University of Warwick

Team

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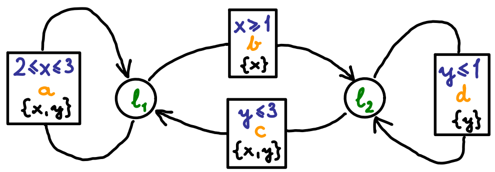
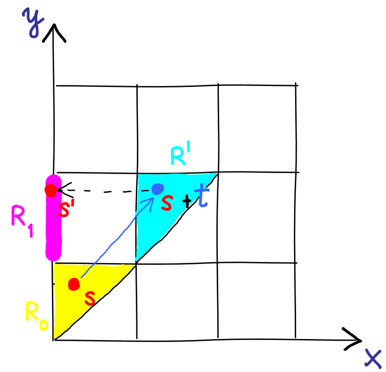
Topics

- Optimization and games on timed automata and hybrid systems
- Algorithms for infinite games
- Logics and automata for semi-structured data

Topic 1A: Optimization on timed automata

Concavely Priced Timed Automata





$\pi : S \times (\mathbb{R}_{\oplus} \times A) \longrightarrow \mathbb{R}$ is concavely priced if

$$(s,t) \longmapsto \pi(s,(t,a))$$

is concave on $\{(s,t) : s \in R_0 \text{ and } s+t \in R'\}$

Optimization problems on priced TA

Run: $r = \langle s_0 \xrightarrow[a_1]{t_1} s_1 \xrightarrow[a_2]{t_2} s_2 \longrightarrow \cdots \rangle$

Cost function: $\mathrm{Cost} : \mathrm{Runs} \longrightarrow \mathbb{R}$

Cost minimization: $\boxed{\begin{array}{l} \mathrm{Cost}_* : \mathrm{States} \longrightarrow \mathbb{R} \\ s \longmapsto \inf_{r \in \mathrm{Runs}} \mathrm{Cost}(r) \end{array}}$

- Reachability time: $r \longmapsto \sum_{i=1}^{\mathrm{STOP}(r)} t_i$
- Reachability price: $r \longmapsto \sum_{i=1}^{\mathrm{STOP}(r)} \pi(s_{i-1},(t_i,a_i))$
- Price-per-time average: $r \longmapsto \lim_{N\to\infty} \left(\sum_{i=1}^{N} \pi(s_{i-1},(t_i,a_i)) \Big/ \sum_{i=1}^{N} t_i \right)$
- Price-per-reward average: $r \longmapsto \lim_{N\to\infty} \left(\sum_{i=1}^{N} \pi(s_{i-1},(t_i,a_i)) \Big/ \sum_{i=1}^{N} \rho(s_{i-1},(t_i,a_i)) \right)$

Optimization problems on PTA

Given: PTA $\mathcal{T}$
initial state s
number D
decide: if $Cost_*(s) \leq D$

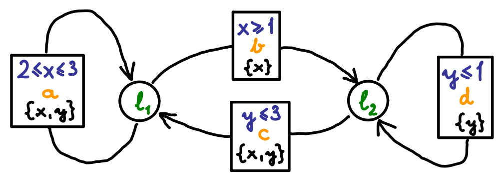
THM [BBBR'07]
Min reachability price problem is PSPACE-complete.

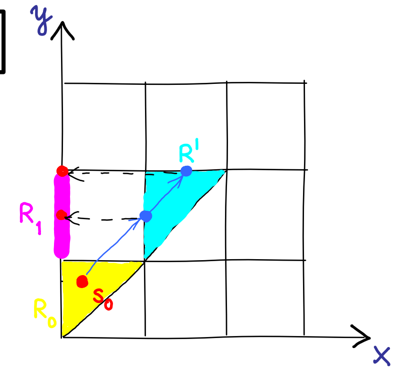
THM [BBL'08]
Min price-per-reward average problem is PSPACE-complete.

THM [JT'08]
Min cost problem is PSPACE-complete for all concave-regular cost functions on PTA.

