Improved Approximation Bounds for Edge Dominating Set in Dense Graphs

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Abstract. We analyze the simple greedy algorithm that iteratively removes the endpoints of a maximum-degree edge in a graph, where the degree of an edge is the sum of the degrees of its endpoints. This algorithm provides a 2-approximation to the minimum edge dominating set and minimum maximal matching problems. We refine its analysis and give an expression of the approximation ratio that is strictly less than 2 in the cases where the input graph has n vertices and at least $\epsilon\binom{n}{2}$ edges, for $\epsilon > 1/2$. This ratio is shown to be asymptotically tight for $\epsilon > 1/2$.

1 Introduction

While there exist sophisticated methods yielding approximate solutions to many NP-hard combinatorial optimization problems, the methods that are the simplest to implement are often the most widely used. Among these methods, greedy strategies are extremely popular and certainly deserve thorough analyses.

We study the worst-case approximation factor of a simple greedy algorithm for the following two NP-hard problems.

Definition 1 (MINIMUM EDGE DOMINATING SET)**.**
INPUT: *A graph* $G = (V, E)$.
SOLUTION: *A subset* $M \subseteq E$ *of edges such that each edge in* E *shares an endpoint with some edge in* M.
MEASURE: $|M|$.

Definition 2 (MINIMUM MAXIMAL MATCHING)**.**
INPUT: *A graph* $G = (V, E)$.
SOLUTION: *A subset* $M \subseteq E$ *of disjoint edges such that each edge in* E *shares an endpoint with some edge in* M.
MEASURE: $|M|$.

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It has been noted since long that MINIMUM EDGE DOMINATING SET (EDS) and MINIMUM MAXIMAL MATCHING (MMM) admit optimal solutions of the same size and that an optimal solution to EDS can be transformed in polynomial time into an optimal solution to MMM [13], the converse transformation being trivial.

The algorithm uses the degree of the edges, with the degree of an edge being the sum of the degrees of its endpoints. It iteratively removes the highest-degree edge and updates the graph accordingly, as shown in Algorithm 1. The algorithm

Algorithm 1 The greedy algorithm.

```
res ← ∅
while E(G) ≠ ∅ do
    e ← arg max_{e∈E(G)} deg_G(e)
    res ← res ∪ {e}
    for each edge f adjacent to e do
        E(G) ← E(G)\{f}
    end for
    E(G) ← E(G)\{e}
end while
return res
```

returns a maximal matching, which provides a solution to both our problems. The algorithm therefore guarantees exactly the same approximation ratios for the two problems.

It is well-known that any maximal matching M provides a 2-approximation for MMM, as each edge in the optimal solution can cover at most two edges of M. Our algorithm is thus clearly a 2-approximation algorithm and is expected to return small matchings as the greedy step always selects a high-degree edge. We however refine this analysis, and provide a tight approximation factor as a function of the density of the graph.

Our contributions

We provide a new bound on the approximation ratio of the greedy heuristic for our problems in graphs with at least $\epsilon\binom{n}{2}$ edges (ϵ-dense graphs). This bound is asymptotic to $1/(1-\sqrt{(1-\epsilon)/2})$, which is smaller than 2 when ϵ is greater than $1/2$. We further provide a family of tight examples for our bound. No algorithm for ϵ-dense graphs with a better approximation ratio than the one shown in this paper seems to be known.

Related works

The MMM and EDS problems go back a long way. Both problems are already referred to in the classical work of Garey and Johnson [6] on NP-completeness. Yannakakis and Gavril [13] then showed that EDS remains NP-hard when restricted to planar or bipartite graphs of maximum degree 3, and gave a polynomial-time algorithm for MMM in trees. Later, Horton et al. [8] and Srinivasan et al. [12] gave additional hard and polynomially solvable classes of graphs. More recently, Carr et al. [2] gave a $2\frac{1}{10}$-approximation algorithm for the weighted edge dominating set problem, a result which was later improved to 2 by Fujito et al. [5]. Finally, Chlebìk and Chlebìkovà [3] showed that it is NP-Hard to approximate EDS(and hence also MMM) within any factor better than $7/6$.

