{i=1}^{12} \pi_{II}(i) P(20, j+10+12-i) + \sum_{i=1}^{10} \pi_I(i) P(21, j+10+12+10-i), \quad j = 1..12 \\
&\quad \sum_{i=1}^{10} \pi_I(i) + \sum_{i=1}^{12} \pi_{II} = 1.
\end{aligned}$$

Note that no more equations are needed.

Solving this system leads to

$$\begin{aligned}
p_I &= \sum_{i=1}^{10} \pi_I(i) = 0.3633246365, \\
p_{II} &= \sum_{i=1}^{12} \pi_{II}(i) = 0.6366753635.
\end{aligned}$$

Now the stationary variance is given by

$$p_I \tilde{\mathbb{V}}(21) + p_{II} \tilde{\mathbb{V}}(20),$$

and this gives

$$s^2 = \frac{p_I \tilde{\mathbb{V}}(21) + p_{II} \tilde{\mathbb{V}}(20)}{m^3} = 0.2112800054.$$

By the renewal theorem for Markov chain, the asymptotic number of rounds to reach n is Gaussian, with mean n/m and variance ns^2. This will be analyzed in Section 6.

4.3 Effect of the alternating sequence

Let us check the effect of the alternating $\bar{k} = 20, \bar{k} = 21$ sequence: we have tried the following Strategy 4: for $n = 1, \ldots, 250$, we use $\bar{k}(n)$ and $\bar{E}(n)$ as computed previously. At the end of this range, the alternating $\bar{k} = 20, \bar{k} = 21$ sequence appears already. For $n = 251, \ldots, 1000$, we always use $\bar{k}(n) = 20$. For $\bar{E}(n)$, the behaviour is similar to Figures 2 and 3. Figure 10 shows $\bar{E}(n) - n/m$ in the neighbourhood of $n = 1000$. An asymptotic period of 22 is still present, due to the alternating behaviour of $\bar{k}(n)$ at the end of the range 1..250. There appears a constant K_3, such that $0.2291 < K_3 < 0.2300$ and $K_1 < K_2 < K_3$.

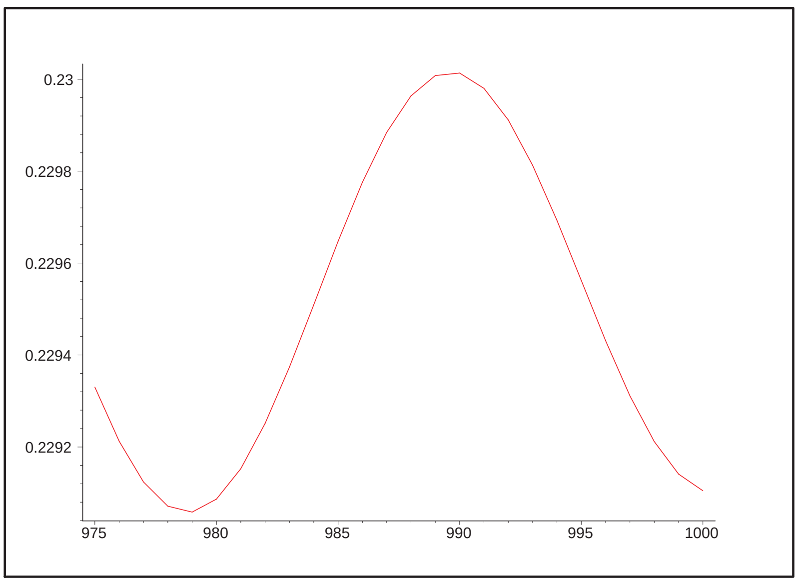

Figure 10: $\bar{E}(n) - n/m$, for Strategy 4

Concerning the variance, the behaviour is different: Figures 11 and 12 give $\mathbb{V}(n) - n\tilde{\mathbb{V}}(20)/m^3$. Now some damping oscillations are apparent: the effect of the alternating behaviour of $\bar{k}(n)$ at the end of the range 1..250 is present later on, but gradually amortized.