Topic 1B: Games on timed automata

Boundary region automaton





In timed automaton $\mathcal{T}$: $s_0 \xrightarrow{\boxed{t}} s_0+t \xrightarrow[X]{b} s_1$

In **boundary** region automaton $\hat{\mathcal{T}}$: $R_0 \xrightarrow[X]{s_0+t \in \mathbf{BD}(R'),\ b} R_1$

Controller synthesis on TA: zero-sum games

Payoff function: $\text{Payoff}: \text{Runs} \to \mathbb{R}$

Lower value: $Val_*(s) = \sup_{\chi} \inf_{\mu} \text{Payoff}(\text{Run}(s,\mu,\chi))$

Upper value: $Val^*(s) = \inf_{\mu} \sup_{\chi} \text{Payoff}(\text{Run}(s,\mu,\chi))$

Determinacy: $Val := \boxed{Val_* = Val^*}$

"Competitive" optimization problems on PTA

Given: PTA $\mathcal{T}$
initial state s
number D
decide: if **Val** $(s) \leq D$

Thm [AM'99, BHPR'07, JT'07]
Reachability-time game problem is **EXPTIME-complete** on TA with ≥ 2 clocks.

Thm [BBM'06]
Existence of optimal strategies in reachability-price games, on PTA with ≥ 3 clocks, is **undecidable**.

Topic 2: Algorithms for infinite games

Average-time games

Thm [JT'08]
The **average-time** game problem is **EXPTIME-complete** on TA with ≥ 2 clocks.

Thm [JT'08]
In **average-time** games on TA both players have **ε-optimal strategies** that either
- are positional; or
- have **finite "regional" memory**.

Technique
Average-reward games on the boundary region automaton.

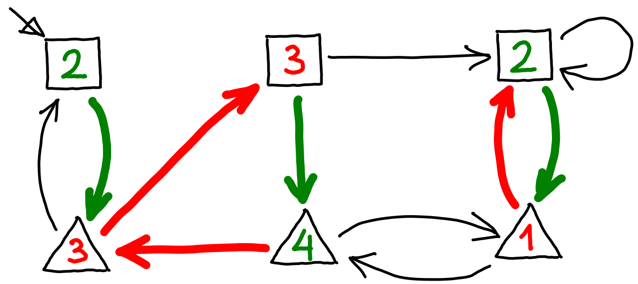
Parity games

Players:

n vertices, m edges, d priorities

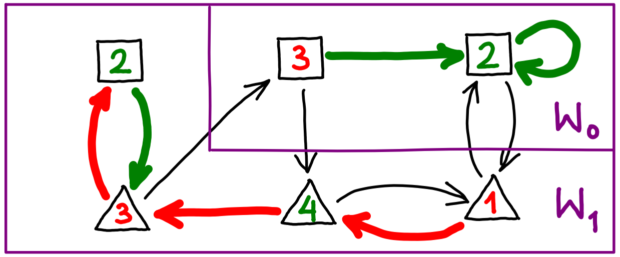
Even $= \square = 0$

Odd $= \triangle = 1$



Winner of an infinite play:
parity of the **highest priority** occurring **infinitely often**

Positional Determinacy



<u>Thm</u> Parity games are positionally determined.

<u>Cor</u> (Deciding the winner in) parity games is in NP $\cap$ co-NP

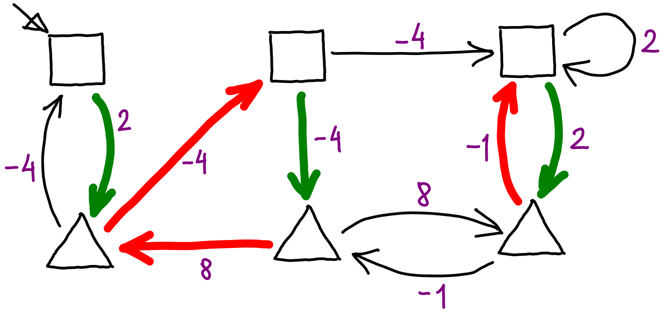
<u>"Solving" parity games</u>: computing $W_0 \cup W_1 = V$

Average-reward Games

Players:

Max = $\square$

Min = $\triangle$



Winner of an infinite play $\pi = \langle v_0, v_1, v_2, \dots \rangle$

Max if $\lim_{n \to \infty} \left(\frac{1}{n} \cdot \sum_{i=0}^{n-1} r(v_i, v_{i+1}) \right) \geqslant 0$