Another recent trend of research on approximation algorithms deals with expressing approximation ratios as functions of some density parameters [4, 7, 9], related to the number of edges, or the minimum and maximum degrees. Not many such results have yet been obtained for our problems. It was nevertheless shown in [1] that MMM and EDS are approximable within ratios that are asymptotic to $\min\{2, 1/(1-\sqrt{1-\epsilon})\}$ for graphs having at least $\epsilon\binom{n}{2}$ edges, and to $\min\{2, 1/\epsilon\}$ for graphs having minimum degree at least ϵn.

2 Analysis of algorithm 1

Definitions and notations

Let $G = (V, E)$ be a (simple, loopless, undirected) graph, with $V = \{v_1, \ldots, v_n\}$. Let OPT be a fixed optimal solution to MMM in G and let T be the set of endpoints in OPT. Let $M = \{e_1, \ldots, e_\mu\}$ be a set of μ edges returned by an execution of the greedy algorithm on G. We assume that these edges are ordered according to the order in which they were chosen by the algorithm.

The definition of the algorithm ensures that M is a maximal matching. Since M is a matching, at least one endpoint of each edge e_i belongs to T. Let us call $\{v_1, \ldots, v_{2\mu}\}$ the endpoints of the edges of M, with $e_i = v_{2i-1}v_{2i}$ and $v_{2i-1} \in T$. Since the matching M is maximal, the set of vertices $\{v_{2\mu+1}, \ldots, v_n\}$ forms a stable set. The set of vertices $V \backslash T$ also forms a stable set as the vertices in T are the endpoints of a maximal matching. Figure 1 shows an example with $\mu = 6$ and $|OPT| = 5$. Our assumptions on the ordering of the vertices ensure that a vertex has a higher index when it is included later (or never) in the heuristic solution and that the vertex with lowest index in e_i belongs to T.

As can be seen in Figure 1, there are two types of edges in M. Edges of the first type have only one endpoint in T. We let X be the set of these endpoints. Edges of the second type have both endpoints in T. Let a be the number of such

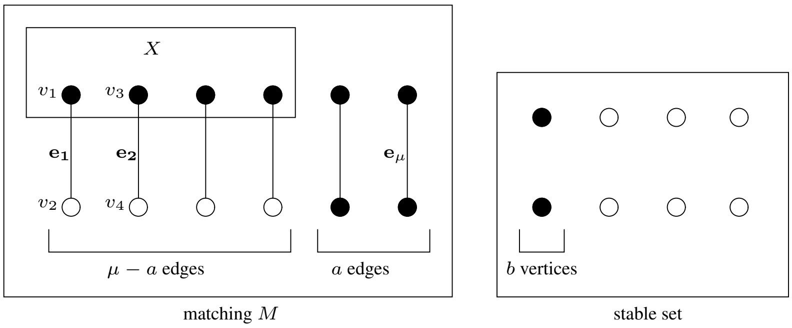


Fig. 1. An example with $\mu = 6$ and $|OPT| = 5$. Black vertices are the endpoints of the minimum maximal matching.

edges. Let finally b be the set of vertices of T outside M. Figure 1 also illustrates X, a and b. Note that in practice the two types of edges can be interleaved in M, whereas they are shown separated in the figure for the sake of clarity.

The approximation ratio is $\beta = \mu/|OPT|$. This quantity is fixed when M and OPT are given. In order to give an upper bound on β, we prove an upper bound on the number of edges in a graph when M and OPT are fixed. This bound is then inverted in order to obtain an upper bound on β as a function of the number of edges. Our theorems are expressed in terms of the *density* of our graphs, according to the following definitions. We define an *ϵ-dense graph* as a graph with at least $\epsilon\binom{n}{2}$ edges.