4.4 Computation of the periodic part

Let us return to the general case with optimal $\bar{k}(n)$. Actually, we can compute the asymptotic periodic contribution in equilibrium as follows. Assume that, asymptotically, we have

$$\bar{E}(n) = \frac{n}{m} + K, \qquad \text{no oscillations,}$$
$$\mathbb{V}(n) = D(n) - \bar{E}(n)^2 = ns^2 + U(n),$$

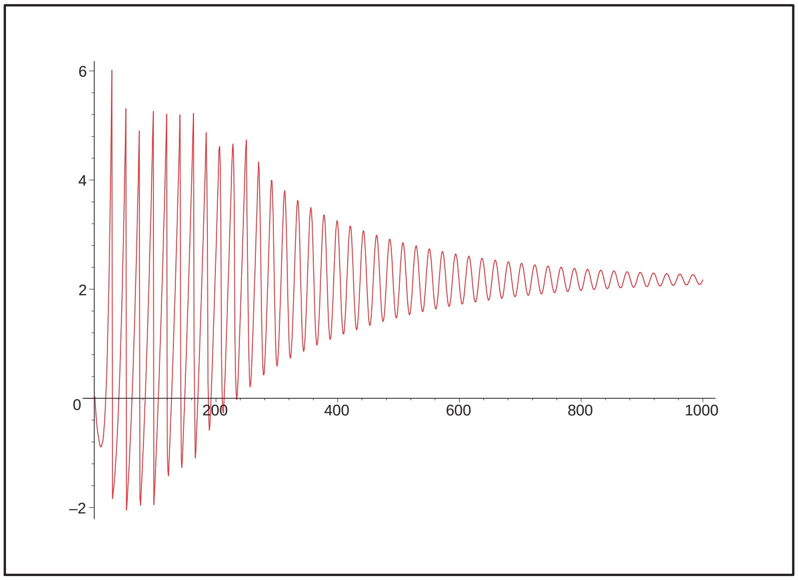

Figure 11: $\mathbb{V}(n) - n\tilde{\mathbb{V}}(20)/m^3$, for Strategy 4

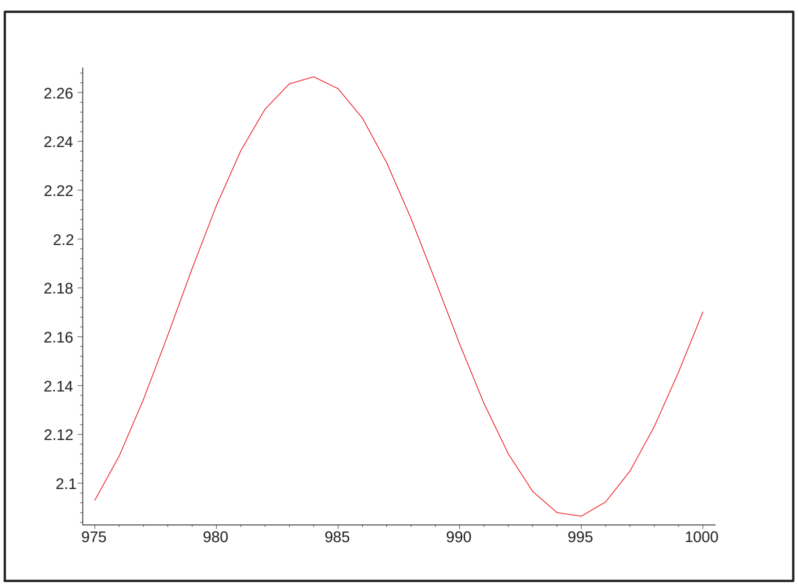

Figure 12: $\mathbb{V}(n) - n\tilde{\mathbb{V}}(20)/m^3$, for Strategy 4

for some constant s^2 and $U(n)$ is periodic with period 22. This gives, from (2),

