Lazy learning indirect control for discrete-time non-linear systems

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Although non linearity characterizes most real control problems, methods for analysis and control design are considerably more powerful and theoretically founded for linear systems than for non linear ones. However, a wide variety of approaches are present in literature which aim to extend linear methodologies to non linear cases. In this paper we propose a hybrid architecture for the indirect control of non linear discrete time plants from their observed input-output behaviour. This approach combines a memory based learning identification procedure with control techniques borrowed from conventional adaptive control. We show how memory-based methods are an effective way to employ linear methods in non linear cases and we discuss the stability properties of the resulting controller in a simplified case. Experimental results in the control of some non linear benchmarks are presented.

1. Problem formulation

This paper focuses on the design of indirect adaptive control algorithms for a class of discrete time dynamic systems whose equations of motion can be expressed in the form

\[ y(k) = f(y(k-1), \ldots, y(k-n_y), u(k-d), \ldots, u(k-d-n_u), e(k-1), \ldots, e(k-n_e)) + e(k) \]  \hspace{1cm} (1)

where \( y(k) \) is the system output, \( u(k) \) the input, \( e(k) \) is a zero-mean disturbance term, \( d \) is the relative degree and \( f(\cdot) \) is some non linear function. This model is known as the NARMAX model (Leontaritis and Billings, 1985).

Let us assume we have no physical description of the function \( f \) but a limited amount of pairs \([u(k), y(k)]\) from the observed input-output behaviour.

Defining the vector

\[ \varphi(k-1) = [y(k-1), \ldots, y(k-n_y), u(k-d), \ldots, u(k-d-n_u), e(k-1), \ldots, e(k-n_e)]^T \]

the system (1) can be written in the form

\[ y(k) = f(\varphi(k-1)) + e(k) \]

Let us now represent the unknown plant dynamics with a parametric mapping \( y(k) = f(\varphi(k-1), \theta^*) \), where \( \theta^* \) is the set of parameters of the real plant. An indirect control scheme (Astrom, Wittenmark, 1989) (Narendra, Annaswamy, 1989) combines a parameter estimator, which computes an estimate \( \theta \) of the unknown parameters, with a control law \( u(k) = K(\varphi(k), \theta^*) \) implemented as a function of the plant parameters. In the adaptive version, the estimator generates the estimate \( \theta(k) \) at each instant \( k \).
by processing the observed input-output behaviour. This estimate is then assumed to be a
specification of the real plant and used to compute the control law \( u(k)=K(q(k),\theta(k)) \) (certainty
equivalence paradigm).

In conventional adaptive control theory, to make the problem analytically tractable, the plant is
assumed to be a linear time invariant system with unknown parameters. The identification procedure
is then a recursive least square algorithm which updates at each time step the estimates of the linear
transfer function parameters. Given the linear transfer function polynomials, well established
techniques can be applied for control. In a generic non linear setting, both the identification and the
control procedures result much more complex. Let us now review some existing approaches for the
identification of a non linear discrete systems.

- **Linearization about an equilibrium point.** A point \( q_0 \) is called an *equilibrium point* of the plant (1)
  if \( q_0=f(q_0) \). Assuming that \( f \) is continuously differentiable at \( q_0 \), we can linearize the equation (1) by
  performing a multivariable Taylor series expansion. The outcome is a linear time invariant system
  that describes locally the non linear dynamics. Under some conditions (Slotine,Li, 1991) this linear
  model provides informations about the local stability properties of the global system. Further,
  starting from its parametric form, a linear controller can be designed to stabilize (1) around \( q_0 \). A
  major drawback of this procedure consists in a not accurate modeling when the system is operating
  away from the equilibrium point. An alternative is provided by the linearization along a trajectory.

- **Linearization about a trajectory.** The objective of this linearization is to study the behaviour of the
  system near a prescribed trajectory. Let \( q^*(k) \) satisfy the equation \( q^*(k+1)=f(q^*(k)) \). Assuming that
  \( f \) is continuously differentiable, the system (1) may be approximated near the trajectory \( q^*(k) \) by a
  linear time-varying system. Let us remark that the time varying property makes the control design
  process more difficult. Moreover, this approach requires the knowledge of the trajectory in advance,
  condition that, unfortunately, is not always satisfied.