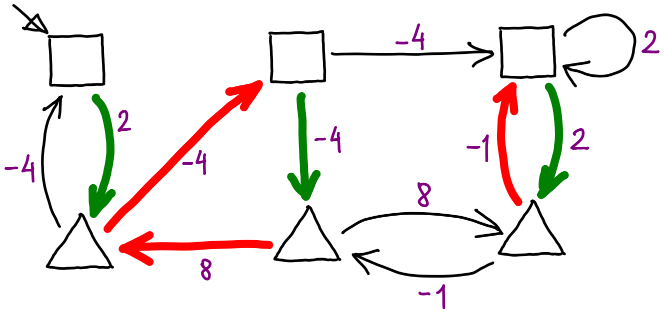
Discounted Games

$0 < \delta < 1$

Players:

Max = $\square$

Min = $\triangle$



Winner of an infinite play $\pi = \langle v_0, v_1, v_2, \dots \rangle$

Max if $\sum_{i=0}^{\infty} \delta^i \cdot r(v_i, v_{i+1}) \geqslant 0$

Bellman equations for discounted games

Opt(Γ):

$$v_s = \begin{cases} \max\limits_{(s,t) \in E} \left(r_{(s,t)} + \delta \cdot v_t \right) & \text{if } s \in S_{Max} \\ \min\limits_{(s,t) \in E} \left(r_{(s,t)} + \delta \cdot v_t \right) & \text{if } s \in S_{Min} \end{cases}$$

<u>Lemma</u> If $V \models$ Opt(Γ) then $V = Val^{\Gamma}$ and positional strategies choosing optimal successor are optimal

Linear Complementarity Problem

Given: $M \in \mathbb{R}^{n \times n}$

$q \in \mathbb{R}^n$

Find: $z, w \in \mathbb{R}^n$

such that

linear $\begin{cases} z \geq 0 \\ w \geq 0 \\ w = Mz + q \end{cases}$

complementarity $\{ \; z \perp w$ i.e., $z^T \cdot w = 0$

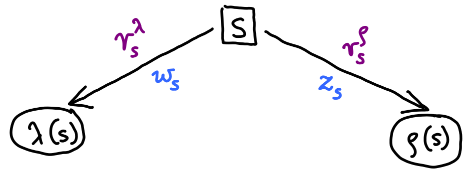
Fact

If $q \geq 0$

then $(\underset{z}{0}, \underset{w}{q}) \models LCP(M, q)$

Bellman equations and comlementarity

$$v_s = \max \left\{ r_s^\lambda + \delta \cdot v_{\lambda(s)} \, , \, r_s^\rho + \delta \cdot v_{\rho(s)} \right\}$$



slacks

$$\boxed{\begin{aligned} v_s &= w_s + r_s^\lambda + \delta \cdot v_{\lambda(s)} \\ v_s &= z_s + r_s^\rho + \delta \cdot v_{\rho(s)} \end{aligned}}$$

$$\boxed{z_s, w_s \geq 0 \qquad z_s \cdot w_s = 0}$$

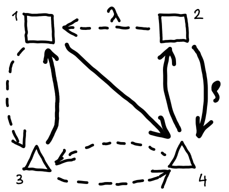
complementarity

Reduction to LCP: eliminate v

$$\boxed{\begin{aligned} & w = Mz + q \\ & w \geq 0 \perp z \geq 0 \end{aligned}}$$

$$\boxed{\begin{aligned} M &= (\hat{I} - \delta \cdot \widehat{T^\lambda}) \cdot (\hat{I} - \delta \cdot \widehat{T^\rho})^{-1} \\ q &= M(\hat{I} r^\rho) - (\hat{I} r^\lambda) \end{aligned}}$$

Example



$$\hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\widehat{T^\rho} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$(\hat{I} - \delta \cdot \widehat{T^\rho}) = \begin{pmatrix} 1 & 0 & 0 & -\delta \\ 0 & 1 & 0 & -\delta \\ \delta & 0 & -1 & 0 \\ 0 & \delta & 0 & -1 \end{pmatrix}$$

What is q ?