$$\left(\frac{n}{m}+K\right)^2+ns^2+U(n)=P(\bar{k}(n),0)\left[1+2\left(\frac{n}{m}+K\right)+\left(\frac{n}{m}+K\right)^2+ns^2+U(n)\right]$$
$$+\sum_{j\in[\bar{k}(n),\ldots,\bar{k}(n)+5]}P(\bar{k}(n),j)\left[1+2\left(\frac{n-j}{m}+K\right)+(n-j)s^2+U(n-j)+\left(\frac{n-j}{m}+K\right)^2\right],$$

or

$$U(n)=P(\bar{k}(n),0)U(n)+\sum_{j\in[\bar{k}(n),\ldots,\bar{k}(n)+5]}P(\bar{k}(n),j)\left[U(n-j)+\frac{j^2}{m^2}\right]-1-ms^2$$
$$=P(\bar{k}(n),0)U(n)+\frac{\tilde{\mathbb{V}}(\bar{k}(n))}{m^2}+\sum_{j\in[\bar{k}(n),\ldots,\bar{k}(n)+5]}P(\bar{k}(n),j)U(n-j)-ms^2.$$

For $n=1,\ldots,22$, $\bar{k}(n)$ alternatively $20,21$ (with periods $12,10$) and $(n-j)$ computed as $(n-j-1+44) \pmod{22}+1$, this gives a set of 22 linear equations, which, together with

$$\sum_1^{22}U(n)=0,$$

give $U(1),\ldots,U(22)$ and $s^2=0.2112800055\ldots$. This fits well with s^2 as computed in Section 4.2. Actually, the best numerical fit between $\mathbb{V}(n)$ and $ns^2+U(n)$ (with $U(n)$ just computed) appears with $s_3^2=0.2137$ and is given in Figure 13. Given that we only used $n=1000$, this is quite good.

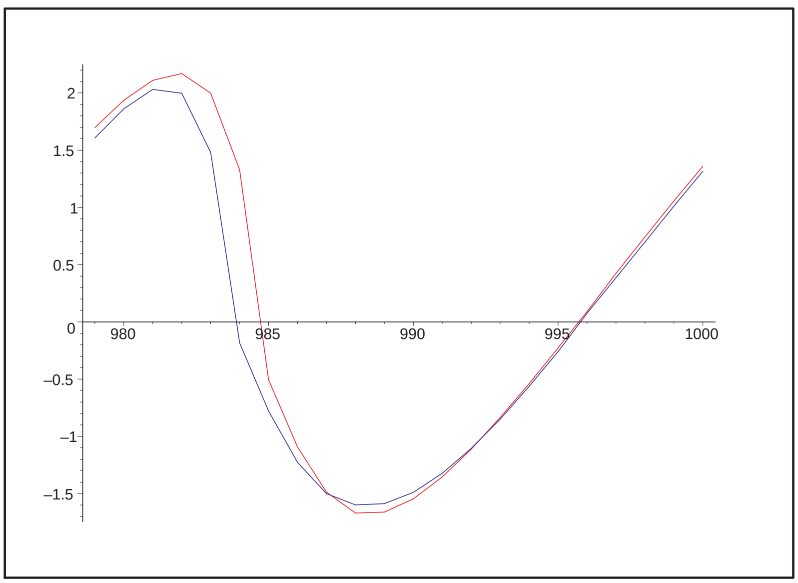

Figure 13: $\mathbb{V}(n)-ns_3^2$ (red), $U(n)$ (blue)

5 Strategy 1: total number of runs and total number of tosses.

If we want to minimize the mean number of tosses, always using the constant threshold k, this amounts to minimize $\frac{n}{\bar{J}(k)}.\bar{R}(k)$, minimum for $k=1$, with value $0.3n$. This can be explained as follows: $\bar{J}(k)$

is slowly increasing up to $k = 20$ and decreasing afterward, $\bar{R}(k)$ is more rapidly increasing to its asymptotic value 6.