The following additional graph-theoretic notations will be useful. For any vertex set $W \subseteq V$ and vertex v, let $N_W(v)$ be the set of neighbors of v in set W and let $d_W(v) = |N_W(v)|$. Let an *anti-edge xy* be a pair of vertices x and y sharing no edge. Let $N_W^<(v_j)$ be the set of neighbors v_i of v_j with $i < j$ and $v_i \in W$, and let $d_W^<(v_j) = |N_W^<(v_j)|$. For any of these notations, the subscript W may be omitted when $W = V$. We also use the classical notation $G[X]$ for the subgraph of G induced by a vertex set X. Let $\bar{m}(G) = \binom{n}{2} - m(G)$ be the number of anti-edges in G. We omit the parameter G when it is clear from context. We define $G \times G'$, the *join* of graphs $G = (V, E)$ and $G' = (V', E')$ as a new graph that contains all the vertices and edges of G ang G' as well as all the possible edges joining both sets of vertices.

Upper bound

Our first lemma shows that a certain set of vertices have degree at most $|T|$. This result is then used by our second lemma in order to find an upper bound on the number of edges in the graph.

Lemma 1. *If $d_X^<(v_j) > 0$ for some vertex v_j, then $d(v_j) \leq |T|$.*

Proof. We call the vertices of T black vertices and the vertices outside of T white vertices. Let i be the smallest index such that $v_i \in X$ and $v_i v_j \in E$. Let $V_a^b = (v_a \dots v_b)$. Figure 2 illustrates these notations. We can express the degree

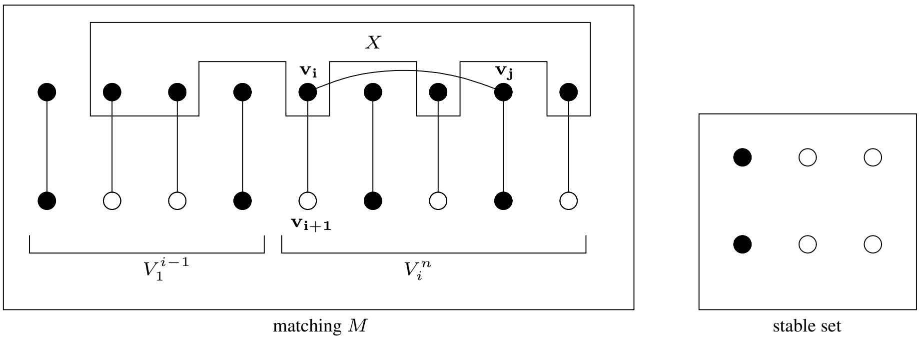


Fig. 2. Structure of the matching M. In this example, v_j was chosen inside the matching and outside X. Note that the lemma also allows v_j to be in the stable set or in X.

of v_j as:

$$d(v_j) = d_{V_1^{i-1}}(v_j) + d_{V_i^n}(v_j).$$

Since v_j has no neighbor in $V_1^{i-1} \cap X$, we have $d_{V_1^{i-1}}(v_j) \leq |V_1^{i-1} \backslash X|$ and therefore

$$d(v_j) \leq |V_1^{i-1} \backslash X| + d_{V_i^n}(v_j). \tag{1}$$

It can easily be seen that $|V_1^{i-1} \backslash X| = |V_1^{i-1} \cap T|$ and therefore

$$d(v_j) \leq |V_1^{i-1} \cap T| + d_{V_i^n}(v_j).$$

The greedy algorithm ensures that edge $v_i v_{i+1}$ has maximum degree in $G[V_i^n]$, and therefore

$$d(v_j) \leq |V_1^{i-1} \cap T| + d_{V_i^n}(v_{i+1}). \tag{2}$$

It is worth noticing that this is the only place in the whole proof of theorem 1 where this property is used. Finally, since v_{i+1} is a white vertex, it can only be adjacent to vertices in T, as the white vertices form a stable set. Therefore

$$\begin{aligned} d(v_j) &\leq |V_1^{i-1} \cap T| + |V_i^n \cap T| \\ &= |T|. \end{aligned}$$

□

The following result provides a lower bound on the number of anti-edges in the graph, hence an upper bound on the number of edges. Its proof uses counting arguments that heavily rely on the bound given in the previous lemma. Recall that a is the number of edges of M having both endpoints in T, and that b is the number of vertices of T that are outside M.