- **Gain scheduling.** This method breaks the control design process in two steps. First, one designs
  local linear controllers based on linearizations at several different equilibrium points. Then, a scheme
  is implemented for interpolating (scheduling) the parameters at intermediate conditions. For a formal
  analysis of this approach see (Rugh, 1991).

- **Neural network modeling.** This approach treats the input-output function as a black box, modeled
  by a neural network which is generally trained by a procedure of backpropagation. There is in
  literature a huge amount of neurocontrol examples. An extensive introduction to this subject is
  provided by Narendra (Narendra, Parthasarathy, 1992), (Narendra, Mukhopadyhay, 1996).

- **Local model networks (LMN).** They extend the concept of operating point by introducing the
  notion of operating regime. An operating regime is a set of operating points where the system
behaves approximately linearly (Johansen, Foss, 1993) (Johansen, Foss, 1995). To each of them a validity region and a local description of the system behaviour are associated. The function f is then approximated with a set of interpolated local models. The use of local model networks in identification and control has been proposed by several authors (Murray-Smith, Hunt, 1995). One major advantage of this approach is the possibility to integrate a priori knowledge with parametric learning procedures. Related non-linear modeling approaches are Takagi Sugeno (1985) fuzzy inference systems and radial basis functions (Moody, Darken, 1989).

-Memory-based approach: This method does not build a global model of the unknown function f but defers any processing of the data until it is required a local description around an operating point \( q^* \) (lazy learning). The identification of the unknown function is performed giving the whole attention to the region surrounding the point where the estimation is required. The procedure consists essentially of the following steps:

- (i) for each query point \( q^* \), a set of neighbors is defined, each weighted according to some relevance criterion (e.g. the distance)
- (ii) a regression function f is chosen in a restricted family of parametric functions (in our case we will limit to linear functions)
- (iii) a local linear approximation of the function f in a neighborhood of \( q^* \) is obtained through a local weighted regression.

By defining the modeling problem as a local linear regression, tools and techniques can be easily imported from the field of linear statistical analysis. The most important and effective of these tools is the PRESS statistic (Myers, 1990), which is a simple, well-founded and economical way to perform leave-one-out cross validation (Efron, Tibshirani, 1993) and therefore to assess the performance in generalization of local linear models. Due to its short computation time which allows its intensive use, it is the key element of the memory-based approach to model identification. In fact, since the PRESS statistic can assess the performance of a given linear model, alternative models, each with a different configuration, can be validated and compared in order to select the best one. This same selection strategy is indeed exploited to select the training sub-set among the neighbors, as well as various structural aspects like the order of the model and the number of relevant features for the identification (Bersini et al., 1997). In two companion papers (Atkeson et al, 1997) present the local weighted regression as an effective tool for modeling from data and for non-linear control. In this paper the memory based approach is used not simply as a technique for identification, but as an effective linearization tool for non-linear control. Without any knowledge of the physical model, but with a limited amount of data, the lazy learning approach returns for each operating condition a linear model, that can be easily employed for stabilization and control design.
Let us now remark the main differences between the above mentioned approaches and the lazy learning technique.

Lazy learning vs. linearization: A linearization approach requires an a priori knowledge of the system, in order to have an analytical characterization of the equilibrium points. Lazy learning does not require an analytical model, but returns the best linear approximation that can be extracted from observed data. Linearization techniques return a local linear model whose range of validity is restricted to a neighborhood of the linearization points. Memory based technique can adaptively change the local description as a function of the current system state.

Lazy learning vs. gain scheduling. Here, the same remarks made about the linearization approach are valid. A further major difficulty in the gain scheduling approach is the selection of appropriate scheduling variables. The lazy learning makes instead an automatic and adaptive selection of the input features that best model the system dynamics in a certain operating condition.

Lazy learning vs. neural networks. The black box neural approach treats the issue of control design as a non linear problem, making difficult a formal analysis of the control system. The memory based approach reduces the problem to a case of linear control design where well-established theoretical results can be employed. Let us consider for example the problem of non minimum phase systems. The concept of minimum phase is significantly more complex in the non linear case and there are no available techniques to diagnostic and treat this configuration. However, we will show how this problem can be managed at a local level.

Lazy learning vs. local model networks. The two approaches share the common idea of decomposition of a difficult problem in simpler local problems. The main differences concern the model identification procedure. Local model networks aim to estimate a global description to cover the whole system operating domain, while memory based techniques focus simply on the current operating point. LMN result more time consuming in identification but faster in prediction. Further, when new data are observed, model update may require to perform the whole modeling process from scratch. On this matter, lazy learning takes an advantage from the absence of a global model: once a new input-output pair is observed, it is enough to update the database which stores the set of input-output pairs.