Assume we always use $k = 20$. We know that the the asymptotic distribution of T_J, the total number of runs, is gaussian, with mean n/m_J, with $m_J = \bar{J}(20)$ and variance $n\mathbb{V}_J/m_J^3$, with $\mathbb{V}_J = \mathbb{V}_J(20)$ or, with Z_1 a classical Normal random variable,

$$T_J \sim n/m_J + \sqrt{n\mathbb{V}_J/m_J^3}Z_1.$$

Now, with $m_R = \bar{R}(20)$ and $\mathbb{V}_R = \mathbb{V}_R(20)$, we have that T_R, the total number of tosses, is asymptotically given by

$$T_R \sim [n/m_J + \sqrt{n\mathbb{V}_J/m_J^3}Z_1]m_R = \sqrt{n/m_J + \sqrt{n\mathbb{V}_J/m_J^3}Z_1}\sqrt{\mathbb{V}_R}Z_2,$$
$$\sim nm_R/m_J + \sqrt{n\mathbb{V}_J/m_J^3}m_R Z_1 + \sqrt{n/m_J}\sqrt{\mathbb{V}_R}Z_2 + \frac{1}{2}\sqrt{n\mathbb{V}_J/m_J^3}\sqrt{\mathbb{V}_R}Z_1 Z_2,$$

with again, Z_2 a classical Normal random variable, and by Sec.2.3, the correlation coefficient of Z_1, Z_2 is given by $C(20) = .6764271127$.

6 Two players. The asymptotic winning probability for the first player without taking the adversary's situation into account.

The distribution of the optimal cost $C(n)$ (number of runs) is given by

$$\Pi(n,j) = \mathbb{P}(C(n) = j) = \sum_{i \in [0, \bar{k}(n), \ldots, \bar{k}(n)+5]} P(\bar{k}(n), i)\Pi(n-i, j-1), \quad j \geq 1,$$
$$\Pi(n,0) = 1, k \leq 0, \quad \Pi(n,0) = 0, n \geq 1.$$

We have computed this probability for n up to 1000.

The comparison between $\Pi(1000, j)$ and $e^{-\left(\frac{j-1000/m}{\sqrt{1000}s}\right)^2/2}/\sqrt{2\pi 1000 s^2}$ (s^2 as computed in Section 4.2) is given in Figure 14.

OPEN PROBLEM 3: explain the slight discrepancy with the Gaussian.

The shape is identical when we use $\bar{E}(1000)$ and $\mathbb{V}(1000)$ instead of n/m and $1000s^2$.

The winning probability Pw for the first player, if both players use the same threshold strategy, *without* looking at each other position, is given by

$$Pw := \sum_k \Pi(n,k) \sum_{i \geq k} \Pi(n,i) = \frac{1}{2}\left[1 + \sum_k \Pi(n,k)^2\right], \tag{3}$$

and, by Euler-Maclaurin

$$Pw \sim \frac{1}{2}\left[1 + \int_{-\infty}^{\infty} e^{-2\left(\frac{k-n/m}{\sqrt{n}s}\right)^2/2}/(2\pi n s^2)dk\right] = \frac{1}{2}\left[1 + \frac{1}{2\sqrt{n\pi}s}\right] = \frac{1}{2}\left[1 + \frac{0.613713\ldots}{\sqrt{n}}\right].$$

For $n = 1000$, $Pw = 0.5097043\ldots$ as given by (3), and our asymptotics gives $0.5097037\ldots$