Lemma 2.

$$\bar{m} \geq 2\binom{n/2 - a - b}{2}$$

Proof. Let $\bar{d}_W(v)$, the *anti-degree* of v, be the number of anti-edges between v and vertices of W. Thus

$$\bar{d}_W(v) = \begin{cases} |W| - d_W(v) & \text{if } v \notin W \\ |W| - 1 - d_W(v) & \text{otherwise.} \end{cases}$$

We first define a family of vertex sets $\{X_i\}$ and show a lower bound on $\bar{m}$ as a function of the sizes of these sets. We call the vertices in (resp. outside) T *black* (resp. *white*) vertices.

The sets of vertices are the following (see Figure 3): X_1 and X_2 are defined as the black and white endpoints of $\mu - a - b$ arbitrary black-white edges of M. Sets X_3 and X_4 are obtained by splitting the b black vertices outside of M into two sets of equal sizes (rounding if necessary). Sets X_5 and X_6 are obtained by splitting the $n - 2\mu - b$ white vertices outside of M into two sets of equal sizes. Finally, X_9 and X_{10} are obtained by dividing the remaining b vertices of the matching into sets of equal sizes. We define $x_i = |X_i|$ for each set X_i.

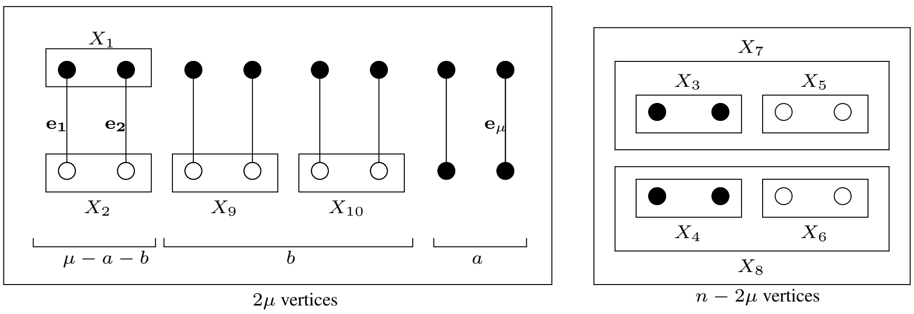


Fig. 3. Notations for the vertex sets.

We first show

$$\bar{m} \geq \binom{x_1}{2} + \binom{x_2}{2} + \binom{x_7}{2} + \binom{x_8}{2} + x_2x_9 + x_2x_5 + x_2x_{10} + x_2x_6 \quad (3)$$

Note that each set X_i except X_1 is stable, because it either contains only white vertices or only vertices outside M. This explains the second,third and fourth terms in the above sum. For each term of the form $x_i x_j$ in the sum, both X_i and X_j contain only white vertices, and therefore share no edge, since any set of white vertices in G is stable. Note that no anti-edge is counted twice, since our anti-edges involve vertices taken in and between disjoint vertex sets.

Concerning the additional number of $\binom{x_1}{2}$ anti-edges required, we use lemma 1 to prove that every edge inside X_1 is compensated for by an anti-edge between a vertex in X_1 and a vertex outside X_1. For each $v_j \in X_1$, we have:

$$\begin{aligned} d^{<}_{X_1}(v_j) &\leq d_{X_1}(v_j) \\ &= d(v_j) - d_{V \setminus X_1}(v_j). \end{aligned}$$

Applying lemma 1 yields:

$$d^{<}_{X_1}(v_j) \leq |T| - d_{V \setminus X_1}(v_j).$$

Using $|T| = \mu + a + b$ and $\mu \leq n/2$ yields

$$d^{<}_{X_1}(v_j) \leq n - (\mu - a - b) - d_{V \setminus X_1}(v_j).$$

Finally, since $|V \setminus X_1| = n - (\mu - a - b)$, the definition of the anti-degree yields

$$d^{<}_{X_1}(v_j) \leq \bar{d}_{V \setminus X_1}(v_j).$$

We now take sums over the elements of X_1:

$$\sum_{v_j \in X_1} d^{<}_{X_1}(v_j) \leq \sum_{v_j \in X_1} \bar{d}_{V \setminus X_1}(v_j).$$

Since the sets $N^{<}_{X_1}(v_j)$ corresponding to the values $d^{<}_{X_1}(v_j)$ in the above sum form a partition of the edges of $G[X_1]$, we have

$$m(G[X_1]) \leq \sum_{v_j \in X_1} \bar{d}_{V \setminus X_1}(v_j).$$

From the definition of $\bar{m}$, we have:

$$\binom{x_1}{2} \leq \bar{m}(G[X_1]) + \sum_{v_j \in X_1} \bar{d}_{V \setminus X_1}(v_j).$$

The above relation thus implies the existence of at least $\binom{x_1}{2}$ anti-edges involving vertices of X_1.