2. The LL-MVG indirect control architecture

Given a linear system, well-established techniques exist for control design. We have extensively discussed into the previous section how lazy learning is a way to reuse these techniques in a non linear context. In this paper we will use a minimum-variance (MV) control technique, generally adopted in linear self-tuning controllers (Astrom, Wittenmark, 1989). The MV control algorithm was
first formulated in (Astrom, 1967). Since then, the MV technique has had many practical applications and a significant theoretical development. The reasons for MV popularity lies in its simplicity and ease of interpretation and implementation. Let us consider a linear discrete-time process described in input-output form by the equation

\[ A(z)y(k) = z^{-d}B(z)u(k) + C(z)e(k) \]  

which we want to regulate about the steady state \( y=0 \).

The MV control problem can be stated as finding that control law which minimizes the variance of the output. The MV controlled closed loop system is stable only if \( B \) has all of its roots inside the unit circle (minimum phase). However, more complex formulation are available in the case of a tracking problem or in configuration of non minimum phase systems (Generalized MV or MVG). In these cases it is possible to select properly the controller parameters in order to make the closed loop system asymptotically stable.

Once the parameters of the process are available the controller is easily implemented. In Self Tuning Regulators, the estimation of the unknown parameters is performed by recursive least squares algorithm. In our architecture the estimation of the parameters is performed by the lazy learning (LL) estimator. Our proposed control algorithm (from now LL-MVG) is then as follows

<table>
<thead>
<tr>
<th>for each instant time ( k^* )</th>
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<tbody>
<tr>
<td>- the vector ( \varphi(k^*) ) is obtained from the observed input output behaviour</td>
</tr>
<tr>
<td>- the function ( f ) is linearized about ( \varphi(k^*) ) by the lazy learning algorithm</td>
</tr>
<tr>
<td>- the polynomials ( A,B,C ) in (2) are derived from the linearized model</td>
</tr>
<tr>
<td>- a MVG controller for (2) is designed to guarantee a closed loop behaviour with desired properties (stability, accuracy, speed..)</td>
</tr>
<tr>
<td>- the control law is applied to the original system (1)</td>
</tr>
<tr>
<td>- the LL database is updated by adding the new observation ( \varphi(k^*) )</td>
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</table>

Applications of this method to some benchmark control problems are presented in section 4.

3. The non linear model in a linear perspective

The non linear system (1) is then reduced to a linear configuration where parameters are changing with the state of the system. This means that the non-linear model can be written as an apparently linear model where parameters vary with the state of the system. This configuration recalls the linear parameter varying configuration introduced by (Shamma, Athans, 1992) in his analytical analysis of gain scheduling controllers, or the state-dependent models presented by Priestley (1981).

These models can be represented in the following form:

\[ A(\varphi^*)y(k) = z^{-d}B(\varphi^*)u(k) + C(\varphi^*)e(k) \]  

(3)
Let us now assume that there exists a LPV model which represents in a sufficiently accurate manner the non linear system (1). When a stable MVG control law is applied to (3), the closed loop transfer function becomes

\[ y(k) = A_c(\phi^*) e(k) \]  

(4)

where \( A_c \) is a stable transfer function.

If we consider the unforced system and we make the hypothesis of complete controllability and observability, the closed-loop system (4) may be put into the state-space form

\[ x(k+1) = F(\phi^*) x(k). \]  

(5)

This representation allows us to analyze the stability of our non linear controller.

Let us first report some results from Lyapunov theory on stabilization.

**Theorem 1** (Kuo, 1980). Consider a discrete system described by

\[ x(k+1) = f(x(k)) \]

where \( x(k) \in \mathbb{R}^n \), \( f(x(k)) \) is a \((n \times 1)\) function vector with the property that \( f(0) = 0 \) for all \( k \).

Suppose that there exists a scalar function \( V(x(k)) \) continuous in \( x(k) \) such that

(a) \( V(0) = 0 \)  
(b) \( V(x(k)) > 0 \) for \( x(k) \neq 0 \)  
(c) \( V(x(k)) \) approaches infinity as \( ||x(k)|| \to \infty \)  
(d) \( \Delta V(x(k)) < 0 \)

for \( x(k) \neq 0 \)

Then the equilibrium state \( x(k) = 0 \) for all \( k \) is asymptotically stable in the large and \( V(x(k)) \) is a Lyapunov function.