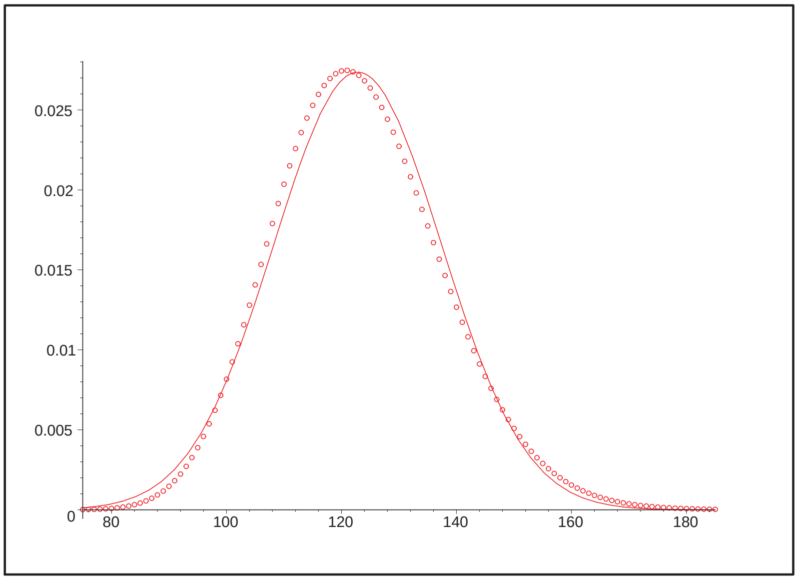

Figure 14: $\Pi(1000, j)$ (circle), $e^{-\left(\frac{j-1000/m}{\sqrt{1000s}}\right)^2/2}/\sqrt{2\pi 1000 s^2}$ (line)

7 Two players. Taking the adversary's situation into account

Let $P(i, j, k)$ be the player's probability of winning if the player's total score is i, the opponent's total score is j, and the player's present turn total is k. In the case where $i + k \geq n, P(i, j, k) = 1$ because the player can simply hold and win. In the general case where $0 \leq i, j < n$ and $k < n - i$, the probability of an optimal player winning is

$$\begin{aligned} P(i,j,k) &= \max(P(i,j,k,r), P(i,j,k,h)), \\ P(i,n,0) &= 0, \\ P(i,j,k) &= 1, i+k \geq n, \forall j, \\ P(n,j,0) &= 1. \end{aligned}$$

where $P(i, j, k, r)$ and $P(i, j, k, h)$ are the probabilities of winning if one tosses and holds, respectively. These probabilities are given by:

$$\begin{aligned} P(i,j,k,r) &= \frac{1}{6}\left((1 - P(j,i,0)) + \sum_{u=2}^{6} P(i,j,k+u)\right), \\ P(i,j,k,h) &= 1 - P(j,i+k,0). \end{aligned}$$

The probability of winning after rolling a 1 or holding is the probability that the other player will not win beginning with the next turn. Using iteration techniques, Nellner and Presser, [11] and [12], computed numerically, for $n = 100$, the solution as a surface which is the boundary between states where player 1 should roll (below the surface) and states where player 1 should hold (above the surface). This is given in Figure 15.

OPEN PROBLEM 4: Obtain a direct numerical solution for $P(i, j, k)$ or better an explicit form.

