Resampling techniques in statistical modeling

INFO 154

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Combination of two estimators

Consider two unbiased estimators $\hat{\boldsymbol{\theta}}_1$ and $\hat{\boldsymbol{\theta}}_2$ of the same parameter θ

$$E[\hat{\boldsymbol{\theta}}_1] = \theta \qquad E[\hat{\boldsymbol{\theta}}_2] = \theta$$

having the same variance

$$\text{Var}\left[\hat{\boldsymbol{\theta}}_1\right] = \text{Var}\left[\hat{\boldsymbol{\theta}}_2\right] = v$$

and being uncorrelated, i.e. $\text{Cov}[\hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2] = 0$.

Combination of two estimators(II)

Let $\hat{\boldsymbol{\theta}}_{\text{cm}}$ be the combined estimator

$$\hat{\boldsymbol{\theta}}_{\text{cm}} = \frac{\hat{\boldsymbol{\theta}}_1 + \hat{\boldsymbol{\theta}}_2}{2}$$

This estimator has the nice properties of being unbiased

$$E[\hat{\boldsymbol{\theta}}_{\text{cm}}] = \frac{E[\hat{\boldsymbol{\theta}}_1] + E[\hat{\boldsymbol{\theta}}_2]}{2} = \theta$$

and with a reduced variance

$$\text{Var}\left[\hat{\boldsymbol{\theta}}_{\text{cm}}\right] = \frac{\text{Var}\left[\hat{\boldsymbol{\theta}}_1\right] + \text{Var}\left[\hat{\boldsymbol{\theta}}_2\right]}{4} = \frac{v}{2}$$

This trivial statistic computation shows that the simple average of two unbiased estimators with a non zero variance returns a combined estimator with reduced variance.

Combination of m estimators

Here, we report the general formula of the linear combination of a number m of estimators. Assume we want to estimate the unknown parameter θ by combining of a set of m estimators $\{\hat{\boldsymbol{\theta}}_j\}$, $j = 1, \ldots, m$.
Let

$$E[\hat{\boldsymbol{\theta}}_j] = \mu_j \quad \text{Var}\left[\hat{\boldsymbol{\theta}}_j\right] = v_j \quad \text{Bias}[\hat{\boldsymbol{\theta}}_j] = b_j$$

be the expected values, the variances and the bias of the m estimators, respectively.
We are interested in estimating θ by forming a linear combination

$$\hat{\boldsymbol{\theta}}_{\text{cm}} = \sum_{j=1}^{m} w_j \hat{\boldsymbol{\theta}}_j = w^T \hat{\boldsymbol{\theta}} \tag{1}$$

where $\hat{\boldsymbol{\theta}} = [\hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_m]^T$ is the vector of estimators and $w = [w_1, \ldots, w_m]^T$ is the weighting vector.

The mean-squared error of the combined system is

$$\text{MSE} = E[(\hat{\boldsymbol{\theta}}_{\text{cm}} - \theta)^2] = E[(w^T\hat{\boldsymbol{\theta}} - E[w^T\hat{\boldsymbol{\theta}}])^2] + (E[w^T\hat{\boldsymbol{\theta}}] - \theta)^2$$
$$= E[(w^T(\hat{\boldsymbol{\theta}} - E[\hat{\boldsymbol{\theta}}]))^2] + (w^T\mu - \theta)^2 =$$
$$= w^T\Omega w + (w^T\mu - \theta)^2$$

where Ω is a $[m \times m]$ covariance matrix whose ij^{th} term is

$$\Omega_{ij} = E[(\hat{\boldsymbol{\theta}}_i - \mu_i)(\hat{\boldsymbol{\theta}}_j - \mu_j)]$$

and $\mu = (\mu_1, \dots, \mu_m)^T$ is the vector of expected values.
The MSE of the combined estimator is minimized for

$$w^* = (\mu\mu^T + \Omega)^{-1}\theta\mu$$

Linear constrained combination

A commonly used constraint is

$$\sum_{j=1}^{m} w_j = 1, \quad w_j \geq 0, \quad j = 1, \ldots, m$$

Let us write w as

$$w = (u^T g)^{-1} g$$

where $u = (1, \ldots, 1)^T$ is an m-dimensional vector of ones,
$g = (g_1, \ldots, g_m)^T$ and $g_j > 0, \forall j = 1, \ldots, m$.
The constraint can be enforced in minimizing the MSE by using the Lagrangian function

$$L = w^T \Omega w + (w^T \mu - \theta)^2 + \lambda(w^T u - 1)$$

with λ Lagrange multiplier.

The optimum is achieved if we set

$$g^* = [\Omega + (\mu - \theta u)(m - \theta u)^T]^{-1} u$$

For these optimal weights

$$\min_w E[(\hat{\boldsymbol{\theta}}_{\mathsf{cm}} - \theta)^2] = \frac{1}{u^T(\Omega + (\mu - \theta u)(\mu - \theta u)^T)^{-1} u}$$

Note that the combined estimator is unbiased if the individual estimators are unbiased, which is the main reason for employing the constraint $\sum_{j=1}^{m} w_j = 1$. With unbiased estimators we obtain

$$g^* = \Omega^{-1} u$$

and with uncorrelated estimators

$$g_j^* = \frac{1}{\mathsf{v}_j} \qquad j = 1, \ldots, m$$