There remains to show that bound 3 is greater than $2\binom{n/2-a-b}{2}$. Plugging $x_2 = x_1$, $x_9 = x_3$, and $x_{10} = x_4$ into 3 yields:

$$\bar{m} \geq \binom{x_1}{2} + \binom{x_2}{2} + \binom{x_7}{2} + \binom{x_8}{2} + x_1x_3 + x_1x_5 + x_2x_4 + x_2x_6.$$

and therefore

$$\bar{m} \geq \binom{x_1}{2} + \binom{x_2}{2} + \binom{x_7}{2} + \binom{x_8}{2} + x_1x_7 + x_2x_8.$$

The desired result follows from repeated applications of the relation $\binom{x+y}{2} = \binom{x}{2} + \binom{y}{2} + xy$:

$$\begin{aligned}
\bar{m} &\geq \binom{x_1}{2} + \binom{x_2}{2} + \binom{x_7}{2} + \binom{x_8}{2} + x_1(x_7) + x_2(x_8) \\
&= \binom{|X_1 \cup X_7|}{2} + \binom{|X_2 \cup X_8|}{2} \\
&= \binom{x_1 + x_7}{2} + \binom{x_2 + x_8}{2} \\
&= \binom{\lfloor n/2 - a - b \rfloor}{2} + \binom{\lceil n/2 - a - b \rceil}{2} \\
&=^* \begin{cases} 2\binom{n/2-a-b}{2} & \text{if } n \text{ is even} \\ 2\binom{n/2-a-b}{2} + 1/4 & \text{otherwise.} \end{cases} \\
&\geq 2\binom{n/2 - a - b}{2}.
\end{aligned}$$

□

The theorem is essentially a consequence of this upper bound on the number of edges.

Theorem 1. *The approximation ratio of the greedy heuristic in ϵ-dense graphs with n vertices is at most*

$$\begin{cases} 2 & \textit{if } \epsilon \leq \frac{1}{2} + \frac{1}{n-1} \\ \left[1 - \frac{1}{2n} - \sqrt{\frac{1}{4n^2} + \left(1 - \frac{1}{n}\right)\frac{(1-\epsilon)}{2}}\right]^{-1} & \textit{otherwise.} \end{cases}$$

$$\longrightarrow_{n\to\infty} \begin{cases} 2 & \textit{if } \epsilon \leq \frac{1}{2} \\ \left[1 - \sqrt{\frac{1-\epsilon}{2}}\right]^{-1} & \textit{otherwise.} \end{cases}$$

Proof. We know from lemma 2 that $\bar{m} \geq 2\binom{n/2-a-b}{2}$. Simple algebra using $\beta = \mu/|OPT|$, $2|OPT| = \mu + a + b$ and $\mu \leq n/2$ implies

$$a + b \leq \frac{n}{2}\left[\frac{2-\beta}{\beta}\right].$$

and therefore

$$\bar{m} \geq 2\binom{n/2 - \frac{n}{2}\left[\frac{2-\beta}{\beta}\right]}{2} = 2\binom{n\left(\frac{\beta-1}{\beta}\right)}{2}. \tag{4}$$