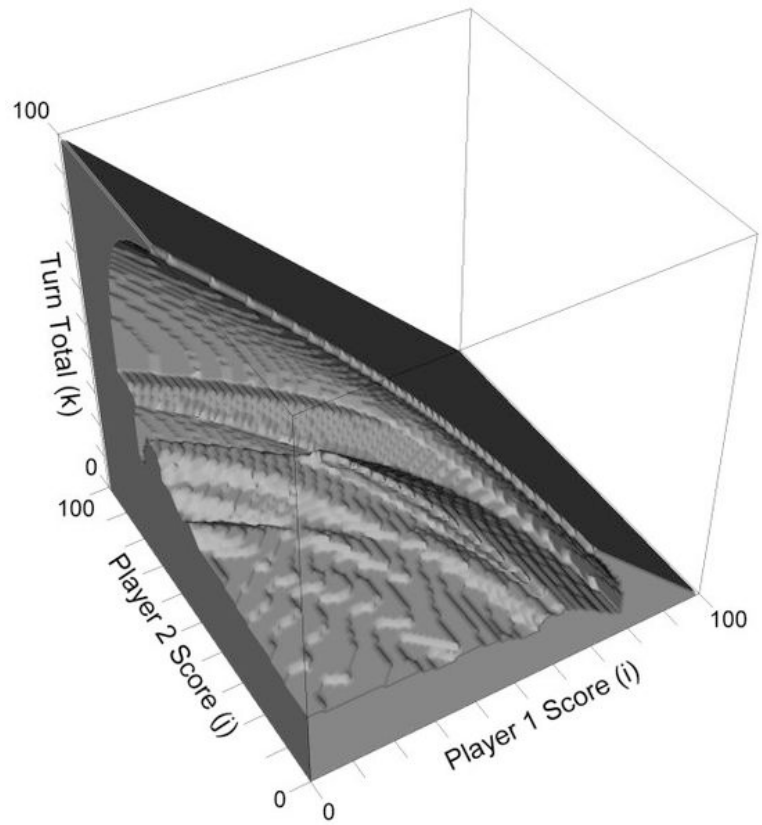

Figure 15: Two players optimal strategy

Let us start with

$$P(i,j,0,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{1}{6}\left(\sum_{u=2}^{6} P(i,j,u)\right),$$
$$P(i,j,0,h) = 1 - P(j,i,0).$$

For $i \geq n-2$, we have

$$P(i,j,0,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{5}{6},$$
$$P(i,j,0,h) = 1 - P(j,i,0).$$

and, for $i \geq n-2, \forall j$,

$$P(i,j,0) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{5}{6},$$

so, for $i, j \geq n-2$,

$$P(i,j,0) = \frac{6}{7}.$$

Now, consider the case $k = 1$. We have

$$P(i,j,1,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{1}{6}\left(\sum_{u=2}^{6} P(i,j,1+u)\right),$$
$$P(i,j,1,h) = 1 - P(j,i+1,0).$$

For $i \geq n-3$, we have

$$P(i,j,1,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{5}{6},$$
$$P(i,j,1,h) = 1 - P(j,i+1,0).$$

This allows computing $P(i,j,1)$ for $i,j \geq n-2$.

For $i = n-3$, we have

$$P(n-3,j,1,r) = \frac{1}{6}\left(1 - P(j,n-3,0)\right) + \frac{5}{6},$$
$$P(n-3,j,1,h) = 1 - P(j,n-2,0).$$

For $j \geq n-2$, we still need $P(j,n-3,0)$, this is computed below.

Next, consider the case $k=2$. We have

$$P(i,j,2,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{1}{6}\left(\sum_{u=2}^{6} P(i,j,2+u)\right),$$
$$P(i,j,2,h) = 1 - P(j,i+2,0).$$

For $i \geq n-4$, we have

$$P(i,j,2,r) = \frac{1}{6}\left(1 - P(j,i,0)\right) + \frac{5}{6},$$
$$P(i,j,2,h) = 1 - P(j,i+2,0).$$

This allows computing $P(i,j,2)$ for $i,j \geq n-2$.

For $i = n-3, j \geq n-2$, we have

$$P(n-3,j,2,r) = \frac{1}{6}\left(1 - P(j,n-3,0)\right) + \frac{5}{6},$$
$$P(n-3,j,2,h) = 1 - P(j,n-1,0).$$

So we need $P(j,n-3,0)$. Hence we consider

$$P(n-3,j,0,r) = \frac{1}{6}\left(1 - P(j,n-3,0)\right) + \frac{1}{6}P(n-3,j,2) + \frac{4}{6},$$
$$P(n-3,j,0,h) = 1 - P(j,n-3,0).$$

and, for $j \geq n-2$,

$$P(j,n-3,0,r) = \frac{1}{6}\left(1 - P(n-3,j,0)\right) + \frac{5}{6},$$
$$P(j,n-3,0,h) = 1 - P(n-3,j,0).$$

We have already three pairs of equations and three max-operators!

OPEN PROBLEM 5: Obtain an asymptotic form for $P(i,j,k)$.

Let us finally mention a report by Crocce and Mordecki [5], where another viewpoint is developed.