**Theorem 2** (Slotine, Li, 1991). A necessary and sufficient condition for a LTI system \( x(k+1) = Fx(k) \) to be strictly stable is that, for any symmetric positive definite matrix \( Q \), the unique matrix \( P \) solution of the Lyapunov equation \( A^T P + PA = -Q \) be symmetric positive definite.

We derive now a theorem for the stability property of a LL-MVG controlled system in the sense of Lyapunov, inspired to the results of (Tanaka, Sugeno, 1992) for the fuzzy control systems. Our demonstration is valid only for a simplified case but we consider them as a starting point for a stability theory of linear interpolated systems.

**Theorem 3.** Let us suppose that (i) the system (1) is completely controllable and observable, (ii) the system (1) can be put in the form (3), (iii) the approximation error of the lazy learning identifier can be neglected.

Then the equilibrium of the non linear system (1) controlled by the LL-MVG controller is globally asymptotically stable.

**Proof.**

Consider the scalar function \( V(x(k)) \) such that \( V(x(k)) = x^T(k)Px(k) \)
where $P$ is a positive definite matrix. This function satisfies the properties (a),(b),(c) of the Th. 1. Next,
\[
\Delta V(x(k)) = V(x(k+1)) - V(x(k)) = x^T(k+1)Px(k+1) - x^T(k)Px(k) = 
\]
From (5) we have for each $k$
\[
= x^T(k)F(\phi^*(k))PF(\phi^*(k)) x^T(k) - x^T(k)Px(k) = x^T(k)(F(\phi^*(k))PF(\phi^*(k))-P)F(k) = 
\]
with $F(\phi^*(k))$ asymptotically stable matrix for each $k$. From Th. 2 we have
\[
= -x^T(k)(Q(\phi^*(k))x(k) 
\]
with $Q(\phi^*(k))$ positive definite matrix for each $k$. Then we obtain (d) $\Delta V(x(k)) < 0$ for $x(k) \neq 0$

4. Simulation results

Example 1. We now apply our indirect control system on a non linear process of delay $d=2$ (Narendra, Parthasarathy, 1992):
\[
y(k+2) = \frac{1}{5} \left\{ y(k)y(k+1)\left[ y(k+1) + 2.5 \right] \right\} 
\]
The system dynamics is identified by a model having the following structure
\[
y(k+2) = f(y(k+1),y(k),u(k)) 
\]
The task is the tracking of a reference sinusoidal function. We employ an initial database of 400 points, obtained exciting the system with an uniformly random distributed input. In the first simulation the database of the lazy learning estimator is updated at each time instant (Fig. 1a). In the second simulation the database is the same for the whole simulation (Fig. 1b). It can be seen from the figures how the performance of the adaptive controller is largely superior. However, also the second control law results in a stable closed loop.

![Fig.1 a) LL-MVG adaptive control (database update) (ref solid line; system dotted line)](image1a)

![b) LL-MVG control with no database update (ref solid line; system dotted line)](image1b)
Example 2. Here we apply our control procedure to the following system
\[ y(k+1) = 2y(k) + u(k) + x(k)u(k). \]
This system is presented in (Atkeson et. al., 1995) as an example of non linear model where the linearized controller about the origin fails to control the system for \( x \) larger than 0.95, due to actions \( u \) too large.

In our simulation we regulate the system along a sinusoidal trajectory with an amplitude of 1.8. This shows how the approach is able to control non linear systems in regions far from the equilibrium points. We employed an initial database of 300 points which is adaptively updated during the simulation.

![Fig. 2. a) Output of the LL.MVG controlled system (ref solid line; system dotted line) b) control input](image)

5. Conclusion and future developments
The paper shows how the combination of model based techniques with techniques from conventional adaptive control is a promising approach for non linear control. The paper presents also a proof of stability in a simplified case. Further theoretical results should derive from the adoption of the linear parameter varying model as a formalism for local based approaches. Future developments should also concentrate on property of robustness of the approach. In fact, memory based techniques allow to have, along with a local model estimation, also a statistic specification of the uncertainty affecting the estimate. Since this happens at no additional computational cost, we intend to use this outcome to adapt on line the degree of robustness of the control system.

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