Lemma

Let v^ρ be the vector of values of strategy pair ρ, i.e.,

$$\boxed{v^\rho = r^\rho + \delta \cdot T^\rho \cdot v^\rho}$$

Then:

$$\boxed{q = \hat{I} \Big(\underbrace{(r^\rho + \delta \cdot T^\rho \cdot v^\rho)}_{\text{take "right" edge and then follow } \rho} - \underbrace{(r^\lambda + \delta \cdot T^\lambda \cdot v^\rho)}_{\text{take "left" edge and then follow } \rho} \Big)}$$

Algorithm 1: Strategy improvement revisited

Def. $s \in S_{\text{Max}}$ is *switchable* (for strategy pair ρ) if

$$\boxed{r_s^\rho + \delta \cdot v^\rho_{\rho(s)} \;<\; r_s^\lambda + \delta \cdot v^\rho_{\lambda(s)}}$$

1. Start with arbitrary ρ, λ.
2. <u>While</u> **there is** a *switchable* vertex (for ρ) <u>do</u>
3. Find ρ' such that:
   a) $\rho' \restriction S_{\text{Max}} = \rho \restriction S_{\text{Max}}$
   b) **no** $s \in S_{\text{Min}}$ is *switchable* for ρ'

   } $\rho' \restriction S_{\text{Min}}$ is a best response to $\rho \restriction S_{\text{Max}}$
4. Switch all (or some) switchable vertices in S_{Max}

Algorithm 2: Murty's "least-index" method

1. Fix a permutation ("indexing") of states.
2. <u>While</u> there is a *switchable* vertex <u>do</u>
3. Switch the *switchable* vertex with **least index**.

Thm Murty's algorithm terminates (in $\leq 2^n$ steps).

"Nested" strategy improvement: $^1\triangle \;\; ^2\triangle \;\; ^3\triangle \;\; ^4\square \;\; ^5\square$

New algorithm: $^1\square \;\; ^2\triangle \;\; ^3\triangle \;\; ^4\square \;\; ^5\triangle$

Algorithm 3: Cottle-Danzig

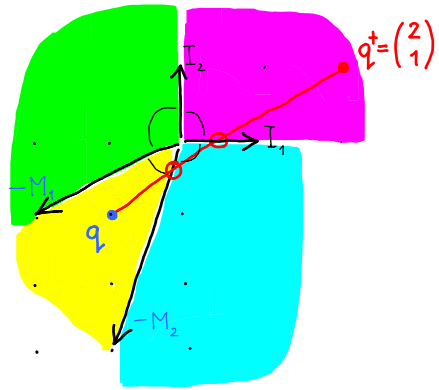
1. <u>For</u> $s = 1, 2, \dots, n$ <u>do</u>
2. <u>If</u> s is *switchable* <u>then</u>
3. Drive r_s^ρ until s becomes **indifferent**
4. <u>While</u> driving r_s^ρ, switch a vertex in $\{1, 2, \dots, s-1\}$ when it becomes **indifferent**
5. Switch s and restore r_s^ρ

Fact Cottle-Danzig terminates (in $\leq 2^n$ steps)

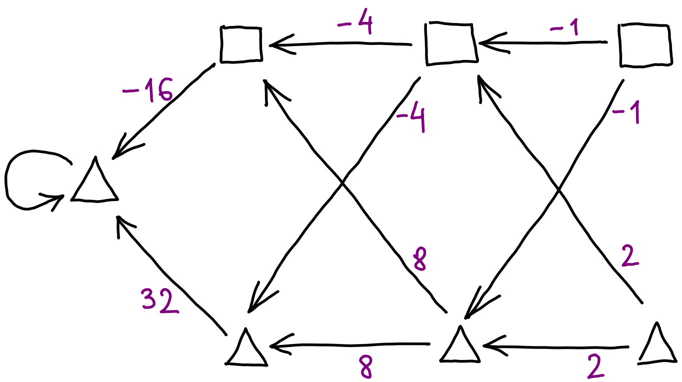
Algorithm 4: Lemke

1. Drive r^λ ("linearly"), until no vertex is switchable.
2. Drive r^λ ("linearly") back, until original r^λ is restored.
   <u>While</u> driving r^λ back, switch vertices when they become indifferent.

$$q = M(\hat{I}\, r^\rho) - (\hat{I}\, r^\lambda)$$



LEMKE IS EXPONENTIAL



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