Let $x = (\beta - 1)/\beta$. We would like to express the above inequality as an upper bound on β, i.e. on x. The inequality can now be written as

$$f(x) = n^2x^2 - nx - \bar{m} \leq 0.$$

Differentiating f with respect to x shows that f decreases when $x < \frac{1}{2n}$ and increases when $x > \frac{1}{2n}$. The value of $f(x)$ can therefore only be negative when $x^- \leq x \leq x^+$, where x^- and x^+ are the roots of $f(x)$. Solving the second-order equation $f(x) = 0$ yields

$$x^- = \frac{1}{2n} - \sqrt{\frac{1}{4n^2} + \frac{\bar{m}}{n^2}}$$

and

$$x^+ = \frac{1}{2n} + \sqrt{\frac{1}{4n^2} + \frac{\bar{m}}{n^2}}.$$

The value of x^- is always negative and thus $x^- \leq x$ brings us no additional knowledge on the ratio. Rewriting inequality $x \leq x^+$ yields

$$\frac{\beta - 1}{\beta} \leq \frac{1}{2n} + \sqrt{\frac{1}{4n^2} + \frac{\bar{m}}{n^2}}$$

and

$$\beta \leq \left[1 - \frac{1}{2n} - \sqrt{\frac{1}{4n^2} + \frac{\bar{m}}{n^2}}\right]^{-1}.$$

Reverting to m and setting $m \geq \epsilon\binom{n}{2}$ yields the desired result

$$\begin{aligned}\beta &\leq \left[1 - \frac{1}{2n} - \sqrt{\frac{1}{4n^2} + \left(1 - \frac{1}{n}\right)\frac{(1-\epsilon)}{2}}\right]^{-1} \\ &= \left[1 - O\left(\frac{1}{n}\right) - \sqrt{\frac{1-\epsilon}{2} + O\left(\frac{1}{n}\right)}\right]^{-1}.\end{aligned}$$

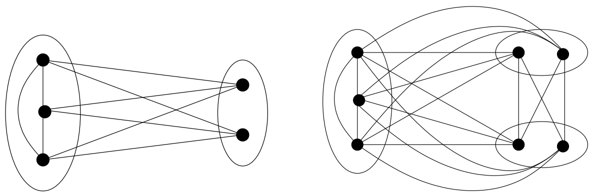
Direct algebraic manipulations show that

$$\left[1-\frac{1}{2n}-\sqrt{\frac{1}{4n^2}+(1-\epsilon)\left(\frac{1}{2}-\frac{1}{2n}\right)}\right]^{-1}<2$$
$$\iff \epsilon > \frac{1}{2}\left(\frac{n}{n-1}\right)$$
$$\iff \epsilon > \frac{1}{2}+\frac{1}{n-1}.$$

□

Tightness

Let $\zeta_{n,k} = K_{n-2k} \times K_{k,k}$, where K_{n-2k} is a complete graph with $n-2k$ vertices and $K_{k,k}$ a complete bipartite graph with two stable sets of size k (see Figure 4(b) for an example). Such a graph can be compared with the *complete split graph* $\Psi_{n,k}$ (see Figure 4(a)), which is defined as the join of a clique of size $n-k$ and an independent set of size k and is a tight example for the simpler greedy algorithm analyzed in [1].



(a) Complete split graph $\Psi_{5,2}$

(b) Tight example $\zeta_{7,2}$

Fig. 4. Tight examples.

Algorithm 1 always finds a perfect matching in $\zeta_{n,k}$. On the other hand, the following matching is clearly maximal: match k vertices of the clique with k vertices of one independent set, and match the remaining vertices of the clique among themselves. This is always possible when k and n are even and $k \leq n/3$ and yields a matching of size $(n-k)/2$. Therefore we have the following bound on the approximation ratio: $\beta = \mu/|OPT| \geq n/(n-k)$.

The number of edges of $\zeta_{n,k}$ is given by $m = \binom{n}{2} - 2\binom{k}{2}$ and therefore $k = \left(1 + \sqrt{1 + 4\left[\binom{n}{2} - m\right]}\right)/2$. We denote by ϵ the ratio $m/\binom{n}{2}$, i.e. the density of $\zeta_{n,k}$. From the above equality, we have $k = \left(1 + \sqrt{1 + 4\binom{n}{2}(1-\epsilon)}\right)/2$.

Plugging this equation into the inequality for β above yields

$$\beta \geq \left[1 - \frac{1}{2n} - \sqrt{\frac{1}{4n^2} + \left(1 - \frac{1}{n}\right)\frac{(1-\epsilon)}{2}}\right]^{-1},$$

which matches the upper bound on the ratio obtained in theorem 1.