8 Another Strategy 6

In [9], Haigh and Roters propose a Strategy 6 slightly different from Strategy 1 (see also Roters [15]). If the player uses k at position n and reaches $k+5$, he continues until reaching at least $k+5+C$ (if possible) for some constant $C > 0$. This amounts to use $\tilde{P}(k,i)$ such that

$$\tilde{P}(k,j) = P(k,j), \quad k \leq j \leq k+4,$$
$$\tilde{P}(k,k+5) = 0,$$
$$\tilde{P}(k,0) = P(k,0) + P(k,k+5)P(C,0),$$
$$\tilde{P}(k,k+5+C+\delta) = P(k,k+5)P(C,C+\delta), \quad \delta = 0,\ldots,5.$$

The authors show that this gives a better value then $\bar{E}(n), \bar{E}_6(n)$, say, if we choose Strategy 6, for $n = 53, k(53) = 17$ (as in Strategy 1), and $C = 4$, and for $n = 75, k(75) = 18$ (as in Strategy 1), and $C = 5$.

They conjecture (using only $n = 200$) that these are the only values where it is better to use Strategy 6. We have checked this conjecture for n up to 1000. If we want to use a Strategy 7 similar to Strategy 1, we observe that the mean $\tilde{G}(k) = \sum_j \tilde{P}(k,j)j$ is maximum for $k = 20, C = 1$ and gives $\tilde{G}(20) = 8.127484455\ldots$. But $\bar{J}(20) - \tilde{G}(20) = 0.014310439\ldots$ So Strategy 7 will never beat Strategy 1 for large n.

9 Ferguson's solution (private communication)

Ferguson proposes to look at the problem as game with some payoff. This will allow an easy proof of the convergence of the numerical iteration procedure. First of all, it is convenient to count down rather than up. More precisely, we assume that both players start with scores of n and are trying to reduce their scores to zero. So let x (resp. y) denote the number of points player A (resp. B) needs to finish. In the notations of Sec. 7, $x = n - i, y = n - j$. Secondly, the formulae look simpler if we use a payoff function of $+1$ for a win and -1 for a loss, and try to maximize the expected payoff. The probability of win is then one-half of one more than the expected payoff, Y say. Indeed,

$$\mathbb{E}(Y) = \mathbb{P}(win) - \mathbb{P}(loss) = 2\mathbb{P}(win) - 1,$$
$$\mathbb{P}(win) = \frac{1 + \mathbb{E}(y)}{2}.$$

Let $v(x,y)$ represent the expected payoff to A under optimal play of both players, if A has x to go, B has y to go and it is A's turn. This corresponds to $v(x,y) = 2P(n-x, n-y, 0) - 1$. Suppose we have a situation with x to go for A and y to go for B, and suppose that A has rolled the dice a few times (without rolling a 1) with a total so far equal to k. If A stops after this, the situation would be changed to one in which A has $x - k$ to go, B has y to go, and it is B's turn. The value to A of this situation is $-v(y, x-k)$. Let $s = x - k$ and let $w(s,x,y)$ denote A's expected payoff in this situation under optimal play of both players. Note that $v(x,y) = w(x,x,y)$ and $w(s,x,y) = 2P(n-x, n-y, x-s) - 1$. Then

$$w(s,x,y) = \max\left(-v(y,s), -\frac{1}{6}v(y,x) + \frac{1}{6}\sum_{j=2}^{6} w(s-j,x,y)\right) \tag{4}$$

for $x \geq 1, y \geq 1$ and $-5 \leq s \leq x$. The boundary conditions on w are

$$w(s,x,y) = +1 \text{ for } x \geq 1, y \geq 1 \text{ and } s \leq 0. \tag{5}$$

We may solve these equations recursively. If we are given $v(x,s)$ for all $s < y$ and $v(s,y)$ for all $s < x$, we can use (4) and the corresponding equations with x and y interchanged to find $v(x,y)$ and $v(y,x)$. One method of accomplishing this is by iteration. Suppose we make a guess at $v(y,x)$. Then we can use (4) to find $w(s,x,y)$ recursively for $s = 1, 2, ..., x$. Then we put $v(x,y) = w(x,x,y)$ and use (4) with x and y interchanged to compute $w(s,y,x)$ recursively for $s = 1, 2, ..., y$. This gives us a new value for $v(y,x)$, namely $w(y,y,x)$, which may be used in the next iteration. To show that this method converges, it is sufficient to show that the mapping $v(x,y) \to w(y,y,x)$ given by this method has a fixed point, and that the derivative of the map is in absolute value less than 1.