The graphs $\zeta_{n,k}$ with n even, k even and $k \leq n/3$ are thus a family of tight examples for our bound on the ratio.

3 Conclusion

Several variants to Algorithm 1 could be devised. For example, one could decide to slightly alter Algorithm 1 by each time selecting the edge that has the highest degree in the original graph rather than the updated graph. This variant is interesting as it can easily be implemented in time $O(n + m)$ using counting sort. Another interesting variant is the one in which one does not select the highest degree edge, but rather the edge defined by the highest degree vertex and its highest degree neighbor. We claim that theorem 1 remains valid for these two variants. One should first notice that the only place in our analysis where explicit use is made of the strategy for choosing an edge is in Lemma 1. It is almost straightforward to adapt its proof for both variants.

Further, it can be checked that the bound of $1/\epsilon$ for graphs with minimum degree at least ϵn obtained for the maximal matching heuristic in [1] is also tight for Algorithm 1.

Finally, Algorithm 1 also provides a 2-approximation for MINIMUM VERTEX COVER by taking the endpoints of the maximal matching returned by the algorithm. The ratio obtained in theorem 1 is also valid for this problem by slight adaptations to the proofs. The analytical form of our asymptotic result compares interestingly with that of both the simplest [1] and the best known approximation algorithm for MINIMUM VERTEX COVER in ϵ-dense graphs [10] : $1/(1 - \sqrt{(1-\epsilon)/2})$ against $1/(1 - \sqrt{(1-\epsilon)})$ and $1/(1 - \sqrt{(1-\epsilon)/4})$.

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References

1. J. Cardinal, M. Labbé, S. Langerman, E. Levy, and H. Mélot. A tight analysis of the maximal matching heuristic. In *Proc. of The Eleventh International Computing and Combinatorics Conference (COCOON)*, LNCS, pages 701–709. Springer–Verlag, 2005.
2. R. Carr, T. Fujito, G. Konjevod, and O. Parekh. A 2 1/10-approximation algorithm for a generalization of the weighted edge-dominating set problem. *Journal of Combinatorial Optimization*, 5:317–326, 2001.
3. M. Chlebìk and J. Chlebìkovà. Approximation hardness of edge dominating set problems. *Journal of Combinatorial Optimization*, 11(3):279–290, 2006.
4. A.V. Eremeev. On some approximation algorithms for dense vertex cover problem. In *Proc. of SOR*, pages 58–62. Springer–Verlag, 1999.
5. T. Fujito and H. Nagamochi. A 2-approximation algorithm for the minimum weight edge dominating set problem. *Discrete Appl. Math.*, 118:199–207, 2002.
6. M. R. Garey and D. S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness*. Freeman and Company, 1979.
7. E. Halperin. Improved approximation algorithms for the vertex cover problem in graphs and hypergraphs. *Siam Journal on Computing*, 31:1608–1623, 2002.
8. J.D. Horton and K. Kilakos. Minimum edge dominating sets. *SIAM J. Discrete Math.*, 6:375–387, 1993.
9. T. Imamura and K. Iwama. Approximating vertex cover on dense graphs. In *Proc. of the 16th ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 582–589, 2005.
10. M. Karpinski and A. Zelikovsky. Approximating dense cases of covering problems. In P. Pardalos and D. Du, editors, *Proc. of the DIMACS Workshop on Network Design: Connectivity and Facilites Location*, volume 40 of *DIMACS series in Disc. Math. and Theor. Comp. Sci.*, pages 169–178, 1997.
11. H. Mélot. Facets Defining Inequalities among Graph Invariants: the system GraPHedron. Submitted, 2005.
12. A. Srinivasan, K. Madhukar, P. Navagamsi, C. Pandu Rangan, and M.-S. Chang. Edge domination on bipartite permutation graphs and cotriangulated graphs. *Inf.Proc. Letters*, 56:165–171, 1995.
13. M. Yannakakis and F. Gavril. Edge dominating sets in graphs. *SIAM J. Appl. Math.*, 38(3):364–372, 1980.