Let $g(z)$ denote the map that maps $z = v(y,x)$ into $z' = w(x,x,y)$ using (4), and let $f(z')$ denote the corresponding map that maps $z' = v(x,y)$ into $w(y,y,x)$. The desired iteration is then $h(z) = f(g(z))$. There will exist a fixed point if the map is continuous and maps $[-1,1]$ into $[-1,1]$.

Let us prove that the map $h(z)$ is continuous non-decreasing from $[-1,1]$ into $[-1,1]$.

Since inductively, each $w(s,x,y)$ is continuous and non-increasing in $z = v(y,x)$, the map $g(z)$ is non-increasing. Similarly the map $f(z')$ is continuous and non-increasing. Therefore the map $h(z)$ is continuous and non-decreasing. Moreover, each $w(s, x, y)$ is equal either to $-v(y, s)$, or to the average of six numbers inductively between -1 and $+1$, and in either case is in the interval $[-1, 1]$.

10 Acknowledgements

We would like to thank F.T. Bruss who suggested this problem and with whom we had many illuminating discussions.

References

[1] J.J.A.M. Brands, F.W Steutel, and R.J.G. Wilms. On the maximum and its uniqueness for geometric random samples. *Statistics and Probability Letters*, 20:209–217, 1994. MR1294106.

[2] F.T. Bruss, M. Drmota, and G. Louchard. The complete solution of the competitive rank selection problem. *Algorithmica*, 22(4):413–447, 1998. MR1701621.

[3] F.T. Bruss and R. Grübel. On the multiplicity of the maximum in a discrete random sample. *Annals of Applied Probability*, 13(4):1252–1263, 2003. MR2023876.

[4] F.T. Bruss and C.A. O'Cinneide. On the number of maxima in a discrete sample. *Journal of Applied Probability*, 27:598–610, 1990. MR1067025.

[5] F. Crocce and E. Mordecki. Optimal minimax strategy in a dice game. Technical report, 2009. arXiv:0912.5518v1.

[6] E.G. Enns and E. Ferenstein. The horse game. *J. Oper. Research Society of Japan*, 28:51–62, 1985. MR0790787.

[7] E.G. Enns and E. Ferenstein. A curious recursion arising in game theory. *Utilitas Mathematica*, 30:219–228, 1986. MR0864823.

[8] E.G. Enns and E. Ferenstein. On a multi-person time-sequential game with priorities. *Sequential Analysis*, 6:239–256, 1987. MR0918908.

[9] J. Haigh and M. Roters. Optimal stopping on a dice game. *Journal of Applied Probability*, 37:1110-1116, 2000. MR1808874.

[10] G. Louchard and H. Prodinger. The number of elements close to near-records in geometric samples. *Quaestiones Mathematicae*, 29,4:447–470, 2006. MR2273945.

[11] T.W. Neller and C.G.M. Presser. Optimal play of the dice game pig. *The UMAP Journal*, 25(1):25–47, 2004.

[12] T.W. Neller and C.G.M. Presser. Pigtail: A pig addendum. *The UMAP Journal*, 26(4):443–458, 2005.

[13] T.W. Neller, I. Russel, and Z. Markov. Solving the dice game pig: an introduction to dynamic programming and value iteration. Technical report, Gettysburg College, 2005.

[14] D. M. Ramsey and K. Szajowski. Selection of a correlated equilibrium in Markov stopping games. *European Journal of Operations Research*, 184(1):185–206, 2008. MR2353354.

[15] M. Roters. Optimal stopping on a dice game. *Journal of Applied Probability*, 35:229–235, 1998. MR1622460.

[16] K. Szajowski. Double stopping by two decision makers. *Advances in Applied Probability*, 25:438–452, 1993. MR1212620.

[17] K. Szajowski. Markov stopping games with random priority. *ZOR- Mathematical Methods of Operations Research*, 39:69–84, 1994. MR1268636.